Superconductors surrounded by normal materials

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We consider a generalized Ginzburg–Landau energy functional modelling a superconductor surrounded by a material in the normal state. In this model, the order parameter is defined in the whole space. We derive existence of a global minimizer in weighted Sobolev spaces for both square-integrable and constant-applied magnetic fields. We then prove boundedness and classical elliptic estimates for the order parameter, in order to study the loss of superconductivity for high applied magnetic fields. In two dimensions for the general case and in three dimensions for the case of constant permeability, we show the existence of an upper critical field above which the only finite-energy weak solutions are the normal states. For the three-dimensional case, we show that as the applied field tends to infinity, finite-energy weak solutions tend to the normal state.

1. Introduction

The Ginzburg–Landau theory describes superconductivity in a body through a complex-valued order parameter, \( \psi \), and a vector field \( A \), the magnetic potential. The standard theory introduces an energy functional defined on the domain, \( D_s \subset \mathbb{R}^n \), occupied by the superconducting material, and makes the hypothesis that the body is an equilibrium state of such functional. Classically, the superconductor either fills the whole space or is surrounded by a vacuum.

Following deGennes [5], the situation of a superconductor in contact with a normal material can be modelled by considering a non-zero boundary condition. In this approach, the order parameter is still defined only on the superconducting part of the domain and the important effects of superconducting electron pairs diffusing into the normal part are not represented (see [4]).

In order to model such effects, it is possible to use generalized Ginzburg–Landau equations, justified by physical considerations (see [4,10] and references contained therein), where the order parameter is also defined in the normal part of the domain, thus following the supercurrent through the body.

We start from the formulation of the generalized model as given in [4], and we study the existence and the properties of minimizers in various situations. Our aim is to frame the problem in a mathematical set-up which will allow for the rigorous proof of well-known physical phenomena, in particular, we would like to study the behaviour of the model in the presence of high applied magnetic fields (see [7]).
On account of the work in [10] and [4], we study the following non-dimensional generalized Ginzburg–Landau energy functional:

$$G(\psi, A, H_a) := \int_{D_s} \frac{1}{2} (1 - |\psi|^2)^2 \, dx + \int_{D_e} a|\psi|^2 \, dx$$

$$+ \int_{\mathbb{R}^n} \frac{1}{m} \nabla A |\psi|^2 \, dx + \int_{\mathbb{R}^n} \mu \left| \frac{1}{\mu} \text{curl} A - H_a \right|^2 \, dx, \quad (1.1)$$

where $\nabla A = ((i/\kappa_s) \nabla + A)$. The region $D_s = \mathbb{R}^n \setminus D_e$ is filled with a normal material, and we assume the set $D_s$ to be a bounded simply connected domain with smooth boundary $\partial D_s$. The constant $a > 0$ depends on the physical parameters of the two materials under consideration. The function $m > 0$ is piecewise constant, and also depends on the superconducting and normal parameters. Its value in $D_s$ is 1, and we will denote by $m_e$ its constant value in $D_e$. The permeability density, $\mu > 0$, rescaled to be 1 in $D_s$, is also piecewise constant; we denote by $\mu_e$ its value in the normal part.

The material constant $\kappa_s$ is the Ginzburg–Landau parameter of the superconducting material, while $H_a$ is the external applied magnetic field. Finally, $(1/\mu) \text{curl} A$ in $\mathbb{R}^3$ is the induced magnetic field and

$$j := 1 \left( \frac{i}{2\kappa_s} (\psi^* \nabla \psi - \psi \nabla \psi^*) - A|\psi|^2 \right)$$

in $\mathbb{R}^n$ is the supercurrent density.

Looking at the Euler–Lagrange equations of (1.1), for $H_a \in L^2_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^3)$, we are led to consider pairs $(\psi, A)$ such that

$$\psi \in H^1_{\text{loc}}(\mathbb{R}^n; \mathbb{C}) \equiv H^1_{\text{loc}}(\mathbb{R}^n), \quad A \in H^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$$

and that are weak solutions to

$$\nabla_A \left( \frac{1}{m} \nabla A \psi \right) - (1 - |\psi|^2)\psi \chi_{D_s} + a\psi \chi_{D_s} = 0 \quad \text{in } \mathbb{R}^n,$$

$$\nabla (\frac{1}{\mu} \text{curl} A - H_a) + \frac{1}{m} \left( \frac{i}{2\kappa_s} (\psi^* \nabla \psi - \psi \nabla \psi^*) + A|\psi|^2 \right) = 0 \quad \text{in } \mathbb{R}^n,$$

$$\left( \frac{1}{\mu} \text{curl} A - H_a \right) \in L^2(\mathbb{R}^n; \mathbb{R}^3). \quad (1.2)$$

Here, $\chi_{D_s}, \chi_{D_e}$ represent the characteristic functions for $D_s, D_e$, respectively.

As in the classical Ginzburg–Landau theory, the solutions to (1.2) are invariant under the gauge transformation

$$(\psi, A) \rightarrow (\psi', A'),$$

where

$$\psi' = \psi e^{i\kappa_s \eta}, \quad A' = A + \nabla \eta$$

for an arbitrary real-valued function $\eta \in H^1_{\text{loc}}(\mathbb{R}^n)$. 
We say that a weak solution to (1.2) is in the normal phase, and call it normal state, if \( \psi \equiv 0 \) in \( \mathbb{R}^n \). That is, \((\psi, A) = (0, A_N)\), where \( A_N \) satisfies weakly

\[
\text{curl} \left( \frac{1}{\mu} \text{curl} A_N - H_a \right) = 0 \quad \text{in} \quad \mathbb{R}^n,
\]

\[
\left( \frac{1}{\mu} \text{curl} A_N - H_a \right) \in L^2(\mathbb{R}^n; \mathbb{R}^3).
\]

We are first concerned with the existence of finite-energy weak solutions of (1.2) in the weighted Sobolev spaces, \( W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^n) \), studied in [1] (see also [8]). We consider existence of minimizers for \( H_a = h e \), where \( h \) is a positive constant and \( e \in \mathbb{R}^3 \) is a fixed unit vector, and for \( H_a \in L^2(\mathbb{R}^n; \mathbb{R}^3) \). In particular, we prove the following two statements.

If \( H_a \in L^2(\mathbb{R}^n; \mathbb{R}^3) \), then \( G(\psi, A, H_a) \) has a global minimizer in \( H^{1}_{\text{loc}}(\mathbb{R}^n) \times W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^n) \) (see proposition 2.7).

If \( H_a = h e \), then there exists a global minimizer \((\psi, A)\) of \( G(\psi, A, H_a) \) with \( \psi \in H^{1}_{\text{loc}}(\mathbb{R}^n) \), and \( A - A_N \in W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^n) \) (see lemma 2.4 and theorem 2.8).

We show that the order parameter of finite-energy weak solutions is bounded and decays exponentially at infinity (see lemma 3.1 and proposition 3.3).

We then analyse the behaviour of finite-energy weak solutions for the constant-applied magnetic field case. When \( n = 2 \), in the spirit of the results true for the classical Ginzburg–Landau energy [7], we show the existence of an upper critical field, for which the only finite-energy weak solutions of (1.2) are the normal states. This implies that when \( h \) is large, the only global minimizers of our energy functional are the normal states.

More precisely, we denote by \( \bar{h} \) the so-called upper critical field, defined as

\[
\bar{h} := \inf \{ h' : \text{normal states are the only finite-energy weak solutions to (1.2) for all } h > h' \},
\]

and we obtain the following result.

Let \( n = 2 \). Then, given \( m_e, \kappa_s, a, \mu_e \), we have \( \bar{h}(m_e, \kappa_s, a, \mu_e, D_s) < \infty \) (see theorem 4.7).

For particular values of the parameters, we also derive estimates on the size of \( \bar{h} \) (see (4.24) and (4.25)).

For \( n = 3 \), in the general case, we recover a result analogous to the one for superconducting films presented in [14].

Let \( n = 3 \). Then, given \( m_e, \kappa_s, a, \mu_e \) as \( h \to \infty \), any finite-energy weak solution tends to the normal state (see theorem 4.8).

In the three-dimensional case, we can prove \( \bar{h} < \infty \), and find estimates on its size, if the permeability is constant in \( \mathbb{R}^3 \), that is, \( \mu \equiv 1 \) in \( \mathbb{R}^3 \) (see §4.2).

The paper is organized as follows. The existence results are presented in §2 by modifications of the techniques used in [7, 8, 13, 15]. Section 3 is devoted to the derivation of estimates for finite-energy weak solutions. In §4, we study the case \( H_a = he \) for \( h \) large.
2. Existence of minimizers

The minimizers of our energy functional are weak solutions of a quasi-linear elliptic system defined in $\mathbb{R}^n$, with piecewise-constant coefficients. In general, the magnetic potential component of finite-energy states does not belong to any $W^{p,r}(\mathbb{R}^n;\mathbb{R}^n)$ space; in fact, it only has bounded curl in the $L^2$-norm. The natural space to consider for the physically relevant magnetic potentials appears thus to be related to the one needed to solve the Laplace equation in $\mathbb{R}^n$ (see [1]).

Yang [13, 15] considers the situation of a superconducting material filling the whole space $\mathbb{R}^n$ in the present of an applied field $\mathbf{H}_a \in L^2(\mathbb{R}^n;\mathbb{R}^3)$, by studying questions of existence and regularity of finite-energy minimizers of the classical Ginzburg–Landau energy functional. For the three-dimensional case [13], he shows existence for magnetic potentials in the closure of $C_0^\infty(\mathbb{R}^3;\mathbb{R}^3)$ with respect to the semi-norm $\|\nabla\|_{L^2(\mathbb{R}^2;\mathbb{R}^3)}$. Which is, in fact, the weighted Sobolev space $W_{0,0}^{1,2}(\mathbb{R}^n;\mathbb{R}^n)$ defined in [1]. In two dimensions, Yang [15] uses a different approach since the closure of $C_0^\infty(\mathbb{R}^2;\mathbb{R}^2)$ with respect to $\|\nabla\|_{L^2(\mathbb{R}^2;\mathbb{R}^2)}$ is not a space of distributions. To overcome this difficulty, he considers a direct variational method to prove existence in a subspace of an ad hoc defined Hilbert space. In fact, in [8], we show that this Hilbert space is equivalent to $W_{0,0}^{1,2}(\mathbb{R}^2;\mathbb{R}^2)$, and we prove existence of finite-energy minimizers in $W_{0,0}^{1,2}(\mathbb{R}^2;\mathbb{R}^2)$ using a unified approach between $n = 2$ and $n = 3$, thus showing that essentially the physically relevant states live in the same space as the solutions of the Laplace operator in $\mathbb{R}^n$.

The main difference in the existence proof between the two dimensions is due to the fact that the $L^2$-norm of the curl, in general, does not control the norm in $W_{0,0}^{1,3}(\mathbb{R}^2;\mathbb{R}^2)$ of divergence-free vectors. An extra step of changing the gauge to one for which this is true is then needed for $n = 2$.

The existence proofs for our generalized Ginzburg–Landau energy follow the above ideas, where some refinements are needed. In particular, we need some care when dealing with a constant-applied magnetic field. Also, we need to resolve the difficulty arising from the fact that while a uniform bound for the $L^2$-norm on a fixed bounded domain of the gradients of the order parameters of a minimizing sequence, classically is obtained by using an analogous bound for the $L^4$-norms; in the model under consideration, the energy yields just an $L^2$-bound in the normal part.

We denote by $W_{0,0}^{1,2}(\mathbb{R}^n;\mathbb{R}^n)$ the space of vector functions with components in

$$W_{0,0}^{1,2}(\mathbb{R}^2) = \left\{ u \in \mathcal{D}'(\mathbb{R}^2); \frac{u}{(1 + |x|^2)^{1/2}\ln(2 + |x|^2)} \in L^2(\mathbb{R}^2); \lambda \in \mathbb{N}^2 : |\lambda| = 1, D^\lambda u \in L^2(\mathbb{R}^2) \right\}$$

if $n = 2$, and in

$$W_{0,0}^{1,2}(\mathbb{R}^3) = \left\{ u \in \mathcal{D}'(\mathbb{R}^3); \frac{u}{(1 + |x|^2)^{1/2}} \in L^2(\mathbb{R}^3); \lambda \in \mathbb{N}^3 : |\lambda| = 1, D^\lambda u \in L^2(\mathbb{R}^3) \right\}$$

if $n = 3$. Here, $\mathcal{D}'(\mathbb{R}^n)$ denotes the dual space of $C_0^\infty(\mathbb{R}^n)$ (see [1] for details).
We remark that
\[ W_{0,0}^{1,2}(\mathbb{R}^3;\mathbb{R}^3) \equiv \dot{W}^{1,2}(\mathbb{R}^3) \equiv \dot{H}^1(\mathbb{R}^3), \]
with \( \dot{W}^{1,2}(\mathbb{R}^3) \) defined as in [13] and \( \dot{H}^1(\mathbb{R}^3) \) as in [7]. Further, \( W_{0,0}^{1,2}(\mathbb{R}^2;\mathbb{R}^2) \equiv \mathcal{H} \), with \( \mathcal{H} \) defined as in [15] (see [8]).

The space \( W_{0,0}^{1,2}(\mathbb{R}^n;\mathbb{R}^n) \) is a reflexive Banach space with respect to the norm
\[ \|B\|_{W_{0,0}^{1,2}(\mathbb{R}^n;\mathbb{R}^n)} \equiv \left\{ \left( \frac{B}{(1 + |x|^2)^{1/2}} \right)^2 \|B\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} + \|\nabla B\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \right\}^{1/2}, \]
if \( n = 2 \) and
\[ \|B\|_{W_{0,0}^{1,2}(\mathbb{R}^3;\mathbb{R}^3)} \equiv \left\{ \left( \frac{B}{(1 + |x|^2)^{1/2}} \right)^2 \|B\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \|\nabla B\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \right\}^{1/2}, \]
if \( n = 3 \), and an Hilbert space with respect to the induced scalar product.

Also,
\[ |B|_{W_{0,0}^{1,2}(\mathbb{R}^n;\mathbb{R}^n)} \equiv \|\nabla B\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \]
defines a semi-norm, for which a Poincaré-type inequality holds, that is,
\[ \|B\|_{W_{0,0}^{1,2}(\mathbb{R}^n;\mathbb{R}^n)} \leq C |B|_{W_{0,0}^{1,2}(\mathbb{R}^n;\mathbb{R}^n)}, \tag{2.1} \]
\[ \inf_{b \in \mathbb{R}^2} \|B + b\|_{W_{0,0}^{1,2}(\mathbb{R}^n;\mathbb{R}^n)} \leq C |B|_{W_{0,0}^{1,2}(\mathbb{R}^n;\mathbb{R}^n)}, \tag{2.2} \]

Since the space \( C_0^\infty(\mathbb{R}^n;\mathbb{R}^n) \) is dense in \( W_{0,0}^{1,2}(\mathbb{R}^n;\mathbb{R}^n) \), for \( B \) of \( W_{0,0}^{1,2}(\mathbb{R}^n;\mathbb{R}^n) \), one has
\[ \int_{\mathbb{R}^n} \|\nabla B\|^2 \, dx = \int_{\mathbb{R}^n} (|\text{div } B|^2 + |\text{curl } B|^2) \, dx, \tag{2.3} \]
from which the lemma below follows.

**Lemma 2.1.** Let \( A, B \in W_{0,0}^{1,2}(\mathbb{R}^n;\mathbb{R}^n) \), with \( \text{div } A = \text{div } B = 0 \), and
\[ \int_{\mathbb{R}^n} |\text{curl } A - \text{curl } B|^2 \, dx = 0. \]

Then \( A = B \) in \( W_{0,0}^{1,2}(\mathbb{R}^n;\mathbb{R}^3) \) if \( n = 3 \), while, if \( n = 2 \), we have \( A = B + b \) in \( W_{0,0}^{1,2}(\mathbb{R}^2;\mathbb{R}^2) \), for some constant vector \( b \in \mathbb{R}^2 \).

We consider separately the case \( H_a \in L^2(\mathbb{R}^n;\mathbb{R}^3) \) from the one \( H_a = h e \), and, without loss of generality, we suppose that \( e = e_3 \). We first study existence of normal states.

We recall that, by definition of weak solutions of (1.3), if \( A_N \) is a normal state, then
\[ \int_{\mathbb{R}^n} \left( \frac{1}{\mu} \text{curl } A_N - H_a \right) \cdot \text{curl } B \, dx = 0 \]
for any \( B \in H^1(\mathbb{R}^n;\mathbb{R}^n) \) with bounded support. Moreover, if \( H_a = h e_3 \), any normal state can be written as \( A_N = h a_N \), where \( a_N \) is a weak solution to
\[ \text{curl} \left( \frac{1}{\mu} \text{curl } a_N \right) = 0 \quad \text{in } \mathbb{R}^n, \quad \left( \frac{1}{\mu} \text{curl } a_N - e_3 \right) \in L^2(\mathbb{R}^n;\mathbb{R}^3). \]
In the rest of the paper, to select a divergence-free gauge, we need the following result.

**Lemma 2.2.** If \( g \in L^2(\mathbb{R}^2) \), there exists a \( u \in W_{0,0}^{1,2}(\mathbb{R}^2; \mathbb{R}^2) \) such that \( \text{curl} \, u = g e_3 \) and \( \text{div} \, u = 0 \). Moreover, \( u \) is unique up to a constant vector.

If \( g \in L^2(\mathbb{R}^3; \mathbb{R}^3) \) such that \( \text{div} \, g = 0 \) in \( \mathcal{D}'(\mathbb{R}^3) \), then there exists a unique \( u \in W_{0,0}^{1,2}(\mathbb{R}^3; \mathbb{R}^3) \) such that \( \text{curl} \, u = g \) and \( \text{div} \, u = 0 \).

**Proof.** For \( n = 2 \), the result is [8, lemma 3.1], while for \( n = 3 \) it is [7, lemma 3.1].

**Lemma 2.3.** Let \( H_N \in L^2(\mathbb{R}^n; \mathbb{R}^3) \). Then a normal state such that \( (0, A_N) \in W_{0,0}^{1,2}(\mathbb{R}^n; \mathbb{R}^n) \) and \( \text{div} \, A_N = 0 \) exists. Moreover, \( A_N \) is unique if \( n = 3 \), while \( A_N \) is unique up to a constant vector if \( n = 2 \).

**Proof.** Define the functional

\[
E(A) = \int_{\mathbb{R}^3} \left( \mu \left| \frac{1}{\mu} \text{curl} \, A - H_a \right|^2 + \text{(div} \, A)^2 \right) \, dx
\]

for \( A \in W_{0,0}^{1,2}(\mathbb{R}^3; \mathbb{R}^3) \). The functional \( E \) is non-negative, bounded above and convex (strictly convex for \( n = 3 \)) in \( W_{0,0}^{1,2}(\mathbb{R}^n; \mathbb{R}^n) \). Let \( A_j \) be a minimizing sequence. For \( \varepsilon > 0 \) fixed, there exists \( j_\varepsilon \) such that, for any \( j > j_\varepsilon \), it holds that

\[
E(A_j) \leq \inf E + \varepsilon.
\]

By (2.1), (2.2), since \( \mu \) is strictly positive, we have

\[
\|A_j\|_{W_{0,0}^{1,2}(\mathbb{R}^3; \mathbb{R}^3)} \leq C \sqrt{\inf E + \varepsilon} + C \|H_a\|_{L^2(\mathbb{R}^3; \mathbb{R}^3)}
\]

and

\[
\inf_{b \in \mathbb{R}^3} \|A_j + b\|_{W_{0,0}^{1,2}(\mathbb{R}^3; \mathbb{R}^3)} \leq C \sqrt{\inf E + \varepsilon} + C \|H_a\|_{L^2(\mathbb{R}^3; \mathbb{R}^3)}.
\]

If \( n = 3 \), we can then find a subsequence, still denoted by \( A_j \), that converges weakly to an element \( A \in W_{0,0}^{1,2}(\mathbb{R}^3; \mathbb{R}^3) \). Therefore, \( \partial A_j / \partial x_k \) converges weakly to \( \partial A / \partial x_k \) in \( L^2(\mathbb{R}^3; \mathbb{R}^3) \) for every \( k = 1, 2, 3 \), from which we conclude that

\[
E(A) \leq \lim \inf E(A_j) \leq \inf E + \varepsilon
\]

for every \( \varepsilon > 0 \), that is, \( A \) is a global minimizer for the energy functional \( E \).

If \( n = 2 \), we can find a sequence \( \{b_j\} \subset \mathbb{R}^2 \) such that

\[
\|A_j + b_j\|_{W_{0,0}^{1,2}(\mathbb{R}^2; \mathbb{R}^2)} \leq C \sqrt{\inf E + \varepsilon} + C \|H_a\|_{L^2(\mathbb{R}^2; \mathbb{R}^2)} + 1.
\]

We claim that the new sequence

\[
\{A_j := A_j + b_j\}_{j > j_\varepsilon}
\]

is still a minimizing sequence, since \( A_j \) is an element of \( W_{0,0}^{1,2}(\mathbb{R}^2; \mathbb{R}^2) \) and \( E(A_j) = E(A_j) \). Moreover, from the previous inequality, we have that \( \{A_j\}_{j > j_\varepsilon} \) is uniformly bounded in \( W_{0,0}^{1,2}(\mathbb{R}^2; \mathbb{R}^2) \). We can then proceed as for the case \( n = 3 \).

Consider a global minimizer \( A \in W_{0,0}^{1,2}(\mathbb{R}^n; \mathbb{R}^n) \) of the energy \( E \), and suppose \( \text{div} \, A \neq 0 \). If \( n = 2 \), we apply lemma 2.2 with \( g = \text{curl} \, A \cdot e_3 \), while if \( n = 3 \),
we take \( g = \text{curl} A \), and find \( \tilde{A} \in W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^n) \) such that \( \text{curl} \tilde{A} = \text{curl} A \) and \( \text{div} \tilde{A} = 0 \). But this would imply \( E(\tilde{A}) < E(A) \), which is in contradiction with the choice of \( A \). As a consequence, \( \text{div} A = 0 \), and since \( A \) satisfies (1.3), we conclude that \((0, A)\) is a normal state.

Uniqueness (up to a constant for \( n = 2 \)) follows from lemma 2.1, since if \( A, \tilde{A} \in W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^n) \) are two divergence-free normal states, then

\[
\int_{\mathbb{R}^n} \mu \left\| \frac{1}{\mu} \text{curl} A - \frac{1}{\mu} \text{curl} \tilde{A} \right\|^2 dx = 0
\]
due to the weak formulation of (1.3) and the properties of the weighted Sobolev spaces \( W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^n) \), and since \( \mu \) is strictly positive. Note that, for \( n = 3 \), uniqueness also follows from the strict convexity of the functional \( E \).

\[\text{Lemma 2.4. There exists a finite-energy normal state } (0, h a_N), \text{ with } a_N - \frac{1}{2} \mu \epsilon a_a \in W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^n), \text{ and } \text{div} a_N = 0 \text{ in } \mathbb{R}^n, \text{ where } a_a = (-x_2, x_1) \text{ for } n = 2 \text{ and } a_a = (-x_2, x_1, 0) \text{ for } n = 3. \text{ Moreover, } a_N \text{ is unique if } n = 3, \text{ while } a_N \text{ is unique up to a constant vector if } n = 2.\]

\[\text{Proof. If } n = 2, \text{ lemma 2.2 allows us to find a vector } v \in W^{1,2}_{0,0}(\mathbb{R}^2, \mathbb{R}^2), \text{ with } \text{curl} v = (\mu - \mu_e) \epsilon_3 \text{ and } \text{div} v = 0. \text{ Set } a_N = v + \frac{1}{2} \mu \epsilon a_a. \text{ Then } (0, h a_N) \text{ is a normal state that verifies the conclusions of the lemma. Uniqueness up to a constant vector follows similarly to what done in lemma 2.3. If } n = 3, \text{ our claim is proven in } [7, \text{ lemma } 3.3].\]

\[\text{Remark 2.5. The proof of lemma 2.4 implies that any normal state } (0, h a_N), \text{ with } a_N - \frac{1}{2} \mu \epsilon a_a \in W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2) \text{ and } \text{div} a_N = 0 \text{ in } \mathbb{R}^2, \text{ also satisfies } \text{curl} a_N = \mu \epsilon_3.\]

In § 4, we will use the following proprieties of the three-dimensional normal states, presented in [7, lemma 3.4].

\[\text{Lemma 2.6. For } n = 3, \text{ let } (0, a_N) \text{ be a normal state. Then } \text{curl} a_N \text{ is uniquely determined, } \text{curl} a_N \text{ is harmonic in } \mathbb{R}^3 \setminus \partial D_a, \text{ and } \text{curl} a_N \in C^{1,0}(D_a) \cup C^{1,0}(D_a). \text{ Moreover, if } \text{div} a_N = 0, \text{ then } a_N \in C^{1,0}(D_a) \cup C^{1,0}(D_a).\]

Existence of global minimizers of the energy functional \( G(\psi, A, H_a) \) in the space \( H^1_{loc}(\mathbb{R}^n) \times W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^n) \) can be proven if \( H_a \in L^2(\mathbb{R}^n; \mathbb{R}^3) \). In particular, one has the following result.

\[\text{Proposition 2.7. Suppose } H_a \in L^2(\mathbb{R}^n; \mathbb{R}^3). \text{ The energy functional } G(\psi, A, H_a) \text{ has at least one global minimizer } (\psi, A) \text{ in } H^1_{loc}(\mathbb{R}^n) \times W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^n).\]

\[\text{Proof. We provide a detailed proof only for the more involving case } n = 2, \text{ as following the proof for } n = 2, \text{ it is straightforward to see how theorem 2.1 in } [13] \text{ can be modified to derive the theorem for } n = 3.\]

For \( H_a \in L^2(\mathbb{R}^n; \mathbb{R}^3) \), our non-negative energy functional is bounded above, since

\[
\inf_{H^1_{loc}(\mathbb{R}^n) \times W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^n)} G(\psi, A, H_a) \leq G(0, 0, H_a) < \infty.
\]
Consider a sequence \((\psi_j, A_j)\) with \(\psi_j \in \mathcal{H}^1_{\text{loc}}(\mathbb{R}^2)\) and \(A_j \in W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2)\), and for which
\[
0 \leq G(\psi_j, A_j, H_a) \xrightarrow{j \to \infty} \inf G < \infty.
\]
We fix \(\varepsilon > 0\) and pick a \(j_\varepsilon\) such that, for any \(j > j_\varepsilon\),
\[
0 \leq G(\psi_j, A_j, H_a) < \inf G + \varepsilon.
\]
For each \(A_j\), we know that \(\text{curl } A_j \cdot e_3 \in L^2(\mathbb{R}^2)\). Thus we can apply lemma 2.2 and find an \(\tilde{A}_j \in W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2)\) such that \(\text{curl } \tilde{A}_j = \text{curl } A_j\) and \(\text{div } \tilde{A}_j = 0\).

Following the argument of theorem 2.9 in [9, p. 31], where we pick \(\omega_m \subset \mathbb{R}^2\) to be a ball of centre zero and radius \(m \in \mathbb{N}\), we conclude that there exists a function \(\eta_j \in H^1_{\text{loc}}(\mathbb{R}^2)\), independent of \(\omega_m\), such that \(\tilde{A}_j - A_j = \nabla \eta_j\). From \(\nabla \eta_j \in W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2)\), we also have \(\eta_j \in H^2_{\text{loc}}(\mathbb{R}^2)\), and thus \((\psi e^{i\eta_j}, \tilde{A}_j)\) is gauge equivalent to \((\psi_j, A_j)\), where now \(\text{div } \tilde{A}_j = 0\). Therefore, without loss of generality, from now on we will assume that \(\text{div } A_j = 0\).

By (2.2) and (2.3), we then have
\[
\inf_{b \in \mathbb{R}^2} \|A_j + b\|_{W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2)} \leq C \left( \int_{\mathbb{R}^2} |\text{curl } A_j|^2 \, dx \right)^{1/2},
\]
which, being \(\mu\) strictly positive, implies
\[
\inf_{b \in \mathbb{R}^2} \|A + b\|_{W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2)} \leq C(\mu) \left( \int_{\mathbb{R}^2} \frac{1}{\mu} |\text{curl } A_j|^2 \, dx \right)^{1/2}.
\]

We can then find a sequence \(b_j = (b^1_j, b^2_j) \in \mathbb{R}^2\) such that
\[
\|A_j + b_j\|_{W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2)} \leq C_1(\mu) \left[ \int_{\mathbb{R}^2} \frac{1}{\mu} |\text{curl } A_j - H_a|^2 \, dx \right]^{1/2} + \|H_a\|_{L^2(\mathbb{R}^2; \mathbb{R}^2)} + 1.
\]

Consider the function \(\eta_j = (b^1_j x_1, b^2_j x_2)\). Then \(\tilde{A}_j = A_j + \nabla \eta_j\), \(\tilde{\psi} = \psi_j e^{i\eta_j}\) is gauge equivalent to \((A_j, \psi_j)\), \(\text{div } \tilde{A}_j = 0\) and
\[
\|\tilde{A}_j\|_{W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2)} \leq C_1(\mu) \sqrt{\inf G + \varepsilon} + C_1(\mu) \|H_a\|_{L^2(\mathbb{R}^2; \mathbb{R}^2)} + 1. \tag{2.4}
\]
Hence \(\{\tilde{A}_j\}_{j \geq j_\varepsilon}\) is uniformly bounded in \(W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2)\) and (up to a subsequence)
\[
\tilde{A}_j \rightharpoonup \tilde{A} \in W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2) \text{ weakly in } W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2). \tag{2.5}
\]
Further, equation (2.4) tells us that \(\{\tilde{A}_j\}_{j \geq j_\varepsilon}\) is uniformly bounded in \(H^1(\omega; \mathbb{R}^2)\), for any \(\omega \subset \mathbb{R}^2\) bounded domain, from which we gather a uniform bound for \(\{A_j\}_{j \geq j_\varepsilon}\) in \(L^p\) for any \(p \geq 1\).

To continue our proof, we need a uniform bound for the \(H^1(\omega)\)-norm of \(\{\tilde{\psi}_j\}_{j \geq j_\varepsilon}\).

From the definition of the energy and for every \(j > j_\varepsilon\), we have that
\[
\int_{D_\varepsilon} \frac{1}{2} (1 - |\tilde{\psi}|^2)^2 \, dx + \int_{D_\varepsilon} a|\tilde{\psi}|^2 \, dx < \inf G + \varepsilon,
\]
and hence, for any bounded domain \( \omega \subset \mathbb{R}^2 \), using \(|\tilde{\psi}|^4 \leq 2(|\tilde{\psi}|^2 - 1)^2 + 1 \) and the fact that \( a \) is piecewise constant, we derive

\[
\int_\omega |\tilde{\psi}|^2 \, dx < C(|\omega \cap D_a|)(\sqrt{\inf G} + \varepsilon) + |\omega \cap D_a| + C(\inf G + \varepsilon).
\]

To find an analogous uniform bound for the gradients, we follow the classical approach, and, from the inequality

\[
|\nabla A\tilde{\psi}|^2 \geq \frac{1}{2}\frac{i}{\kappa_s} |\nabla \tilde{\psi}|^2 - |A\tilde{\psi}|^2 \geq \frac{1}{2}\frac{i}{\kappa_s} |\nabla \tilde{\psi}|^2 - |\tilde{\psi}|^4,
\]

we gather

\[
\int_\omega \frac{1}{2}\frac{i}{\kappa_s} |\nabla \tilde{\psi}|^2 \leq \int_\Omega |\nabla A\tilde{\psi}|^2 + \int_\omega |A\tilde{\psi}|^2 + \int_{\omega \cap D_a} (2(|\tilde{\psi}|^2 - 1)^2 + 1) + \int_{\omega \cap D_a} |\tilde{\psi}|^4.
\]

All terms on the above right-hand side are clearly uniformly bounded, either from the definition of the energy functional or from the bound for \( \{A_j\}_{j \geq 2} \), previously derived, except for the last integral. Since any finite-energy solutions belongs to \( H^1_{\text{loc}}(\mathbb{R}^n) \times H^1_{\text{loc}}(\mathbb{R}^n_n, \mathbb{R}^n) \), we observe that, for \( x \) with \( \psi(x) \neq 0 \), we have

\[
\frac{1}{m} |\nabla A\psi|^2 = \frac{1}{m} \left[ \frac{i}{2\kappa_s} (\psi^* \nabla \psi - \psi \nabla \psi^*) + A|\psi|^2 \right] |\psi|^{-1} \geq \frac{1}{m} |\nabla \psi| \geq 2(|\tilde{\psi}|^2 - 1)^2 + 1 + \int_{\omega \cap D_a} |\tilde{\psi}|^4.
\]

and \( |\nabla \psi| = 0 \) a.e. in the set where \( \psi = 0 \). We define the right-hand side of (2.6) to be equal to zero on \( \{ \psi = 0 \} \), obtaining

\[
\int_\Omega \frac{1}{m} |\nabla A\psi|^2 \, dx = \int_{\Omega} \frac{1}{m} \left[ \frac{i}{2\kappa_s} (\psi^* \nabla \psi - \psi \nabla \psi^*) + A|\psi|^2 \right] |\psi|^{-1} \, dx + \int_{\Omega} \frac{1}{m} |\nabla \psi|^2 \, dx.
\]

(2.7)

With \( m \) being piecewise constant, equation (2.7) together with the bound on the energy yields a uniform bound in \( H^1(\mathbb{R}^n) \) for \( |\psi_j| \). The Rellich–Kondrachov theorem then implies a uniform bound in \( L^4(\omega) \) for \( |\psi_j| \).

Considering a subsequence if necessary, we then conclude that, for any \( \omega \subset \mathbb{R}^2 \) bounded domain, it holds \( (\psi_j, A_j) \to (\tilde{\psi}, A) \) weakly in \( H^1(\omega) \times H^1(\omega; \mathbb{R}^2) \) and \( H^1_{\text{loc}}(\mathbb{R}^2) \times W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2) \), and strongly in \( L^p(\omega; \mathbb{C}) \times L^p(\omega; \mathbb{R}^2) \), for any \( p \geq 1 \).

Let \( G_\omega(\tilde{\psi}, A, H_a) \) be the restriction of \( G(\psi, A, H_a) \) to the domain \( \omega \). Then, for some \( j_\varepsilon^*(\omega) > j_\varepsilon \), we have

\[
G_\omega(\tilde{\psi}, A, H_a) \leq \liminf_{j \to \infty} G_\omega(\tilde{\psi}_j, A_j, H_a) = G_\omega(\tilde{\psi}_j^*(\omega), A_j^*(\omega), H_a) + \varepsilon = G_\omega(\psi_j^*(\omega), A_j^*(\omega), H_a) + \varepsilon \leq G(\psi_j^*(\omega), A_j^*(\omega), H_a) + \varepsilon \leq \inf G + 2\varepsilon.
\]

Letting \( \omega \to \mathbb{R}^2 \) yields the theorem. \( \square \)
Next, we state the existence result for a constant-applied magnetic field, \( H_a = h e_3 \). As in proposition 2.7, we sketch the proof only for the case \( n = 2 \).

**Theorem 2.8.** Let \( H_a = h e_3 \), \( h > 0 \). The energy functional \( G(\psi, A, H_a) \) has at least one global minimizer \( (\psi, A) \) such that \( \psi \in H^1_{\text{loc}}(\mathbb{R}^n) \) and \( A - \frac{1}{2} h \mu_e a_a \in W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^n) \), where \( a_a \) is defined as in lemma 2.4.

**Proof.** The energy functional \( G \) is bounded below, and, by lemma 2.4, the infimum of \( G \) for \( \psi \in H^1_{\text{loc}}(\mathbb{R}^n) \) and \( A - \frac{1}{2} h \mu_e a_a \in W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^n) \) is bounded above by \( \frac{1}{2} |D_a| \).

Let \( n = 2 \) and consider a sequence \( (\psi_j, A_j) \) such that \( \psi_j \in H^1_{\text{loc}}(\mathbb{R}^2) \), and \( A_j - \frac{1}{2} h \mu_e a_a \in W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2) \) for which

\[
0 \leq G(\psi_j, A_j, H_a) \xrightarrow{j \to \infty} \inf G < \infty.
\]

For every fixed \( \varepsilon > 0 \), there exists a \( j_\varepsilon \) such that, for any \( j > j_\varepsilon \),

\[
0 \leq G(\psi_j, A_j, H_a) < \inf G + \varepsilon.
\]

For each \( A_j \), we now know \( \text{curl}(A - \frac{1}{2} h \mu_e a_a) \cdot e_3 \in L^2(\mathbb{R}^2) \), and thus, by lemma 2.2, we can choose a \( v_j \in W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2) \) such that \( \text{curl} v_j = \text{curl}(A - \frac{1}{2} h \mu_e a_a) \) and \( \text{div} v_j = 0 \).

We define

\[
A_j = v_j + \frac{1}{2} h \mu_e a_a
\]

and notice that \( \text{curl} A_j = \text{curl} A_j, \text{div} A_j = 0 \) and \( A_j - \frac{1}{2} h \mu_e a_a \in W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2) \). Thus, again, we will assume \( \text{div} A_j = 0 \).

Following the steps of proposition 2.7, we can find a solution \( (\tilde{A}_j, \tilde{\psi}) \) to \( (A_j, \psi_j) \), still with \( \text{div} \tilde{A}_j = 0 \), and such that

\[
\|(\tilde{A}_j - \frac{1}{2} h \mu_e a_a)\|_{W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2)} \leq C(\mu) \sqrt{\inf G + \varepsilon} + 1. \tag{2.8}
\]

Hence \( \{\tilde{A}_j - \frac{1}{2} h \mu_e a_a\}_{j > j_\varepsilon} \) is uniformly bounded in \( W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2) \) and (up to a subsequence)

\[
\tilde{A}_j - \frac{1}{2} h \mu_e a_a \rightharpoonup B \in W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2) \quad \text{weakly in } W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2). \tag{2.9}
\]

Equation (2.8) also tells us that \( \{\tilde{A}_j\}_{j > j_\varepsilon} \) is uniformly bounded in \( H^1(\omega; \mathbb{R}^2) \), for any \( \omega \subset \mathbb{R}^2 \) bounded domain, from which we gather a uniform bound for \( \{\tilde{A}_j\}_{j > j_\varepsilon} \) in \( L^p \) for any \( p \geq 1 \).

Again, as in proposition 2.7, we can derive a uniform bound for the \( H^1(\omega) \)-norm of \( \{\tilde{A}_j\}_{j > j_\varepsilon} \) and conclude that up to subsequences \( (\tilde{\psi}_j, \tilde{A}_j) \rightharpoonup (\tilde{\psi}, \tilde{A}) \) weakly in \( H^1(\omega) \times H^1(\omega; \mathbb{R}^2) \) and strongly in \( L^p(\omega; \mathbb{C}) \times L^p(\omega; \mathbb{R}^2) \) for any \( p \geq 1 \), where \( \omega \subset \mathbb{R}^2 \) is a bounded domain.

Note that, since (2.9) implies

\[
\tilde{A}_j - \frac{1}{2} h \mu_e a_a \rightharpoonup B \in H^1(\omega; \mathbb{R}^2) \quad \text{weakly in } H^1(\omega; \mathbb{R}^2),
\]

for any \( \omega \) bounded domain, we have \( B = \tilde{A} - \frac{1}{2} h \mu_e a_a \). The theorem follows as in proposition 2.7.

We conclude the section with two lemmas, which will allow us to work with divergence-free weak solutions.
Lemma 2.9. Let \((\zeta, B)\) be a finite-energy weak solution to (1.2) and let \(H_a = he_3\). There is a gauge-equivalent solution \((\psi, A)\) and a normal state \(a_N\) such that \(\text{div } A = \text{div } a_N = 0\) and \(A - ha_N \in W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^n)\).

Proof. If \(n = 2\), set \(g = \text{curl}(B - \frac{1}{2}\mu_e a_a) \cdot e_3\), with \(a_a\) defined as in lemma 2.4. Then, by lemma 3.1 in [8], we can consider a vector \(v \in W^{1,2}_{0,0}(\mathbb{R}^2; \mathbb{R}^2)\) with \(\text{curl } v = \text{curl}(B - \frac{1}{2}\mu_e h a_a)\) and \(\text{div } v = 0\). The vector \(A = v + \frac{1}{2}\mu_e a_a\) has \(\text{div } A = 0\). Moreover, we have \(\text{curl } A = \text{curl } B\). Thus, following a standard argument (already used at the beginning of proposition 2.7), we can find a function \(\eta \in H^2_{\text{loc}}(\mathbb{R}^n)\) such that \(A - B = \nabla \eta\). Thus \((\psi = e^{i\kappa_0 \eta}, A)\) is gauge equivalent to \((\zeta, B)\). The lemma follows if we recall lemma 2.4.

If \(n = 3\), set \(g = \text{curl}(B - \frac{1}{2}\mu_e a_a)\) and proceed as above using lemma 3.1 in [7].

Lemma 2.10. Let \((\zeta, B)\) be a finite-energy weak solution to (1.2) and let \(H_a \in L^2(\mathbb{R}^n; \mathbb{R}^3)\). Then there exists a gauge-equivalent solution \((\psi, A)\) such that \(\text{div } A = 0\) and \(A \in W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^n)\).

Proof. Since \((\zeta, B)\) is of finite energy and \(\mu\) is strictly positive, we have \(\text{curl } B \in L^2(\mathbb{R}^n; \mathbb{R}^3)\). Therefore, we can set \(g = \text{curl } B \cdot e_3\) and apply lemma 3.1 in [8], if \(n = 2\). When \(n = 3\), we set \(g = \text{curl } B\) and apply lemma 3.1 in [7]. We then argue as in lemma 2.9 and find \(v \in W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^n)\), with \(\text{div } v = 0\) and \(\text{curl } v = \text{curl } B\). For \(A := v\), we can show (again as in theorem 2.8) that \(A = B + \nabla \eta\) with \(\eta \in H^2_{\text{loc}}(\mathbb{R}^n)\). Thus \((\psi = e^{i\kappa_0 \eta}, A)\) is the desired gauge-equivalent solution.

3. Bounds on finite-energy weak solutions

The weak formulation of (1.2) reads as follows:

\[
\int_{\mathbb{R}^n} \left( \frac{1}{m} (\nabla A \psi) \cdot (\nabla A \phi^*) - \frac{1}{2} |A|^2 \right) \, dx
\]

\[
+ \int_{D_a} \left( |\psi|^2 - 1 \right) \psi \phi^* \, dx
\]

\[
+ \int_{D_a} a \psi \phi^* \, dx = 0
\]

for any \(\phi \in H^1(\mathbb{R}^n; \mathbb{C})\) with bounded support, (3.1a)

\[
\int_{\mathbb{R}^n} \left[ \frac{1}{\mu} (\text{curl } A - H_a) \cdot \text{curl } B + \frac{1}{m} \left( \frac{i}{2\kappa_0} (\psi^* \nabla \psi - \psi \nabla \psi^*) + |A|^2 \right) \cdot B \right] \, dx = 0
\]

for any \(B \in H^1(\mathbb{R}^n; \mathbb{R}^n)\) with bounded support.

The physics behind the problem suggests that, for a minimizer \((\psi, A)\), the modulus of the order parameter \(\psi\) (being a renormalized density function) has to be at most one. Mathematically, this is a classical result that can also be proven for the generalized model we are considering.

Lemma 3.1. If \((\psi, A)\) is a finite-energy weak solution to (1.2), then \(|\psi| \leq 1\) a.e.

Proof. It is not difficult to see that the proof of lemma 3.1 in [13, p. 152] can be modified to derive the lemma. Due to the fact that the functions \(a\) and \(m\) are piecewise constant, \(D_a\) is bounded and \(|\psi| \in L^2(\mathbb{R}^2)\), for any finite-energy weak solution \((\psi, A)\).
In order to study the properties of the weak solutions of our model, we need to have control on the behaviour at infinity of the order parameter. We follow the standard approach and start by showing higher integrability of $\nabla A \psi$.

**Proposition 3.2.** If $(\psi, A)$ is a finite-energy weak solution to (1.2), with $H_0 = h e_3$ or $H_0 \in L^2(\mathbb{R}^n; \mathbb{R}^3)$, then $|\nabla A \psi| \in H^1(D_0)$. Moreover, $|\psi|^2 \to 0$ as $|x| \to \infty$.

**Proof.** We start by defining some suitable cut-off functions, which we will use in our proof. Let $d_\rho$ be the radius of a ball of centre zero, $B(0, d_\rho)$, for which $D_0 \subset B(0, d_\rho)$. Denote by $\eta \in C_0^\infty(\mathbb{R})$ a function with $0 \leq \eta(t) \leq 1$, $\eta(t) = 1$ for $t \leq 1$ and $\eta(t) = 0$ for $t \geq 2$, and by $\eta_1 \in C_0^\infty$ a function with $0 \leq \eta_1(t) \leq 1$, $\eta_1(t) = 0$ for $t \leq 2$ and $\eta_1(t) = 1$ for $t \geq 3$. For a fixed $\rho > 0$, we define

$$\eta_\rho(x) \equiv \eta \left( \frac{|x|}{\rho} \right) \eta_1 \left( \frac{|x|}{d_\rho} \right).$$

Note that $\eta_\rho \in C_0^\infty(\mathbb{R}^n)$ and $0 \leq \eta_\rho(x) \leq 1$. If $\rho > 3d_\rho$, we have $\eta_\rho(x) = 1$ for $3d_\rho \leq |x| \leq \rho$ and $\eta_\rho(x) = 0$ for $|x| \leq 2d_\rho$ and $|x| \geq 2\rho$. Thus, as $\rho \to \infty$, we have $\eta_\rho(x) \to 0$ if $|x| \leq 3d_\rho$, $\eta_\rho(x) = \eta_1(|x|/d_\rho)$ if $2d_\rho \leq |x| \leq 3d_\rho$ and $\eta_\rho(x) \to 1$ if $|x| \geq 3d_\rho$, point-wise. We also have the bound

$$|\nabla \eta_\rho| \leq \left( \frac{c_1}{\rho} + \frac{c_2}{d_\rho} \right),$$

where $c_1$, $c_2$ are constants independent of $\rho$ and $d_\rho$.

The definition of the energy and the hypotheses on $m$ and $D_0$ imply $|\psi|, |\nabla A \psi| \in L^2(\mathbb{R}^n)$. In addition, since $A \in H^1_{\text{loc}}(\mathbb{R}^n)$, $\psi \in H^1_{\text{loc}}(\mathbb{R}^n)$ and $|\psi| \leq 1$ a.e., by a standard difference quotients argument on compact sets, one has that $\frac{\partial}{\partial x_k}(\nabla A \psi)$ exists in $D_0 \cup D_\rho$ and belongs to $L^2(D_0; \mathbb{C}) \cap L^2_{\text{loc}}(D_\rho; \mathbb{C})$ for $k = 1, \ldots, n$. As a consequence, we have

$$\nabla^2 A \psi \equiv \nabla A (\nabla A \psi) \in L^2(D_0; \mathbb{C}) \cap L^2_{\text{loc}}(D_\rho; \mathbb{C}).$$

Let

$$\partial A_k := \left( \frac{i}{\kappa_s} \frac{\partial}{\partial x_k} + A_k \right)$$

for $k$ fixed, and consider as a test function $\partial A_k \phi$, where $\phi$ is any complex-valued function that verifies $\phi \in C_0^\infty(\mathbb{R}^n; \mathbb{C})$ and $\text{supp} \phi \subset D_\rho$. The weak formulation (3.1a) yields

$$\int \frac{1}{m} \left( \partial A_k (\nabla A \psi) \cdot (\nabla A \phi)^* + \frac{i}{\kappa_s} \sum_k \left( \frac{\partial A_k}{\partial x_k} - \frac{\partial A_k}{\partial x_k} \right) (\partial A_k \psi) \phi^* \right) + a \psi (\partial A_k \phi)^* \, dx = 0,$$

for any $k$ fixed and $\phi \in C_0^\infty(\mathbb{R}^n; \mathbb{C})$ with $\text{supp} \phi \subset D_\rho$.

By either lemma 2.9 or lemma 2.10, we can consider a gauge-equivalent weak solution $(\tilde{\psi}, \tilde{A})$ such that $\text{div} \tilde{A} = 0$. Since $|\nabla \tilde{A} \psi| = |\nabla \tilde{A} \psi|$ and $|\tilde{\psi}| = |\psi|$, proving the desired results for this gauge-equivalent solution will then imply them for the original one.

Since $(\tilde{\psi}, \tilde{A})$ is a divergence-free finite-energy weak solution, we have $\tilde{A} \in L^p_{\text{loc}}$ for any $p$ if $n = 2$ and $\tilde{A} \in L^q_{\text{loc}}$ if $n = 3$. Hence the analogue of (3.2) for $(\tilde{\psi}, \tilde{A})$ can
be extended by density to any $\phi \in H^1(\mathbb{R}^n; \mathbb{C})$ with $\text{supp } \phi \subseteq D_\varepsilon$. To proceed, we choose $\phi = \eta_\rho^2 \partial_{A_k} \tilde{\psi}$, for $k$ and $\rho > 3d_\varepsilon$ fixed, and we obtain

$$\int |(\partial_{A_k} \nabla_A \tilde{\psi})|^2 \eta_\rho^2 \, dx$$

$$\leq \int \left[ C \left( \sum_i \left| \frac{\partial A_i}{\partial x_k} - \frac{\partial A_k}{\partial x_i} \right| (|(\partial_{A_k} \partial_{A_i} \tilde{\psi})| + |(\partial_{A_k} \tilde{\psi})(\partial_{A_i} \tilde{\psi})|) \right) \eta_\rho^2 + |(\partial_{A_k} \tilde{\psi})^* \partial_{A_k} \partial_{A_k} \partial_{A_k} \tilde{\psi}| \eta_\rho^2 + |\tilde{\psi} \eta_\rho \nabla \eta_\rho \cdot (\nabla_A \tilde{\psi})^*| \right] \, dx,$$

for some positive constant $C = C(m_e, \kappa_s, a)$. Therefore, for any $\varepsilon > 0$, we have

$$\int |(\partial_{A_k} \nabla_A \tilde{\psi})|^2 \eta_\rho^2 \, dx$$

$$\leq \int C_1 \left[ \left( \frac{1}{\varepsilon} \sum_i \left| \frac{\partial A_i}{\partial x_k} - \frac{\partial A_k}{\partial x_i} \right| |\tilde{\psi}|^2 \right) \eta_\rho^2 + \varepsilon |\partial_{A_k} \nabla_A \tilde{\psi}|^2 \eta_\rho^2$$

$$+ \sum_i \left( \frac{\partial A_i}{\partial x_k} - \frac{\partial A_k}{\partial x_i} \right) |(\partial_{A_k} \tilde{\psi})(\partial_{A_i} \tilde{\psi})|^2 \eta_\rho^2 + \varepsilon |\partial_{A_k} \tilde{\psi}|^2 \eta_\rho^2$$

$$+ \frac{1}{\varepsilon} |\tilde{\psi}|^2 \eta_\rho^2 + \frac{1}{\varepsilon} \left( \frac{1}{\rho^2} + 1 \right) |\partial_{A_k} \tilde{\psi}|^2 + \varepsilon |\partial_{A_k} \tilde{\psi}|^2 \eta_\rho^2$$

$$+ |\tilde{\psi}|^2 \eta_\rho^2 + \left( \frac{1}{\rho^2} + 1 \right) |\nabla_A \tilde{\psi}|^2 \right] \, dx.$$
In the above estimate, we control the term $\int |H_a|^2 \psi^2$ either by $h \int |\psi|^2$ or by $\|H_a\|_{L^2}^2$, and the term $\int |H_a||\nabla A\psi|^2 \eta_p^2$ either by $h \int |\nabla A\psi|^2 \eta_p^2$ or by

$$
\|H_a\|_{L^2}(\|\nabla A\psi\|_{L^2}^{1/2} (\|\nabla A\psi\|_{L^2}))^{3/2}.
$$

We now need to distinguish the case $n = 3$ from $n = 2$. For $n = 3$, we have the relation

$$
\|(\nabla A\psi)\eta_p\|_{L^6} = \|\nabla A\psi|\eta_p\|_{L^6} \leq c\|\nabla(\nabla A\psi|\eta_p)\|_{L^2} \leq c\|\nabla A\psi|\eta_p\|_{L^2} + c\|\nabla A\psi|\eta_p\|_{L^2},
$$

where $c$ does not depend on $\rho$. Moreover, since, away from the zeros of $\nabla A\psi$,

$$
|\nabla^2 A\psi|^2 = \left[\left(\frac{1}{2\kappa_a}(\nabla A\psi)^* \nabla(\nabla A\psi) - \nabla A\psi \nabla(\nabla A\psi)^*) + \tilde{A}|\nabla A\psi|^2\right) |\nabla A\psi|^{-1} \right]^2 + \frac{1}{\kappa_a} \nabla|\nabla A\psi|, \tag{3.4}
$$

we obtain (similarly to what was done in proposition 2.7 and using $|\nabla A\psi| \in L^2(\mathbb{R}^3)$ and $|\eta_p| \leq 1$)

$$
\|(\nabla A\psi)\eta_p\|_{L^6} \leq c_2\|\nabla A\psi\|_{L^2} + c_3(\kappa_a)\|(\nabla A\psi)\eta_p\|_{L^2},
$$

with $c_2, c_3$ independent of $\rho > 1$. Substituting the previous inequality in (3.3), since $|\partial^2 A_{\kappa}\psi|^2 \leq |(\partial_{A_{\kappa}} \nabla A\psi)|^2$, we derive

$$
\|(\nabla^2 A\psi)\eta_p\|_{L^2} \leq C_5(1 + \|(\nabla^2 A\psi)\eta_p\|_{L^2}^{1/2}),
$$

with $C_5 = C_5(m, \kappa_a, a, G(\tilde{\psi}, \tilde{A}, H_a), \mu, H_a)$, which implies

$$
\|(\nabla^2 A\psi)\eta_p\|_{L^2} \leq C_6, \tag{3.5}
$$

for some constant $C_6 = C_6(m, \kappa_a, a, G(\tilde{\psi}, \tilde{A}, H_a), \mu, H_a)$. Letting $\rho \to \infty$, we conclude that

$$
|\nabla^2 A\psi| \in L^2(\{|x| \geq 3d_a\}).
$$

Using (3.4) again, we then have $|\nabla A\psi| \in H^1(\{|x| \geq 3d_a\})$, as a consequence the first part of our theorem. To obtain the second part, we look at $\eta_1|\nabla A\psi| \in H^1(\mathbb{R}^4)$. By the Sobolev embedding theorem, we know that $\eta_1|\nabla A\psi| \in L^6(\mathbb{R}^3)$, and thus $|\nabla A\psi| \in L^6(\{|x| \geq 3d_a\})$. A simple calculation shows that

$$
|\nabla|\psi|^2| = \kappa_a|\psi^*\nabla A\psi - \psi(\nabla A\psi)^*| \leq 2|\nabla A\psi|,
$$

and we conclude that $|\nabla|\psi|^2| \in L^2(\{|x| \geq 3d_a\}) \cap L^6(\{|x| \geq 3d_a\})$. By looking, if one wishes, at the function $\eta_1|\psi|^2$, we see that $|\psi|^2 \in L^6(\{|x| \geq 3d_a\})$, and hence $|\psi|^2 \in W^{1,6}(\{|x| \geq 3d_a\})$, which, in turns, implies that $|\psi| \to 0$ as $x \to \infty$ (see remark 13 in [3, p. 167]).
For $n = 2$, since $H^1(\mathbb{R}^2) \subset L^q(\mathbb{R}^2)$ for $q \geq 2$ with continuous injection, we know that
\[
\|\nabla \tilde{A} \tilde{\psi}\|_{L^q} = \|\nabla \tilde{A} \tilde{\psi}\|_{L^q} 
\leq c_4 \|\nabla \tilde{A} \tilde{\psi}\|_{H^1} 
\leq c_5 \|\nabla \tilde{A} \tilde{\psi}\|_{L^q} + c_6 \|\nabla (\nabla \tilde{A} \tilde{\psi})\|_{L^q},
\]
where $c_5, c_6$ do not depend on $\rho$. We then proceed as for the case $n = 3$ and recover the analogue of (3.5),
\[
\|\nabla \tilde{A} \tilde{\psi}\|_{L^q}^2 \leq C_T, \tag{3.6}
\]
for some constant $C_T = C_T(m_0, \kappa_0, a, G, \tilde{A}, H_0, \mu, H_0)$ and any $q \geq 2$. Letting $\rho \to \infty$ gives $\|\nabla \tilde{A} \tilde{\psi}\| \in L^q(\{|\mathbf{x}| \geq 3d_0\})$ for $q \geq 2$, and we have the first part of our theorem. A repetition of the argument presented for $n = 3$ implies now that $\|\nabla \tilde{A} \tilde{\psi}\| \in H^1(\{|\mathbf{x}| \geq 3d_0\})$. Therefore, $\eta_1 |\psi|^2 \in W^{1,q}(\mathbb{R}^2)$ for any $q \geq 2$, and, by remark 13 in [3, p. 167], we have $|\psi| \to 0$ as $\mathbf{x} \to \infty$. Note that $|\psi| \in L^2(\mathbb{R}^2)$ implies $|\psi|^2 \in L^2(\mathbb{R}^2)$ if $q \geq 2$ as $|\psi| \leq 1$ a.e.

The knowledge given by the previous proposition that the order parameter tends to zero far enough from $D_0$ is not sufficient for our analysis. As it will be clear in the next section, we need to know the speed at which this decay happens. In particular, since the magnetic potential has a polynomial or polynomial-log behaviour at infinity (depending on $n$), we need to prove that the order parameter decays faster than polynomially or log-polynomially. In the following, we show that, as can be expected, the order parameter in fact decays at an exponential rate.

**Proposition 3.3.** Let $(\psi, \mathbf{A})$ be a finite-energy weak solution to (1.2), for $H_0 = he_3$ or $H_0 \in L^2(\mathbb{R}^n; \mathbb{R}^3)$. Then there exists a constant $\alpha_0 = \alpha_0(m_0, \kappa_0, a, d_0)$ such that
\[
|\psi(\mathbf{x})|^2 \leq \beta e^{\alpha d_0} e^{-\alpha |\mathbf{x}|} \quad \text{for a.e. } \mathbf{x} \in \mathbb{R}^n \setminus B(\mathbf{0}, d_0),
\]
for any $\beta > 1$ and $0 < \alpha \leq \alpha_0$. Here, $B(\mathbf{0}, d_0)$ is a ball of centre zero and radius $d_0$. In particular, we can choose
\[
2\alpha_0 d_0 = \sqrt{1 + 8am_0\kappa_0^2 d_0^2} - 1.
\]

**Proof.** We denote by $\alpha$ a positive constant to be determined later and we consider the function $\sigma(\mathbf{x}) = \beta e^{\alpha d_0} e^{-\alpha |\mathbf{x}|}$ for $\beta > 1$ fixed. Using the first equation of (1.2), we derive, by direct calculation,
\[
\Delta (\sigma - |\psi|^2) = \Delta \sigma - 2am_0\kappa_0^2 |\psi|^2 - 2\kappa_0^2 \nabla \psi^2 \quad \text{in } \mathbb{R}^n \setminus B(\mathbf{0}, d_0),
\]
that is,
\[
\Delta (\sigma - |\psi|^2) = \beta e^{\alpha d_0} e^{-\alpha |\mathbf{x}|} \left( \frac{\alpha}{|\mathbf{x}|} - \frac{n \alpha}{|\mathbf{x}|} + \alpha^2 \right) - 2am_0\kappa_0^2 |\psi|^2 - 2\kappa_0^2 \nabla \psi^2.
\]
We define $c(\mathbf{x}) = (\alpha/|\mathbf{x}| + \alpha^2)$ and obtain in $\mathbb{R}^n \setminus B(\mathbf{0}, d_0),$
\[
\Delta (\sigma - |\psi|^2) - c(\sigma - |\psi|^2)
= -\beta e^{\alpha d_0} \frac{n \alpha}{|\mathbf{x}|} e^{-\alpha |\mathbf{x}|} + \left( \frac{\alpha}{|\mathbf{x}|} + \alpha^2 - 2am_0\kappa_0^2 \right) |\psi|^2 - 2\kappa_0^2 \nabla \psi^2. \quad (3.7)
\]
Let us consider
\[
\frac{1}{|x|} \leq \frac{1}{d_s}
\]
if we choose \(0 < \alpha \leq \alpha_0\) with
\[\alpha_0 = \frac{1}{2d_s} \left( \sqrt{1 + 8\mu_a \kappa_s^2 d_s^2} - 1 \right),\]
we have
\[
\left( \alpha \frac{|x|}{|x|} + \alpha^2 - 2\mu_a \kappa_s^2 \right) \leq 0,
\]
and (3.7) yields
\[
\Delta (\sigma - |\psi|^2) - c(\sigma - |\psi|^2) \leq 0.
\]
The function \(c(x)\) is non-negative, and, by lemma 3.1, when \(\beta > 1\), we have
\[
(\sigma(x) - |\psi(x)|^2) > 0 \quad \text{for} \quad |x| = d_s.
\]
Using the fact that proposition 3.2 implies \((\sigma(x) - |\psi(x)|^2) \to 0\) as \(|x| \to \infty\), we apply the maximum principle to conclude
\[
\inf_{|x| \geq d_s} (\sigma(x) - |\psi(x)|^2) \geq 0.
\]

The growth at infinity of the functions in our weighted Sobolev spaces is controlled by the exponential decay of the order parameter. In terms of the physical parameters involved in our model, we have the following bound, which is an easy consequence of the previous proposition and lemma 3.1.

**Corollary 3.4.** If \((\psi, A)\) is a finite-energy weak solution to (1.2), with \(H_a = h e_3\) or \(H_a \in L^2(\mathbb{R}^n; \mathbb{R}^3)\), then a.e.,
\[
|\psi(x)|^2 \left(1 + |x|^2 \right)^{1/2} (\ln(2 + |x|^2))^{3-n} \leq C_0.
\]
with \(C_0 = C_0(m_e, \kappa_s, a, d_s)\). Moreover, if \(am_a \kappa_s^2 d_s^2 \geq \frac{3}{8}\), then \(C_0 = C_0(d_s)\).

**Lemma 3.5.** If \((\zeta, B)\) is a finite-energy weak solution to (1.2), with \(H_a = h e_3\), then the gauge-equivalent solution \((\psi, A)\) derived in lemma 2.9 is such that \(\psi \in H^1(\mathbb{R}^n)\).

**Proof.** The elementary inequality \(\frac{1}{2}|c|^2 - |b|^2 \leq |c + b|^2\) for \(c, b \in \mathbb{C}\) yields
\[
\left| \left( \frac{1}{\kappa_s} \nabla + A \right) \psi \right|^2 \geq \frac{1}{2} \left| \frac{1}{\kappa_s} \nabla \psi + (A - \frac{1}{2} h \mu_a a_a) \psi \right|^2 - \left| \frac{1}{2} h \mu_a a_a \psi \right|^2.
\]
Since, by the proof of lemma 2.9, we have \((A - \frac{1}{2} h \mu_a a_a) \in W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^3)\), and since \(|a_a|^2 = |x|^2\), proposition 3.3, corollary 3.4 and the finite-energy condition imply that
\[
\int_{\mathbb{R}^n} \frac{1}{m} \left| \frac{1}{\kappa_s} \nabla \psi \right|^2 < \infty.
\]
Recalling that \( m \) is bounded below and \( \psi \in L^2(\mathbb{R}^n; \mathbb{C}) \), we conclude \( \psi \in \mathcal{H}^1(\mathbb{R}^n) \).

4. Upper critical field

We turn our attention exclusively to the case \( H_a = h e_3 \) and study the behaviour of the system for large values of \( h \). We extend to our model the results contained in [7]. When \( n = 2 \), we show the existence of a finite upper critical field \( \bar{h} \) and we derive estimates on its size (see theorem 4.7). When \( n = 3 \), as it can be expected since we are working with an unbounded domain, we have a weaker result for the general case, that is, we prove that, as \( \kappa \) tends to infinity, any finite-energy weak solution tends (in a sense to be specified) to the normal state (see theorem 4.8). In the three-dimensional case, we can prove that \( \bar{h} < \infty \) and find estimates on its size under the additional hypothesis \( \mu \equiv 1 \) (see §4.2).

In the spirit of our previous work [7], we consider the principal eigenvalue of the operator \( i \nabla + h \kappa s a_N \) and prove an upper bound for it in the class of finite-energy weak solutions. From this, we are able to gather information on the behaviour of the order parameter as \( h \) increases.

By lemma 2.9, for any finite-energy weak solution, we can find a gauge-equivalent weak solution \((\psi, A)\) and a normal state \( a_N \) such that \( \text{div} A = \text{div} a_N = 0 \) and \( A - h a_N \in W^{1,2}_0(\mathbb{R}^n; \mathbb{R}^n) \). In the arguments that we will present in this section, we will always work with this choice of gauge.

4.1. Upper bound for the principal eigenvalue in the whole space

We consider in equation (3.1a) the test function \( \phi = \psi \eta_\rho \), with \( \eta_\rho \) defined as in proposition 3.2, and obtain

\[
\int_{\{|x| \geq 2d_s\}} \left[ \frac{1}{m} |\nabla A \psi|^2 \eta_\rho^2 - \frac{2i}{m \kappa s} \psi^* (\nabla A \psi) \cdot \nabla \eta_\rho \eta_\rho + a |\psi|^2 \eta_\rho^2 \right] dx = 0.
\]

Using standard inequalities, we see that

\[
\int_{\{|x| \geq 2d_s\}} \frac{1}{m} |\nabla A \psi|^2 \eta_\rho^2 \, dx \\
\leq \frac{1}{2} \int_{\{|x| \geq 2d_s\}} \frac{1}{m} |\nabla A \psi|^2 \eta_\rho^2 \, dx \\
+ 2 \int_{\{|x| \geq 2d_s\}} \frac{4}{m \kappa s^2} |\psi|^2 |\nabla \eta_\rho|^2 \, dx + \int_{\{|x| \geq 2d_s\}} a |\psi|^2 \eta_\rho^2 \, dx,
\]

and, by the definition of \( \eta_\rho \), we have

\[
\int_{\{|x| \geq 2d_s\}} \frac{1}{m} |\nabla A \psi|^2 \eta_\rho^2 \, dx \\
\leq \int_{\{2d_s \leq |x| \leq 2 \rho\}} \frac{16}{m \kappa s^2} |\psi|^2 \left( \frac{c_1}{\rho} + \frac{c_2}{d_s} \right)^2 \, dx + 2 \int_{\{|x| \geq 2d_s\}} a |\psi|^2 \eta_\rho^2 \, dx.
\]
We then let $\rho \to \infty$ and we conclude that
\[
\int_{|x| \geq 3d_\varepsilon} \frac{1}{m} |\nabla A \psi|^2 \, dx \leq C_1 \int_{\mathbb{R}^n} \left( \frac{1}{m \kappa_s^2 d_s^2} + a \right) |\psi|^2 \, dx. \tag{4.1}
\]
Similarly, if we pick as a test function $\phi(x) = \psi(x)\eta(|x|/3d_\varepsilon)$, with $\eta$ as in proposition 3.2, then, taking into account lemma 3.1, we obtain
\[
\int_{|x| \leq 3d_\varepsilon} \frac{1}{m} |\nabla A \psi|^2 \, dx \leq C_2 \int_{\mathbb{R}^n} \left( \frac{1}{m \kappa_s^2 d_s^2} + \max\{a, 1\} \right) |\psi|^2 \, dx. \tag{4.2}
\]
Equations (4.1), (4.2) then imply that
\[
\int_{\mathbb{R}^n} \frac{1}{m} |\nabla A \psi|^2 \, dx \leq C_3 \int_{\mathbb{R}^n} \left( \frac{1}{m \kappa_s^2 d_s^2} + \max\{a, 1\} \right) |\psi|^2 \, dx. \tag{4.3}
\]
By the properties of $W^{1,2}_{0,0}(\mathbb{R}^n, \mathbb{R}^n)$, we can find a sequence $\{B_j\}$ of functions in $H^1(\mathbb{R}^n; \mathbb{R}^n)$ of bounded support, which converges to $A - hA_N$ in $W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^n)$. We choose $B_j$ as the test function in (3.1b) and in the weak formulation of (1.3a) to obtain
\[
\int_{\mathbb{R}^n} \frac{1}{\mu} \text{curl}(A - hA_N) \cdot \text{curl} B_j \, dx = - \int_{\mathbb{R}^n} \left[ \frac{1}{m} \left( \frac{1}{2\kappa_s} \psi^* \nabla \psi - \psi \nabla \psi^* + A|\psi|^2 \right) \right] B_j \, dx,
\]
and hence, using (2.7) and (4.3), for every $\varepsilon > 0$, we derive
\[
\int_{\mathbb{R}^n} \left[ \frac{1}{\mu} \text{curl}(A - hA_N) \cdot \text{curl} B_j \right] \, dx \leq C_3 \frac{\varepsilon}{\varepsilon} \int_{\mathbb{R}^n} \left( \frac{1}{m \kappa_s^2 d_s^2} + \max\{a, 1\} \right) |\psi|^2 \, dx + \varepsilon \int_{\mathbb{R}^n} |\psi|^2 |B_j|^2 \, dx. \tag{4.4}
\]
The limit for $j \to \infty$ in (4.4), recalling the definition of $\mu$ and corollary 3.4, yields
\[
\int_{\mathbb{R}^n} \frac{1}{\mu} |\text{curl}(A - hA_N)|^2 \, dx \leq C_3 \frac{\varepsilon}{\varepsilon} \int_{\mathbb{R}^n} \left( \frac{1}{m \kappa_s^2 d_s^2} + \max\{a, 1\} \right) |\psi|^2 \, dx + \varepsilon \int_{\mathbb{R}^n} |\psi|^2 |A - hA_N|^2 \, dx. \tag{4.5}
\]
Note that (4.5) holds for any weak solution $(\psi, A)$ with $A - hA_N \in W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^n)$. For $n = 3$, since div $A = 0$, we know that $\int_{\mathbb{R}^n} |\text{curl}(A - hA_N)|^2 \, dx$ is equivalent to the $W^{1,2}_{0,0}(\mathbb{R}^n; \mathbb{R}^n)$-norm of $A - hA_N$. Thus we apply corollary 3.4 to conclude from (4.5) that
\[
\int_{\mathbb{R}^3} |\text{curl}(A - hA_N)|^2 \, dx \leq \tilde{C} \max\{\mu_e, 1\} \int_{\mathbb{R}^3} \left( \frac{1}{m \kappa_s^2 d_s^2} + \max\{a, 1\} \right) |\psi|^2 \, dx, \tag{4.6}
\]
where $\tilde{C} = \tilde{C}(d_s)$ if $am_e\kappa_s^2 d_s^2 \geq \frac{3}{8}$ and $\tilde{C} = \tilde{C}(m_e, \kappa_s, a, d_s)$ otherwise.
The analogue of (4.6) for the two-dimensional case requires, as usual, a slightly different argument. By (2.3), we have

$$\inf_{b \in \mathbb{R}^2} \| A - hA_N + b \|_{W^{1,2}(\mathbb{R}^2;\mathbb{R}^2)} \leq C \left( \int_{\mathbb{R}^2} |\text{curl}(A - hA_N)|^2 \, dx \right)^{1/2}. \quad (4.7)$$

We can find a vector $b \equiv (b_1, b_2) \in \mathbb{R}^2$ for which

$$\| A - hA_N + b \|_{W^{1,2}(\mathbb{R}^2;\mathbb{R}^2)} \leq C \int_{\mathbb{R}^2} |\text{curl}(A - hA_N)|^2 \, dx + \int_{\mathbb{R}^2} \max\{a, 1\} |\psi|^2 \, dx. \quad (4.8)$$

If we consider the gauge-equivalent solution $(\tilde{\psi}, \tilde{A}) = (\psi e^{i\eta}, A + \nabla \eta)$, where $\eta = (b_1 x_1, b_2 x_2) \in H^{1,0}_0(\mathbb{R}^2)$, we have $\text{div} \tilde{A} = 0$ and

$$\| \tilde{A} - hA_N \|_{W^{1,2}(\mathbb{R}^2;\mathbb{R}^2)} \leq C \int_{\mathbb{R}^2} |\text{curl}(\tilde{A} - hA_N)|^2 \, dx + \int_{\mathbb{R}^2} \max\{a, 1\} |\tilde{\psi}|^2 \, dx. \quad (4.9)$$

Thus, since (4.5) holds for $(\tilde{\psi}, \tilde{A})$ as well, using corollary 3.4 and (4.9), we conclude that

$$\int_{\mathbb{R}^2} \frac{1}{\mu} |\text{curl}(\tilde{A} - hA_N)|^2 \, dx$$

$$\leq C_3 \varepsilon \int_{\mathbb{R}^2} \left( \frac{1}{\mu \kappa_s^2 d_s^2} + \max\{a, 1\} \right) |\psi|^2 \, dx$$

$$+ C_0 \varepsilon \int_{\mathbb{R}^2} \frac{|\tilde{A} - hA_N|^2}{(1 + |x|^2)^{1/2} (\ln(2 + |x|^2))} \, dx$$

$$\leq C_3 \varepsilon \int_{\mathbb{R}^2} \left( \frac{1}{\mu \kappa_s^2 d_s^2} + \max\{a, 1\} \right) |\psi|^2 \, dx$$

$$+ C C_0 \varepsilon \int_{\mathbb{R}^2} |\text{curl}(\tilde{A} - hA_N)|^2 \, dx + C_0 \varepsilon \int_{\mathbb{R}^2} \max\{a, 1\} |\tilde{\psi}|^2 \, dx.$$

Taking into account that

$$\int_{\mathbb{R}^2} |\text{curl}(\tilde{A} - hA_N)|^2 \, dx = \int_{\mathbb{R}^2} |\text{curl}(A - hA_N)|^2 \, dx$$

and

$$\int_{\mathbb{R}^2} |\tilde{\psi}|^2 \, dx = \int_{\mathbb{R}^2} |\psi|^2 \, dx,$$

we obtain a bound similar to (4.6), that is,

$$\int_{\mathbb{R}^2} |\text{curl}(A - hA_N)|^2 \, dx \leq \tilde{C} \max\{\mu_e, 1\} \int_{\mathbb{R}^2} \left( \frac{1}{\mu \kappa_s^2 d_s^2} + \max\{a, 1\} \right) |\psi|^2 \, dx, \quad (4.10)$$

where again $\tilde{C} = \hat{C}(d_s)$ if $am_e \kappa_s^2 d_s^2 \geq \frac{3}{8}$ and $\tilde{C} = \hat{C}(m_e, \kappa_s, a, d_s)$ otherwise.
We rewrite $\nabla A \psi = \nabla h a_N \psi + (A - h a_N) \psi$ and use (4.3) to derive
\[
\frac{1}{2} \int_{\mathbb{R}^n} |\nabla h a_N \psi|^2 \, dx \leq C_3 \max\{m_e, 1\} \int_{\mathbb{R}^n} \left( \frac{1}{m \kappa_s^2 d_s^2} + \max\{a, 1\} \right) |\psi|^2 \, dx \\
+ \int_{\mathbb{R}^n} |\psi|^2 |A - h a_N|^2 \, dx. \tag{4.11}
\]

We deal with the second integral in (4.11) similarly to what was done while deriving (4.6) and (4.10) (note that again we need to consider a change of gauge for \( n = 2 \)), and we recover
\[
\frac{1}{2} \int_{\mathbb{R}^n} |\nabla h a_N \psi|^2 \, dx \leq C_4 \max\{m_e, 1\} \int_{\mathbb{R}^n} \left( \frac{1}{m \kappa_s^2 d_s^2} + \max\{a, 1\} \right) |\psi|^2 \, dx \\
+ C_5 \int_{\mathbb{R}^n} |\text{curl}(A - h a_N)|^2 \, dx, \tag{4.12}
\]
where the constants \( C_4, C_5 \) depend only on \( d_s \) if \( a m_e \kappa_s^2 d_s^2 \geq \frac{3}{8} \) and on \( m_e, \kappa_s, a, d_s \) otherwise.

We combine inequalities (4.6) or (4.10) with (4.12), and find the bound
\[
\int_{\mathbb{R}^n} |\nabla h a_N \psi|^2 \, dx \leq C_t \max\{\mu_e, m_e, 1\} \int_{\mathbb{R}^n} \left( \frac{1}{m \kappa_s^2 d_s^2} + \max\{a, 1\} \right) |\psi|^2 \, dx, \tag{4.13}
\]
with \( C_t = C_t(d_s) \) if \( a m_e \kappa_s^2 d_s^2 \geq \frac{3}{8} \) and \( C_t = C_t(m_e, \kappa_s, a, d_s) \) otherwise. Equation (4.13) can be rewritten as
\[
\int_{\mathbb{R}^n} |(i \nabla + h \kappa_s a_N) \psi|^2 \, dx \leq C_t \max\{\mu_e, m_e, 1\} \int_{\mathbb{R}^n} \left( \frac{1}{m d_s^2} + \kappa_s^2 \max\{a, 1\} \right) |\psi|^2 \, dx. \tag{4.14}
\]

**4.2. The case \( \mu_e = 1 \)**

If the permeability \( \mu \) is constant in \( \mathbb{R}^n \), a normal state that verifies the hypotheses of lemmas 2.4 and 2.9 is \( a_N = \frac{1}{2} a_s \) (with \( a_s \) defined as in lemma 2.4). To derive a lower bound for our principal eigenvalue in the whole space, we can use the following proposition from [2].

**Proposition 4.1.** There is a continuous function \( \sigma : [0, \infty) \to \mathbb{R} \), with \( \sigma(t) > 0 \) for \( t > 0 \), for which \( \lim_{t \to \infty} \sigma(t) \) exists with \( 0 < \lim_{t \to \infty} \sigma(t) < 1 \), and such that
\[
\int_{\mathcal{B}(0, r)} |(i \nabla + \frac{1}{2} \omega^2 a_s) \zeta|^2 \, dx \geq \omega^2 \sigma(\omega r) \int_{\mathcal{B}(0, r)} |\zeta|^2 \, dx
\]
for all \( \zeta \in \mathcal{H}^1(\mathcal{B}(0, r)) \) and \( \omega \geq 0 \).

Its three-dimensional analogue is a particular case of the following result presented in [7].

**Proposition 4.2.** Let \( T(x_0, r, v) \) be a cylinder with central axis parallel to \( v \in \mathbb{R}^3 \), with \( |v| = 1 \), height \( 2r > 0 \) and middle cross-section the disc \( \mathcal{B}(x_0, r) \). If \( b \in \mathbb{R}^n \)
Superconductors surrounded by normal materials

\( H^1(T(x_0, r, v); \mathbb{R}^3) \) is such that \( \text{curl} \, b = v \), then

\[
\int_{T(x_0, r, v)} |(i \nabla + \omega^2 a_N)\zeta|^2 \, dx \geq \omega^2 \sigma(\omega r) \int_{T(x_0, r, v)} |\zeta|^2 \, dx
\]

for all \( \zeta \in H^1(T(x_0, r, v)) \) and \( \omega \geq 0 \), where \( \sigma \) is as in proposition 4.1.

Given any finite-energy weak solution \( (\psi, A) \), we know that \( \psi \in H^1_{\text{loc}}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n; \mathbb{C}) \), so we can apply the above propositions, on \( B(0, r) \) or \( T(0, r, e_3) \), for any \( r > 0 \). If we then let \( r \) tend to infinity, we have

\[
\int_{\mathbb{R}^n} |(i \nabla + \omega^2 a_N)\psi|^2 \, dx \geq \omega^2 \sigma_{\infty} \int_{\mathbb{R}^n} |\psi|^2 \, dx,
\]

where \( \sigma_{\infty} = \lim_{t \to \infty} \sigma(t) < 1 \).

**Remark 4.3.** An exact value for the infimum over functions in \( H^1(\mathbb{R}^n) \) of the above eigenvalue problem can be found by the methods presented in [6,11,12].

We select \( \omega^2 = h_2 \kappa_s \) and combine (4.14) and (4.15),

\[
h_2 \kappa_s \sigma_{\infty} \int_{\mathbb{R}^n} |\psi|^2 \, dx \leq C_1 \max\{m_e, 1\} \int_{\mathbb{R}^n} \left( \frac{1}{md_a^2} + \kappa_s^2 \max\{a, 1\} \right) |\psi|^2 \, dx,
\]

where \( C_1 = C_1(d_a) \) if \( am_e \kappa_s^2 d_a^2 \geq \frac{3}{8} \) and \( C_1 = C_1(m_e, \kappa_s, a, d_a) \) otherwise.

If \( \psi \) is not zero a.e., we have

\[
h \leq \frac{C}{\sigma_{\infty}}, \quad \text{where} \quad C = C(m_e, \kappa_s, a, d_a),
\]

from which we see that if \( h \) is large enough, then \( \psi \) has to be zero. This shows that the upper critical field for our model is finite: \( \bar{h} < \infty \).

If \( am_e \kappa_s^2 d_a^2 \geq \frac{3}{8} \), we have the following estimate on the size of the upper critical field:

\[
\bar{h} \leq \frac{C_1(d_a)}{\sigma_{\infty}} \left( \frac{3}{8} a + \max\{a, 1\} \right) \kappa_s \quad \text{if} \quad m_e \leq 1
\]

and

\[
\bar{h} \leq \frac{C_1(d_a)}{\sigma_{\infty}} m_e \left( \frac{3}{8} am_e + \max\{a, 1\} \right) \kappa_s \quad \text{if} \quad m_e > 1.
\]

**4.3. The case \( n = 2 \) and \( m_e \neq 1 \)**

Physically, one expects the permeability \( \mu \) to attain different values in the superconducting and normal parts. As a consequence, the case \( m_e \neq 1 \) is more realistic, but also more involved from the mathematical point of view.

A result analogous to the one presented in § 4.2 can be shown, provided lemma 2.8 in [7] is extended to the case of a domain in the shape of the exterior of a ball \( B(0, r) \subset \mathbb{R}^2 \). In particular, we need the following lemmas.
LEMMA 4.4. Given $\lambda_0 > 0$, there exists a constant $d(\lambda_0, r) > 0$ such that whenever

\[
\int_{\mathbb{R}^n \setminus B(0,r)} |\nabla f|^2 \, dx \leq \lambda^2 \int_{\mathbb{R}^n \setminus B(0,r)} |f|^2 \, dx
\]

for some $f \in H^1(\mathbb{R}^n \setminus B(0,r))$ with $\lambda \geq \lambda_0$, then

\[
\frac{1}{2} \int_{\mathbb{R}^n \setminus B(0,r)} |f|^2 \, dx \leq \int_{\mathbb{R}^n \setminus B(0,r+d/\lambda)} |f|^2 \, dx.
\]

Proof. The proof of lemma 2.6 in [7] applies to this case, since the boundary of our external domain is contained in a bounded set. Therefore, we can find a finite open cover $\bigcup_{k=0}^N F_k$ for $\mathbb{R}^n \setminus B(0,r)$, with $F_0 \subset \mathbb{R}^n \setminus B(0,r)$, and $F_k$, verifying the conditions required to follow the proof in [7].

LEMMA 4.5. Given $l > 0$, there exists a constant $E = E(l, r), 0 < E \leq 1$, such that, if $\omega^2 \geq l$, then

\[
\int_{\mathbb{R}^3 \setminus B(0,r)} |(i\nabla + \omega^2 b)\zeta|^2 \, dx \geq \omega^2 E \int_{\mathbb{R}^3 \setminus B(0,r)} |\zeta|^2 \, dx,
\]

(4.19)

for all $\zeta \in H^1(\mathbb{R}^3 \setminus B(0,r))$ and $b \in H^{1,\text{loc}}(\mathbb{R}^3 \setminus B(0,r); \mathbb{R}^3)$ with $\text{curl } b = e_3$.

Proof. If the integral on the left-hand side of (4.19) is infinity, we have nothing to prove. If it is finite, let $\zeta \in H^1(\mathbb{R}^3 \setminus B(0,r))$ be such that $\int_{\mathbb{R}^3 \setminus B(0,r)} |\zeta|^2 \, dx > 0$ and

\[
\int_{\mathbb{R}^3 \setminus B(0,r)} |(i\nabla + \omega^2 b)\zeta|^2 \, dx \leq \omega^2 \int_{\mathbb{R}^3 \setminus B(0,r)} |\zeta|^2 \, dx
\]

for some $\omega, \omega^2 > l$. If no such $\zeta$ exists, then (4.19) is true for $E = 1$ and we are done. Looking at (2.6), we see that the above inequality implies that

\[
\int_{\mathbb{R}^3 \setminus B(0,r)} |\nabla \zeta|^2 \, dx \leq \omega^2 \int_{\mathbb{R}^3 \setminus B(0,r)} |\zeta|^2 \, dx.
\]

Hence we can apply lemma 4.4, with $\lambda = \omega^2$, to obtain

\[
\frac{1}{2} \int_{\mathbb{R}^3 \setminus B(0,r)} |\zeta|^2 \, dx \leq \int_{\mathbb{R}^3 \setminus B(0,r+d/\omega)} |\zeta|^2 \, dx.
\]

(4.20)

Fix $R > r+d/\omega$ and consider the annulus $B(0,R) \setminus B(0, r+d/\omega)$. We can find a cover for $B(0,R) \setminus B(0, r+d/\omega)$, consisting of a finite number of discs $\{B(\kappa, d/\omega), k = 1, \ldots, N(\omega, R)\}$, each contained in $B(0,2R) \setminus B(0, r)$ and such that

\[
\sum_{\kappa=1}^{N(\omega, R)} \chi_{B(\kappa, d/\omega)} \leq K_1,
\]

where $\chi_{B(\kappa, d/\omega)}$ is the characteristic function of $B(\kappa, d/\omega)$.
with \( K_1 \) independent of \( \omega \) and \( R \). As a consequence, we have

\[
\int_{B(0,2R)\backslash B(0,R)} |(i\nabla + \omega^2 b)\zeta|^2 \, dx \\
\geq \int_{\bigcup_{k=1}^{N(\omega,R)} B(x_k, d/\omega)} |(i\nabla + \omega^2 b)\zeta|^2 \, dx \\
\geq K_1^{-1} \sum_{k=1}^{N(\omega,R)} \int_{B(x_k, d/\omega)} |(i\nabla + \omega^2 b)\zeta|^2 \, dx \\
\geq K_1^{-1} \omega^2 \sigma(d) \sum_{k=1}^{N(\omega,R)} \int_{B(x_k, d/\omega)} |\zeta|^2 \, dx \\
\geq K_1^{-1} \omega^2 \sigma(d) \int_{B(0,R)\backslash B(0,r+d/\omega)} |\zeta|^2 \, dx, \tag{4.21}
\]

where we have used proposition 4.1 to control the integrals on a single ball.

Taking the limit \( R \to \infty \), since \( K_1 \) does not depend on \( R \), we find from (4.21) that

\[
\int_{\mathbb{R}^2 \backslash B(0,r)} |(i\nabla + \omega^2 b)\zeta|^2 \, dx \geq K_1^{-1} \omega^2 \sigma(d) \int_{\mathbb{R}^2 \backslash B(0,r+d/\omega)} |\zeta|^2 \, dx, \tag{4.22}
\]

which, by (4.20), implies the conclusion of our lemma.

The required estimate then follows.

**Lemma 4.6.** Let \( l > 0 \) and \( n = 2 \). There exists a constant \( E_1 = E_1(d_s, \mu_e, l) \), \( 0 < E_1 \leq 1 \), such that, for \( \omega \) with \( \min\{\omega^2, \mu_\omega \omega^2\} \geq l \), we have

\[
\int_{\mathbb{R}^2} |(i\nabla + \omega^2 a_N)\zeta|^2 \, dx \geq \omega^2 E_1 \int_{\mathbb{R}^2} |\zeta|^2 \, dx, \tag{4.23}
\]

for all \( \zeta \in \mathcal{H}^1(\mathbb{R}^2) \).

**Proof.** If the integral on the left-hand side of (4.23) is infinity, we have nothing to prove. Otherwise, we consider \( \int_{\mathbb{R}^2} \) as

\[
\int_{\mathbb{R}^2} = \int_{D_s} + \int_{B(0,d_s)\backslash D_s} + \int_{\mathbb{R}^2 \backslash B(0,d_s)}. \]

Each of these three integrals is then bounded, and we can apply lemma 2.8 in [7] to the first two integrals and lemma 4.5 above to the third one (for the last two integrals, we choose \( \sqrt{\mu_\omega}\omega \) as \( \omega \)). We thus derive

\[
\int_{\mathbb{R}^2} |(i\nabla + \omega^2 a_N)\zeta|^2 \, dx \\
\geq \omega^2 C_2 \int_{D_s} |\zeta|^2 \, dx + \omega^2 \mu_e C_2 \int_{B(0,d_s)\backslash D_s} |\zeta|^2 \, dx + \omega^2 E \int_{\mathbb{R}^2 \backslash B(0,d_s)} |\zeta|^2 \, dx,
\]

for any given \( l > 0 \) and \( \omega \) such that \( \min\{\omega^2, \mu_\omega \omega^2\} \geq l \). Set \( E_1 = \min\{C_2, C_2 \mu_e, E\} \) and note that \( 0 < E_1 \leq 1 \) and \( E_1 = E_1(d_s, \mu_e, l) \), to gather (4.23). \( \square \)
We can now proceed as in §4.2 to derive our main results.

**Theorem 4.7.** Let \( n = 2 \). There exists a constant \( \phi = \phi(m_e, \kappa_s, a, \mu_e, d_s) \) such that, if

\[
h > \max \left\{ \frac{1}{\kappa_s}, \frac{1}{\mu_e \kappa_s}, \phi \right\},
\]

then any finite-energy weak solution for (1.2) is normal.

Moreover, if \( am_e \kappa_s^2 d_s^2 > \frac{3}{8} \), then

\[
\bar{h} \leq \frac{C_t(d_s)}{E_1(a_s, \mu_e)} \max\{\mu_e, 1\} \left( \frac{8}{3} a + \max\{a, 1\} \kappa_s \right) \text{ if } m_e \leq 1 \quad (4.24)
\]

and

\[
\bar{h} \leq \frac{C_t(d_s)}{\sigma_\infty} \max\{\mu_e, m_e\} \left( \frac{8}{3} am_e + \max\{a, 1\} \right) \kappa_s \text{ if } m_e > 1. \quad (4.25)
\]

**Proof.** Let \( \omega^2 = h \kappa_s \). Under our hypothesis, \( \min\{\omega^2, \mu_e \omega^2\} \geq 1 \), so, recalling lemma 3.5, we can apply (4.23) with \( l = 1 \), which, together with (4.14), yields

\[
h \kappa_s E_1 \int_{\mathbb{R}^2} |\psi|^2 \, dx \leq C_t \max\{\mu_e, m_e, 1\} \int_{\mathbb{R}^n} \left( \frac{1}{m d_s^2} + \kappa_s^2 \max\{a, 1\} \right) |\psi|^2 \, dx. \quad (4.26)
\]

The theorem then follows as in §4.2. \( \square \)

**4.4. The case \( n = 3 \) and \( \mu_e \neq 1 \)**

In the three-dimensional setting, we study the asymptotic behaviour of finite-energy weak solutions for large values of the applied field. We are able to show that, as \( h \) increases, finite-energy weak solutions approach the normal state.

**Theorem 4.8.** Let \( n = 3 \). For \( h > 0 \) fixed, denote by \( (\psi_h, A_h) \) a corresponding finite-energy weak solution of (1.2).

If \( h \to \infty \), then \( |\psi_h| \) converges to 0 in \( H^1(\mathbb{R}^3) \) and \( (1/\mu) \text{curl} A_h \) converges to \( h \epsilon_3 \) in \( L^2(\mathbb{R}^3; \mathbb{R}^3) \).

**Proof.** Let \( a_N \) be the normal state of lemma 2.4. Let \( h_j \) a sequence of positive number divergent to infinity. For \( j \) fixed, consider a finite-energy weak solution, denoted by \( (\psi_j, A_j) \), to (1.2), with \( h = h_j \). Without loss of generality, we can assume that \( (\psi_j, A_j) \) and \( a_N \) verify the conclusions of lemma 2.9. Note that, for \( n = 3 \), \( a_N \) does not depend on \( j \).

If \( \lim_{j \to \infty} \int_{\mathbb{R}^3} |\psi_j|^2 \, dx = 0 \), we are done with the first part of our theorem.

If not, then there exists a subsequence (still indicated by the index \( j \)) and an \( \varepsilon_0 > 0 \) such that \( \lim_{j \to \infty} h_j = \infty \) and

\[
\lim_{j \to \infty} \int_{\mathbb{R}^3} |\psi_j|^2 \, dx > \varepsilon_0 > 0. \quad (4.27)
\]

Equations (2.6) and (4.3) imply that

\[
\int_{\mathbb{R}^3} \frac{1}{m} \frac{1}{\kappa_s} \nabla |\psi_j| \, dx \leq C(m_e, \kappa_s, a, d_s) \int_{\mathbb{R}^3} |\psi_j|^2 \, dx, \quad (4.28)
\]
while lemma 3.1 and proposition 3.3 yield
\[
\int_{\mathbb{R}^3} |\psi_j|^2 \, dx \leq |B(0, d_\epsilon)| + \int_{\mathbb{R}^3 \setminus B(0, d_\epsilon)} |\beta e^{ad_\epsilon} e^{-a|\cdot|}| \, dx.
\] (4.29)

Therefore, since $1/m > 0$, the sequence $\{ |\psi_j| \}$ is uniformly bounded in $H^1(\mathbb{R}^3)$.

By proposition 3.3, for any fixed $\delta > 0$, there exists an $R_\delta > d_\epsilon$, $R_\delta$ independent of $j$, such that
\[
\int_{\mathbb{R}^3 \setminus B(0, R_\delta)} |\psi_j|^2 \, dx \leq \delta \quad \text{for every } j.
\] (4.30)

On the other hand, since the sequence $\{ |\psi_j| \}$ is uniformly bounded in $B(0, R_\delta)$, there exists a subsequence $\{ \psi_{j_k} \}$ and a function $\phi_\delta$ such that $|\psi_{j_k}|$ converges to $\phi_\delta$ in $L^2(B(0, R_\delta))$.

There are two possible cases.

**Case 1.** For every $\delta > 0$, one has $\|\phi_\delta\|_{L^2(B(0, R_\delta))} = 0$. But then there exists an $n = n(\delta)$ such that, for every $j_k > n(\delta)$, we have
\[
\int_{B(0, R_\delta)} |\psi_{j_k}|^2 \, dx < \delta,
\]
which, together with (4.30), implies $\int_{\mathbb{R}^3} |\psi_{j_k}|^2 \, dx < 2\delta$ for every $j_k > n(\delta)$, which is in contradiction with (4.27) if we choose $\delta = \frac{1}{2} \varepsilon_0$.

**Case 2.** There exist a $\delta$ and a corresponding $R_\delta$ for which
\[
\|\phi_\delta\|_{L^2(B(0, R_\delta))} = M > 0.
\]

Note that, by construction, $D_\epsilon \subset B(0, R_\delta)$, so that at least one of the following will hold:
\[
\|\phi_\delta\|_{L^2(B(0, R_\delta) \setminus D_\epsilon)} = M_\epsilon > 0 \quad \text{or} \quad \|\phi_\delta\|_{L^2(D_\epsilon)} = M_\epsilon > 0.
\]

Suppose that the former is true. From lemma 2.6, the set of $x$ in $B(0, R_\delta) \setminus D_\epsilon$ for which $\text{curl } a_N(x) = 0$ is a closed set of measure zero. Hence we can find a ball $B_{2\gamma} \subset B(0, R_\delta) \setminus D_\epsilon$ such that $\int_{B_{2\gamma}} |\phi_\delta|^2 \, dx = 2\gamma > 0$ for some $\gamma$, $\inf_{B_{2\gamma}} |\text{curl } a_N| > 0$ and $\int_{B_{2\gamma}} |\psi_{j_k}|^2 \, dx > \gamma > 0$ for $j_k$ large enough. We apply the argument of lemma 3.12 in [7] and get
\[
\int_{B(0, R_\delta) \setminus D_\epsilon} |(i\nabla + h_{j_k} \kappa_\epsilon a_N)\psi_{j_k}|^2 \, dx \geq Ch_{j_k} \kappa_\epsilon \int_{B_{2\gamma}} |\psi_{j_k}|^2 \, dx \geq Ch_{j_k} \kappa_\epsilon \gamma,
\] (4.31)

where $C$ depends on $\inf_{B_{2\gamma}} |\text{curl } a_N|$, $\|a_N\|_{C^{1,1}(\overline{B(0, R_\delta)} \setminus D_\epsilon)}$, and $\gamma$ but not $j$.

On the other hand, from (4.14) and (4.29), we also have
\[
\int_{B(0, R_\delta) \setminus D_\epsilon} |(i\nabla + h_{j_k} \kappa_\epsilon a_N)\psi_{j_k}|^2 \, dx \leq C_1 (m_\epsilon, \kappa_\epsilon, a, d_\epsilon, \mu_\epsilon),
\]
which, together with (4.31), implies, for $j_k$ large enough, that $C_1 \geq Ch_{j_k} \kappa_\epsilon \gamma$, with $C_1$, $C$ independent of $j_k$. Letting $j_k \to \infty$, we obtain a contradiction. If $\|\phi_\delta\|_{L^2(D_\epsilon)} = M_\epsilon > 0$, a similar argument can be applied.

We then show that $\lim_{j_k \to \infty} \int_{\mathbb{R}^3} |\psi_{j_k}|^2 \, dx = 0$. Taking in account (4.28), this implies that $|\psi_j|$ tends to 0 in $H^1(\mathbb{R}^3)$.

The second part of the theorem follows from (4.6).
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References


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