1. Find the principal argument $\text{Arg}(z)$ when

(a) $z = \frac{i}{-2-2i}$

$i = e^{i\frac{\pi}{2}}$, and $-2-2i = 2(-1-i)$. So, $\text{arg } i = \frac{\pi}{2} + 2k\pi$, and $\text{arg } (-2-2i) = -\frac{3\pi}{4} + 2m\pi$

Hence, $\text{arg } \left(\frac{i}{-2-2i}\right) = \text{arg } i - \text{arg } (-2-2i) = \frac{\pi}{2} + 2k\pi - \left(-\frac{3\pi}{4} + 2m\pi\right) = \frac{5\pi}{4} + 2(k-m)\pi$

Finally, since $\text{Arg}(z)$ needs to be between $-\pi$ and $\pi$, $\text{Arg}\left(\frac{i}{-2-2i}\right) = -\frac{3\pi}{4}$

(b) $z = (\sqrt{3} + i)^6$

$|\sqrt{3} + i| = 2$, so $z = 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$ and $\sqrt{3} = \cos \phi$, $\frac{i}{2} = \sin \phi$. So $\phi = \frac{\pi}{6}$ and $\sqrt{3} + i = 2e^{i\frac{\pi}{6}}$

So, $\text{arg } (\sqrt{3} + i)^6 = 6 \text{ arg } (\sqrt{3} + i) = 6\left(\frac{\pi}{6} + 2k\pi\right) = (\pi + 2k\pi)$ and $\text{Arg } (\sqrt{3} + i) = \pi$

1. Show that

(a) $|e^{i\theta}| = 1$

$|e^{i\theta}| = |\cos \theta + i \sin \theta| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$

(b) $e^{i\theta} = e^{-i\theta}$

The left hand side (LHS) is $(\cos \theta + i \sin \theta) = \cos \theta - i \sin \theta = \cos(-\theta) + i \sin(-\theta) = e^{i(-\theta)} = e^{-i\theta}$, which is the right hand side (RHS) of the equality

5. Use de Moivre formula (Sec. 7) to derive the following trig identities

(a) $\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$

De Moivre formula gives

$(\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta$

On the other hand, the binomial formula gives

$(\cos \theta + i \sin \theta)^3 = \cos^3 \theta + i3\cos^2 \theta \sin \theta - 3\cos \theta \sin^2 \theta - i \sin^3 \theta$

$= \cos^3 \theta - 3 \cos \theta \sin^2 \theta + i (3 \cos^2 \theta \sin \theta - \sin^3 \theta)$

Equating real parts we obtain that

$\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$
(b) \( \sin 3\theta = 3\cos^2 \theta \sin \theta - \sin^3 \theta \)

Equating the imaginary parts above we obtain \( b \),

6. By writing the individual factors on the left in exponential form, performing the needed operations and finally changing back to rectangular coordinates, show that

(a) \( i \left( 1 - i\sqrt{3} \right) \left( \sqrt{3} + i \right) = 2 \left( 1 + i\sqrt{3} \right) \)

\[ i = e^{i\frac{\pi}{6}}, \quad |1 - i\sqrt{3}| = 2 = |\sqrt{3} + i|, \text{ so } 1 - i\sqrt{3} = 2 \left( \frac{1}{2} - i\frac{\sqrt{3}}{2} \right) \text{ and } \sqrt{3} + i = 2 \left( \frac{\sqrt{3}}{2} + i\frac{1}{2} \right) \]

So,

\[
i \left( 1 - i\sqrt{3} \right) \left( \sqrt{3} + i \right) = e^{i\frac{\pi}{6}} 2 \left( \frac{1}{2} - i\frac{\sqrt{3}}{2} \right) 2 \left( \frac{\sqrt{3}}{2} + i\frac{1}{2} \right) = \]

\[ = e^{i\frac{\pi}{6}} 2 \left( \cos \frac{\pi}{6} - i \sin \frac{\pi}{6} \right) 2 \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \]

\[ = e^{i\frac{\pi}{6}} 2 e^{-i\frac{\pi}{6}} e^{i\frac{\pi}{6}} = 4 e^{i\frac{\pi}{6}} = 4 \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \]

\[ = 4 \left( \frac{1}{2} + i\frac{\sqrt{3}}{2} \right) = 2 \left( 1 + i\sqrt{3} \right) \]

(b) \( \frac{5i}{2+i} = 1 + 2i \)

Done in class

7. Show that if Re \( z_1 > 0 \) and Re \( z_2 > 0 \) then \( \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \)

Note that if Re \( z_i > 0 \) then \(-\frac{\pi}{2} < \arg(z_i) < \frac{\pi}{2} \). But then, adding both inequalities,

\[ -\frac{\pi}{2} < \arg(z_1) < \frac{\pi}{2} \]

\[ -\frac{\pi}{2} < \arg(z_2) < \frac{\pi}{2} \]

\[ -\pi < \arg(z_1) + \arg(z_2) < \pi \]

Since \( \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \), and \( \arg(z_1 z_2) \) is the unique value or \( \arg(z_1 z_2) \) between \(-\pi\) and \( \pi \), then

\[ \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \]

8. Let \( z \) be a non zero complex number and \( n \) a negative integer (\( n = -1, -2, \ldots \)). Also write \( z = re^{i\theta} \) and \( m = -n = 1, 2, \ldots \). Using the expressions

\[ z^m = r^m e^{im\theta} \quad \text{and} \quad z^{-1} = \frac{1}{r} e^{i(-\theta)} \]

verify that

\[ (z^m)^{-1} = (z^{-1})^m \]
and hence that the definition $z^n = (z^{-1})^m$ in Sec 7 could have been written alternatively as $z^n = (z^m)^{-1}$

Solution:

$$ (z^{-1})^m = \left( \frac{1}{r} e^{i(-\theta)} \right)^m = \left( \frac{1}{r^m} e^{i(-\theta)m} \right) = \frac{1}{r^m} e^{i(-\theta)m} = (r^m)^{-1} (e^{im\theta})^{-1} = (r^m e^{im\theta})^{-1} = (z^m)^{-1} $$

So, $z^n = (z^{-1})^m = (z^m)^{-1}$

**Sections 8 and 9.**

3. Done in a handout

4. Show that if $z_0 = -4\sqrt{2} + 4\sqrt{2}i$, then the principal value of the cube root of $z_0$ is $c_0 = \sqrt{2}(1 + i)$ and

$$ c_1 = c_0 \omega_3 = \frac{-\left(\sqrt{3} + 1\right) + \left(\sqrt{3} + 1\right) i}{\sqrt{2}} \quad c_2 = c_0 \omega_3 = \frac{\left(\sqrt{3} + 1\right) - \left(\sqrt{3} + 1\right) i}{\sqrt{2}} $$

Note that $z_0 = 8 \left( \frac{1}{\sqrt{2}} + \frac{1}{i\sqrt{2}} \right) = 8 \exp i \left( \frac{3\pi}{4} + 2k\pi \right)$. So,

$$ z^{\frac{1}{3}} = \left\{ 8^{\frac{1}{3}} \exp i \left( \frac{\pi}{4} + \frac{2k\pi}{3} \right) \right\}, \quad k = 0, 1, 3 $$

and the principal value of the cube roots of $z_0$ is

$$ c_0 = 2 \exp i \frac{\pi}{4} = 2 \left( \frac{1}{\sqrt{2}} + \frac{1}{i\sqrt{2}} \right) = \sqrt{2}(1 + i) $$

which is the first assertion of the problem.

Now to obtain the rest of the roots, we have

$$ c_1 = 2 \exp i \left( \frac{\pi}{4} + \frac{2\pi}{3} \right) = 2 \exp i \frac{\pi}{4} \exp i \frac{2\pi}{3} $$

But the two exponentials on the RHS are $c_0$ and $\omega_3$, which is the second cube root of $z = 1$. Moreover,

$$ \omega_3 = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = -\frac{1}{2} + \frac{i}{2} \frac{\sqrt{3}}{2} $$

So,

$$ c_1 = c_0 \omega_3 = \sqrt{2}(1 + i) \left( -\frac{1}{2} + \frac{\sqrt{3}}{2} \right) = -\left( \frac{\sqrt{3} + 1}{2} \right) + \frac{\sqrt{3} + 1}{2} i $$

The argument for $c_2$ is similar.
8. (a) Prove that the usual formula solves the quadratic equation $az^2 + bz + c = 0 \ (a \neq 0)$ when the coefficients $a$, $b$, $c$ are complex numbers. (Complete the square on the left hand side to derive the formula)

$$z = \frac{-b \pm (b^2 - 4ac)^{1/2}}{2a}$$

where both square roots are to be considered when $b^2 - 4ac \neq 0$).

Solution: Completing the square we have

$$az^2 + bz + c = a \left( z^2 + \frac{b}{a}z + \frac{c}{a} \right) = a \left[ \left( z + \frac{b}{2a} \right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} \right]$$

So, $az^2 + bz + c = 0$ implies that

$$a \left[ \left( z + \frac{b}{2a} \right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} \right] = 0$$

or

$$\left( z + \frac{b}{2a} \right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$$

Then

$$\left( z + \frac{b}{2a} \right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

$$z + \frac{b}{2a} = \left( \frac{b^2 - 4ac}{4a^2} \right)^{1/2} = \frac{(b^2 - 4ac)^{1/2}}{(4a^2)^{1/2}}$$

But if $w$ is any complex number, then $w^{1/2} = \pm z$ where $z^2 = w$. So,

$$z + \frac{b}{2a} = \pm \frac{(b^2 - 4ac)^{1/2}}{ \pm 2a} = \pm \frac{(b^2 - 4ac)^{1/2}}{2a}$$

So,

$$z = \frac{-b \pm (b^2 - 4ac)^{1/2}}{2a}$$

$$z = \frac{-b \pm (b^2 - 4ac)^{1/2}}{2a}$$