3. Show that \((1 + z)^2 = 1 + 2z + z^2\)

Solution: Using the identification of \(z = x + iy\) with \((x, y)\) and the definition of the product of two complex numbers we have that

\[
(1 + z)^2 = ((1, 0) + (x, y))^2 = (1 + x, y)^2 = ((1 + x)^2 - y^2, 2(1 + x)y) \\
= (1 + 2x + x^2 - y^2, 2y + 2xy) \\
= (1, 0) + (2x, 2y) + (x^2 - y^2, 2xy) \\
= 1 + 2z + z^2
\]

4. Verify that both \(z = 1 \pm i\) satisfy \(z^2 - 2z + 2 = 0\)

This follows from just plugging both values of \(z\) into the equation and checking that they both satisfy it.

7. Use the associative law for addition and the distributive law to show that

\[z(z_1 + z_2 + z_3) = zz_1 + zz_2 + zz_3\]

\[
z(z_1 + z_2 + z_3) = z((z_1 + z_2) + z_3), \text{ by associative law},
\]
\[
= z(z_1 + z_2) + zz_3, \text{ by the distributive law},
\]
\[
= (zz_1 + zz_2) + zz_3, \text{ again by the distributive law},
\]
\[
= zz_1 + zz_2 + zz_3, \text{ by the associative law}.
\]

8. By writing \(i = (0, 1)\) and \(y = (y, 0)\) show that \(-(iy) = (-i)y = i(-y)\)

Solution:

\[-(iy) = -(0, 1)(y, 0) = -(0y - 1 \cdot 0, 1y + 0 \cdot 0) = -(0, y) = (0, -y)
\]
\[-(i)y = (0, -1)(y, 0) = (0y - (-1)0, -y + 0) = (0, -y)
\]
\[i(-y) = (0, 1)(-y, 0) = (0(-y) - 1 \cdot 0, -y + 0 \cdot 0) = (0, -y)
\]

So, all three expressions give the same complex number.

9. (a) Write \((x, y) + (u, v) = (x, y)\) and point out how it follows that the complex number \(0 = (0, 0)\) is the unique additive identity.
Solution: If \((x, y) + (u, v) = (x, y)\), then \((x + u, y + v) = (x, y)\). And two complex numbers cannot be equal unless their real and imaginary parts are equal. So,

\[
\begin{align*}
x + u &= x \\
y + v &= y
\end{align*}
\]

By properties of real numbers these equations can only be solve for \(u\) and \(v\) when \(u = 0\) and \(v = 0\). So, the only complex number that solves the original equation is the number \((u, v) = 0 = (0, 0)\)

(b) Likewise, write \((x, y) (u, v) = (x, y)\) and show that the number \(1 = (1, 0)\) is a unique multiplicative identity

Solution: If \((x, y) (u, v) = (x, y)\), then \((xu - yv, yu + xv) = (x, y)\). So,

\[
\begin{align*}
xu - yv &= x \\
yu + xv &= y
\end{align*}
\]

This must be true for every \((x, y)\) in \(\mathbb{C}\). In particular, if \(x = 1\) and \(y = 0\), we have that

\[
\begin{align*}
1u - 0v &= 1 \\
0u + 1v &= 0
\end{align*}
\]

So \((u, v) = (1, 0) = 1\)

10. Solve the equation \(z^2 + z + 1 = 0\) for \(z = (x, y)\) by writing

\[(x, y) (x, y) + (x, y) + (1, 0) = (0, 0)\]

Solution:

\[
(x, y) (x, y) + (x, y) + (1, 0) = \left(x^2 - y^2, 2xy\right) + (x, y) + (1, 0) = \left(x^2 - y^2 + x + 1, 2xy + y\right)
\]

The last expression equals \((0, 0)\) if and only if both real and imaginary parts equal 0. Then,

\[
x^2 - y^2 + x + 1 = 0
\]

and

\[
2xy + y = (2x + 1) y = 0
\]

But the last equation holds if and only if either

\[
2x + 1 = 0
\]

or

\[
y = 0
\]
(a) Assume first that \( y = 0 \). Then
\[
x^2 - y^2 + x + 1 = x^2 + x + 1 = 0
\]
Solving that equation for \( x \) gives
\[
x = \frac{-1 \pm \sqrt{1 - 4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}
\]
Which is not a real number. So, this could not be true, since \( x \) must be real.

(b) We then have to assume that \( y \neq 0 \). But then \( 2x + 1 = 0 \), which means that \( x = -\frac{1}{2} \). Now plugging \( x \) in the first equation gives
\[
\frac{1}{4} - y^2 - \frac{1}{2} + 1 = \frac{3}{4} - y^2 = 0
\]
\[
y = \pm \frac{\sqrt{3}}{2}
\]
So, \( z = \left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right) \) or \( \left( -\frac{1}{2}, -\frac{\sqrt{3}}{2} \right) \)

Section 3:

5. Derive expression (2) Sec. 3 for the quotient \( \frac{z_1}{z_2} \) by the method described just after it.
\[
\frac{z_1}{z_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{(x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2}
\]
\[
= \frac{x_1x_2}{x_2^2 + y_2^2} + i\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}
\]

7. Use relations (6) and (7) in Sec. 3 to derive (8)
\[
\frac{z_1z_2}{z_3z_4} = (z_1z_2)(z_3z_4)^{-1}, \text{ by (6), where } z_3 \neq 0 \text{ and } z_4 \neq 0.
\]
\[
= (z_1z_2)\left(\frac{1}{z_3}\frac{1}{z_4}\right), \text{ by (7)}
\]
\[
= (z_1\frac{1}{z_3})(z_2\frac{1}{z_4}), \text{ by associativity and commutativity or by exercise 3}
\]
\[
= \frac{z_1z_2}{z_3z_4}, \text{ by (6)}.
\]

8. Use Mathematical induction to verify the binomial formula \( (z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k} z_2^k \) \((n = 1, 2, \ldots)\).

(a) Case \( n = 1 \)
\[
(z_1 + z_2)^1 = \sum_{k=0}^{1} \binom{1}{k} z_1^{1-k}z_2^k = \left( \begin{array}{c} 1 \\ k \end{array} \right) z_1^{1-k} z_2^k = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) z_1^1 z_2^0 + \left( \begin{array}{c} 1 \\ 1 \end{array} \right) z_1^0 z_2^1 = z_1 + z_2
\]

(b) Assume true for \( n = m \), so
\[
(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k} z_2^k
\]
(c) Show that it is true for \( n = m + 1 \):
\[
(z_1 + z_2)^{m+1} = (z_1 + z_2) (z_1 + z_2)^m = (z_1 + z_2) \left( \sum_{k=0}^{m} \binom{m}{k} z_1^{m-k} z_2^k \right)
\]
\[
= \sum_{k=0}^{m} \binom{m}{k} z_1^{m+1-k} z_2^k + \sum_{k=0}^{m} \binom{m}{k} z_1^{m-k} z_2^{k+1}
\]
\[
= \binom{m}{0} z^{m+1} + \sum_{k=1}^{m} \binom{m}{k} z_1^{m+1-k} z_2^k + \sum_{k=1}^{m} \binom{m}{k-1} z_1^{m-k+1} z_2^k
\]
\[
= \binom{m+1}{0} z^{m+1} + \left( \sum_{k=1}^{m} \left[ \binom{m}{k} \binom{m}{k-1} \right] z_1^{m+1-k} z_2^k \right) + \binom{m}{m} z_2^{m+1}
\]
So, we only need to establish that \( \binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k} \)

In fact,
\[
\binom{m}{k} + \binom{m}{k-1} = \frac{m!}{k! (m-k)!} + \frac{m!}{(k-1)! (m-k+1)!}
\]
\[
= \frac{m! (m-k+1)!}{k! (m-k+1)!} = \frac{(m+1)!}{k! (m+1-k)!} = \binom{m+1}{k}
\]

So,
\[
(z_1 + z_2)^{m+1} = \binom{m+1}{0} z^{m+1} + \sum_{k=1}^{m} \binom{m+1}{k} z_1^{m+1-k} z_2^k + \binom{m+1}{m+1} z_2^{m+1}
\]

Section 4.

2. Done in class

3. Verify that \( \sqrt{2} |z| \geq |\text{Re } z| + |\text{Im } z| \). Suggestion: Reduce this inequality to \((|x| - |y|)^2 \geq 0 \).

Let \( z = x + iy \) then

\[
\sqrt{2} |z| \geq |\text{Re } z| + |\text{Im } z|
\]
\[
\iff \sqrt{2} \sqrt{x^2 + y^2} \geq |x| + |y|
\]
\[
\iff \left( \sqrt{2} \sqrt{x^2 + y^2} \right)^2 \geq (|x| + |y|)^2
\]
\[
\iff 2 (x^2 + y^2) \geq (|x| + |y|)^2 = |x|^2 + 2 |x| |y| + |y|^2
\]
\[
\iff 2 x^2 + 2 y^2 \geq |x|^2 + 2 |x| |y| + |y|^2
\]
\[
\iff 2 x^2 + 2 y^2 - |x|^2 - 2 |x| |y| - |y|^2 \geq 0
\]
\[
\iff |x|^2 - 2 |x| |y| + |y|^2 \geq 0
\]
\[
\iff (|x| - |y|)^2 \geq 0
\]

Since the last inequality is always true, then all previous statements are and viceversa. So the first inequality is true also.
5. (a) Done in class

(b) \(|z - 1| = |z + i|\) represents the line through the origin whose slope is \(-1\).

If \(z = x + iy\) satisfies the condition, then \(|x + iy - 1| = |x + iy + i|\)

So

\[
\begin{align*}
|x + iy - 1|^2 &= |x + iy + i|^2 \\
(x - 1)^2 + y^2 &= x^2 + (y + 1)^2 \\
x^2 - 2x + 1 + y^2 &= x^2 + y^2 + 2y + 1 \\
-2x &= 2y \\
x &= -y
\end{align*}
\]

Which describes the line \(y = -x\), through the origin with slope \(-1\).

Section 5

1. c. Straightforward

(a) Straightforward

7. Use established properties of moduli to show that, when \(|z_3| \neq |z_4|\),

\[
\left| \frac{z_1 + z_2}{z_3 + z_4} \right| \leq \frac{|z_1| + |z_2|}{|z_3| - |z_4|}
\]

By triangle inequality \(|z_1 + z_2| \leq |z_1| + |z_2|\) and \(|z_3 + z_4| \geq |z_3| - |z_4|\).

Now, since \(|z_3| \neq |z_4|\), then \(|z_3| - |z_4| \neq 0\) and \(z_3 \neq -z_4\), so \(z_3 + z_4 \neq 0\) and

\[
\frac{1}{|z_1 + z_2|} \leq \frac{1}{|z_3| - |z_4|}
\]

So,

\[
\left| \frac{z_1 + z_2}{z_3 + z_4} \right| = \left| \frac{z_1 + z_2}{z_3 + z_4} \right| \leq \frac{|z_1| + |z_2|}{|z_3| + |z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}
\]

10. Done in class

14. Straightforward

15. Straightforward.