Lecture 12
Bundles over Paracompact Spaces

We have used the following result (to define $C_r(E)$) for example:

**Theorem:** If $E$ is a vector bundle over a paracompact Hausdorff space $X$, then there exists a diagram

$$
\begin{array}{c}
E \\
\downarrow \pi \\
X \\
\downarrow f \\
Gr_n(C^\infty)
\end{array}
$$

The corresponding result holds in the real case as well. [Link: MS proves this without assuming $X$ is Hausdorff!]

**Proof:** First, we claim that it suffices to construct a continuous, linear injection $E \hookrightarrow C^\infty \cong C^\infty \otimes C^\infty$.

Given such a map $j$, we define

$$
\begin{array}{c}
E \\
\downarrow \pi \\
X \\
\downarrow f \\
Gr_n(C^\infty)
\end{array}
$$

by $\tilde{f}(e) = (j(\text{fiber through } e), j(e))$ and $f(x) = j(x^0)$.

Note that locally, $f$ has the form

$\begin{array}{c}
\mathbb{C} \otimes \mathbb{C} \\
\downarrow f \\
Gr_n(C^\infty)
\end{array}$

So $f$ is continuous.
So we must construct a map

$$E \rightarrow C^n \otimes C^n \otimes \cdots$$

Lemma: If $X$ is paracompact and $\{U_i\}_{i \in I}$ is an open cover of $X$, then there exists a countable open cover $\{V_k\}_{k=1}^\infty$ of $X$ such that

1. Each $V_k$ can be written as a disjoint union

$$V_k = \bigsqcup_{j} U_k^j$$

with $U_k^j \subset U_i$ for some $i = i(j)$.

2. There is a partition of unity $\{\varphi_k\}_{k=1}^\infty$ with

$$\sum_{k=1}^\infty \varphi_k \leq 1.$$

Assuming the Lemma, we can easily construct the desired map $j: E \rightarrow C^\infty$.

Let $\{U_i\}_{i \in I}$ be a cover of $X$ over which $E$ is trivial, and let $\{V_k\}_{k=1}^\infty$ be the cover in the lemma.

Note that condition 1) implies that $E|_{V_k}$ is trivial for each $k$. Choose trivializations $\psi_k: E|_{V_k} \rightarrow V_k \times C^n$, and let $\psi_k$ denote $E|_{V_k} \rightarrow V_k \times C^n$.

Letting $\{\varphi_k\}$ denote the partition of unity in 2), we define

$$j(e) = \bigoplus_{k=1}^\infty \varphi_k(e) \cdot \psi_k \in C^n \otimes C^n \otimes \cdots \subseteq C^n.$$ 

Since only finitely many $\varphi_k$ are non-zero at $e$, this point lies in $E|_{V_k}$. 


Also, \( j \) is injective by at each \( x \in X \), some \( \psi_k \) must be non-zero (by \( \sum_{k=1}^{\infty} \psi_k(x) = 1 \) at each \( x \in X \)). This completes the proof of the theorem.

**Proof**: Since \( X \) is paracompact Hausdorff, there exists a (locally finite) part of \( I \) subordinate to \( \{ U_i \}_{i \in I} \).

This means a collection of forms \( \{ \psi_j \}_{j \in J} \) s.t.
- \( \psi_j : X \rightarrow \mathbb{R}_{\geq 0} \)
- \( \text{supp}(\psi_j) = \psi_j^{-1}(\mathbb{R}_{>0}) \) is compact in some \( U_i \)
- For each \( x \in X \), \( \exists \) an open \( W \) with \( x \in W \) s.t. only finitely many \( \psi_j \) are non-zero on \( W \).

Define, for each finite set \( S \subseteq I \),

\[
V_S = \left\{ x \in X \mid \exists s \in S, \forall i \notin S, \psi_S(x) > \psi_i(x) \right\},
\]

Note that if \( x \in X \), \( \exists \) \( W \) s.t. only \( \psi_i, \ldots, \psi_n \) are non-zero on \( W \), so \( V_S \cap W = \bigcap_{s \in S} \bigcup_{j=1}^{n} \psi_j^{-1}(\mathbb{R}_{>0}) \bigcap_{s \in S} (\psi_S - (\psi_i \cdots \psi_n))^{-1}(\mathbb{R}_{>0}) \cap W \)

which is a finite intersection of open sets in \( X \). Hence \( V_S \) is open in \( X \) for each (finite) set \( S \subseteq I \).

Note that \( V_S \subseteq U_i \) if \( \text{supp}(\psi_S) \subseteq U_i \) for some \( s \in S \).
6/4 each \( \varphi_s \) (so \( s \)) is positive on \( V_s \).

Let \( V_k = \bigcup_{l | s \subseteq l} V_s \). We claim that

\[ \{ l | s \subseteq l \} \text{ contains } k \text{ elts} \]

\[ \{ V_k \}_{k=1}^\infty \text{ is the desired cover of } X. \] It's certainly a cover, since for any \( x \in X \), \( x \in V \{ s | \varphi_s(x) > 0 \} \).

Next, we claim that \( V_s \cap V_{s'} = \emptyset \) if \( |s| = |s'|. \) Since \( S' \neq S \), \( S' \neq S \), we can choose \( s \in S \setminus S' \), \( s' \in S \setminus S'. \)

Then any point \( x \in V_s \cap V_{s'} \) would have to satisfy both

\[ \varphi_s(x) > \varphi_{s'}(x) \quad \text{and} \quad \varphi_s(x) > \varphi_s(x), \]

which is impossible.

So \( U_k = \bigcup_{s \subseteq l} V_s \), and each \( V_s \) lies in some \( U_l \), \( l | s \).

Finally, consider a part of 1 such to \( \{ V_k \}_{k=1}^\infty \), say \( \{ \varphi_{s_k} \}_{s_k \subseteq s} \), and let \( \varphi_k = \sum_{s_k \subseteq s} \varphi_{s_k} : \supp(\varphi_{s_k}) \subseteq V_k \) but \( \text{not in } V_{k-1} \).

Then \( \supp(\varphi_k) \subseteq \bigcup_{l | s \subseteq l} V_s \subseteq V_k \), \( \sum_{s | l \subseteq s \subseteq k} \varphi_s(x) = \sum_{\alpha | \alpha \subseteq l} \varphi_\alpha(x) \),

and \( \{ \varphi_k \}_{k=1}^\infty \) is locally finite by \( \{ \varphi_{s_k} \} \) was locally finite, \( \sum \varphi_\alpha \)
and each \( \varphi_\alpha \) appears as a summand in just one \( \varphi_k \).
Some Important Facts about Chern Classes

Theorem: If $X$ is a
then line bundles over $X$ are completely determined
by their first Chern class (cplx case) or their
first Stiefel-Whitney class (real case).

Proof: We need to show that if $c_1(L) = c_1(M)$
then $L = M$. Let $f: X \to CP^\infty$, $g: X \to CP^\infty$
be classifying maps for $L$ and $M$ (resp.). Then
the induced maps
$f^\ast, g^\ast: H^\ast CP^\infty = \mathbb{Z}[\alpha] \to H^\ast X$
are completely determined by the image of $\alpha$,
so $f^\ast = g^\ast$. Hence we need to show that maps
from $\text{cw cplxs}$ into $CP^\infty$ are completely determined
(up to homotopy) by their effect in cohomology (w/ $\mathbb{Z}$ - coeffs).

Theorem: If $Z$ is a space with just one
non-zero homotopy group $\pi_n(Z) = \pi$ (with $\pi$
abelian if $n = 1$) then $[X, Z] \cong H^n(X; \pi)$
for any CW cplx $X$. (unbased hom classes of maps)
Spaces like $\mathbb{Z}$, with one non-zero homology group, are called Eilenberg-MacLane spaces, and are usually denoted $\mathbb{Z} = K(\pi, 1)$. Up to homotopy, there is a unique CW model for $K(\pi, 1)$.

This result applies to both $CP^\infty = K(\mathbb{Z}, 2)$ and $RP^\infty = K(\mathbb{Z}/2, 1)$.

- $CP^\infty = Gr(1, C^\infty) = BU(1) \Rightarrow \pi_{*} CP^\infty = \pi_{*} U(1) = \mathbb{Z}$, $*$ = 2
  $\uparrow$ $\mathbb{Z}$, else
  Note: $U(1) \cong S^1$

- $RP^\infty = Gr(1, R^\infty) = BO(1) \Rightarrow \pi_{*} RP^\infty = \pi_{*} O(1) = \mathbb{Z}/2$, $*$ = 1
  $\uparrow$ $O(1) = \{ \pm I \}$
  $\mathbb{Z}$, else

[Remark: The isomorphism $\pi_{*} B G \cong \pi_{0} G$ is an isomorphism of groups.]

The isomorphism $\pi_{*} B\mathbb{Z} \cong \mathbb{Z}$
The isomorphism
\[ [\Sigma X, K(\pi, n)] \xrightarrow{\simeq} H_n(X) \]
is given by sending \( f : X \to K(\pi, n) \) to \( f^*(e) \)
for a particular "universal class" \( e \in \pi \cdot \Omega^n(K(\pi, n)) \).
Hence if two maps \( f, g : X \to K(\pi, n) \) have the same
effect in cohomology, they are homotopic. This
completes the proof. \( \Box \)

The universal class in \( H^n(K(\pi, n); \pi) \):
This class can be described in terms of the
Hurewicz map
\[ \pi_\ast : \pi_* \mathbb{Z} \to H_n(\mathbb{Z}; \mathbb{Z}) \]
\( \alpha : S^n \to \mathbb{Z} \xrightarrow{\alpha_*} \mathbb{Z}^{\pi^n} \)
Fundamental class in \( H_n(S^n; \mathbb{Z}) \)

Theorem [Hurewicz Thm]: If \( \pi_k \mathbb{Z} = 0 \) for \( k < n \),
then the Hurewicz map \( \pi_\ast : \pi_* \mathbb{Z} \to H_n(\mathbb{Z}; \mathbb{Z}) \) is an isomorphism.
(When \( n=1 \), we must also assume \( \pi_1 \mathbb{Z} \) is abelian).
Now if \( Z \) is a \( K(\pi, n) \), we define \( \mathcal{L} \subset H^n(K(\pi, n), \pi) \) to be the image of \( \text{Id}: \pi \to \pi \) under the maps
\[
\text{Hom}(\pi, \pi) \cong \text{Hom}(\pi_n(K(\pi, n), \pi) \xrightarrow{\text{Hurewicz}} \text{Hom}(H^n(K(\pi, n), \pi)) \xrightarrow{\text{natural}} H^n(K(\pi, n); \pi).
\]

This class \( \mathcal{L} \) depends only on our identification \( \pi \cong \pi_n(K(\pi, n)) \), and if we replace \( \pi \) by the isomorphic group \( \pi_n(K(\pi, n)) \) everywhere, \( \mathcal{L} \) becomes canonical.

For a proof that \( H^n(X; \pi) \cong \left[ X, K(\pi, n) \right] \)
\[
\text{f}^*([1]) \leftarrow f
\]
is an isom., see Hatcher, Chap. 4. (Possibly there will be a HW exercise containing another proof.)

We have defined and studied Chern classes and Stiefel-Whitney classes, and we observed that \( w_i, c_i \) are not always zero (b/c there exist non-trivial line bundles).

**Theorem:** The Chern classes of \( \gamma_n^m \) and the Stiefel-Whitney classes of \( \gamma_n^m \) are all non-zero.
**Proof:** It suffices to show that there exist bundles \( W_k \) such that \( c_k \) is non-zero. We'll work in the complex case; the real case is identical.

Let \( \pi_i : CP^\infty \times \cdots \times CP^\infty \to CP^\infty \) denote the \( i \)th projection, and consider the bundle \( \gamma_1 \times \cdots \times \gamma_l = \pi_1^* \gamma_1 \oplus \cdots \oplus \pi_l^* \gamma_1 \).

We have \( c(\gamma_1 \times \cdots \times \gamma_l) = \pi_1^* c(\gamma_1) = \pi_1^* (1 + c_1 \gamma_1) \),

and we claim that each term in this sum is non-zero, i.e., each Chern class \( c_1, \cdots, c_k \) of \( \gamma_1 \times \cdots \times \gamma_l \) is non-zero. This follows from the Kunneth Theorem, which says that

\[
H^\bullet (CP^\infty \times \cdots \times CP^\infty) \cong \bigotimes_{i=1}^k H^\bullet (CP^\infty)
\]

meaning that there can be no relations among the classes \( \pi_i^* (c_1 \gamma_1) \). Since the degree \( l \) term in \( \pi_1^* (1 + c_1 \gamma_1) \) is a poly. in \( c_1 \gamma_1 \), it must be non-zero. \( \square \)
In fact, more is true:

Then \[ H^0(Gr_n(C^\infty)) \cong \mathbb{Z} [c_1, c_2, \ldots, c_n] \]

where \( c_i = c_i(\gamma_n) \), the Chern classes of
the universal bundle and

\[ H^*(Gr_n(R^n, \mathbb{Z}) \cong \mathbb{Z}/2 [v_1, v_n] \]

where \( w_1 \) and \( w_n = w_i(\gamma_n) \).

This theorem says that up to multiplicative combinations, the Stiefel-Whitney/Chern classes account for all characteristic classes of vector bundles.

Sketch of proof (MS §7):

We have shown that \( H^*(Gr_n(C^\infty, \mathbb{Z}) \) and

\[ H^*(Gr_n(R^n, \mathbb{Z}) \) contain poly. algebras

on \( c_1, \ldots, c_n / w_1, \ldots, w_n \) subject to relations among

these classes would imply relation among the Chern classes of \( \gamma, \lambda, - \lambda \). MS §5 gives a cell structure on the Grassmannians, which provides the corresponding upper bound on \( H^* \).