Math 372B, HW 3

Please turn in at least 3 solutions by Thursday, 3/26.

1. a) Prove the following alternate form of the splitting principle: For any \( n \geq 1 \), the map

\[
\text{Gr}_1(\mathbb{C}^\infty) \times \cdots \times \text{Gr}_n(\mathbb{C}^\infty) \overset{\alpha}{\to} \text{Gr}_n(\mathbb{C}^\infty)
\]

classifying the complex \( n \)-plane bundle \( \gamma_1 \times \cdots \times \gamma_n \) induces an injection on integral cohomology. (Hint: use the projective bundle theorem.)

b) Deduce that there can be no algebraic relations amongst the elementary symmetric polynomials. (Think about \( x^*(C_i(y_n)) \in H^*(\text{Gr}_i(\mathbb{C}^\infty), \mathbb{Z}) \), where \( x \) is the map in (A).

c) Deduce that if \( \sum_{i=1}^n n_i = n \), then the classifying map for \( \gamma_{n_1} \times \cdots \times \gamma_{n_n} \) induces an injection on \( H^*(-, \mathbb{Z}) \).

2. (Some computative)

a) Show that the Stiefel-Whitney classes of the tangent bundle \( T(S^n) \) are all trivial. (Hint: consider the normal bundle, i.e., the orthogonal complement of \( T S^n \subseteq T R^{n+1} \).

b) Show that if \( M \) is an orientable manifold, then \( w_i(T M) = 0 \). (Hint: use our alternative defn of \( w_i \) involving loops.)

c) Show that if \( M \) is a 2-dimensional orientable manifold (so \( M = M^2 = \bigvee_{i=1}^k S^{2i} \)) then \( w_1(T M) = w_2(M) = 0 \).
3. Let $E$ and $F$ be complex 2-plane bundles over $X$.

Compute $c_i(E \otimes F)$ via the Splitting Principle (your answer should be some formula in terms of $c_i E$ and $c_i F$).

2. Problems involving the Chern Character:

4. Show that if $M^{2n+1}$ is an odd dimensional, closed, orientable manifold, then $K^0(M)$ and $K^*(M)$ have the same rank.

5. Show that if $H^{2n-1}(X; \mathbb{Q}) \neq 0$, then there exists a complex vector bundle $E \to X$ with $c_1(E) \neq 0$ such that $H^{2n}(X; \mathbb{Z}) \cong H^{2n-1}(X; \mathbb{Z})$.

6. Prove that if $f: X \to Y$ is a morphism of finite CW-complexes which induces an injection

$$H^n(Y; \mathbb{Z}) \xrightarrow{f^*} H^n(X; \mathbb{Z})$$

then it induces an injection

$$H^*(Y; \mathbb{Q}) \xrightarrow{f^*} H^*(X; \mathbb{Q}).$$

(Hint: show that $f^*$ can be identified (naturally) with

$$H^*(Y; \mathbb{Z}) \otimes \mathbb{Q} \xrightarrow{f_* \otimes \mathbb{Q}} H^*(X; \mathbb{Z}) \otimes \mathbb{Q},$$

the result then follows from the fact that $\mathbb{Q}$ is a "Flat" $\mathbb{Z}$-module.

(Hint: The Unwinding Theorem for chain complexes of finitely generated free objects applies equally well for cochain complexes of such.)