1. Let $\pi_1 \xrightarrow{f} \pi_2$ be a map of vector bundles, and assume that $\text{Rank}_x(f) = \dim \left( f(\pi_1^{-1}(x)) \right) \leq \pi_2^{-1}(x)$ is constant. Prove that $\text{Im}(f)$ and $\text{Ker}(f)$ are (locally trivial) vector bundles over $X$. (The image and kernel are simply the collections of all vectors in $\text{im}(f_x : \pi_1^{-1}(x) \to \pi_2^{-1}(x))$ or $\text{ker}(f_x : \pi_1^{-1}(x) \to \pi_2^{-1}(x))$.)

[Hint: this is similar to the proof, in MS, that the orthogonal complement of a subbundle is locally trivial (MS Thm 3.3), which we used in class to write the pullback $\rho(E_1 \times X)$ as a sum: $\rho^* E = L_E \oplus L_E^\perp$. It's also similar to the proofs, in the notes, that tangent bundles and Stiefel bundles are locally trivial.]

2. a) Let $\pi \xrightarrow{E} X$ be a vector bundle with transition maps $\phi_{ij} : U_i \cap U_j \to \text{GL}_n \mathbb{R}$ (or $\text{GL}_n \mathbb{C}$), where $\{U_i\}$ is an open cover of $X$ over which $E$ is trivial. Calculate the clutching maps of the dual bundle $E^* = \text{Hom}(E, \mathbb{R})$ over the same sets $U_i \cap U_j$. 
b) Let \( E_1 \overset{\varphi}{\to} E_2 \) be as in a), and let \( \varphi_i : U_1 \cap U_i \to GL_m R \) and \( \psi_i : U_i \cap U_j \to GL_m R \) be the transition functions. Describe the transition functions of \( E_1 \otimes E_2 \) in terms of \( \varphi_i \) and \( \psi_i \).

3. (M5 Problem 4-A) Compute the Stiefel-Whitney (or Chern) classes of a Cartesian product \( E \times F \) in terms of the classes of \( E \) and \( F \). (Hint: Find a way to apply the Whitney sum formula.)

4. We say that bundles \( E \overset{\varphi}{\to} X \) and \( F \overset{\psi}{\to} X \) are stably isomorphic if there exists \( n \geq 0 \) s.t. \( E \oplus (X \times \mathbb{R}^n) \cong F \).

How are the Stiefel-Whitney classes of \( E \) and \( F \) related if \( E \) is stably isomorphic to \( F \)? (The cplx case is identical.)

The (reduced) \( K \)-theory group \( \tilde{K}_0(X) \) is the group \( \text{Vect}(X)/(\text{stable isomorphism}) \). Deduce that \( w_i, c_i \) are well-defined functions \( \tilde{K}_0(X) \to H^*(X) \), where in the real case we use \( \mathbb{Z}/2 \)-coeffs and in the cplx case we use \( \mathbb{Z} \)-coeffs.
5. a) Say $A \subseteq X$ is a $CW$ pair (i.e. $A$ is a subcomplex of $X$). Prove that for any space $Y$, the restriction function $\map{(X, Y)}{F}{\map{(A, Y)}}{F|_A}$ is a (Serre) fibration. [Hint: as explained in Hatcher Chapter 0, all $CW$ pairs are cofibrations, i.e. they satisfy the Homotopy Extension Property.]

Remark: You'll need the following fact regarding the compact-open topology on mapping spaces: If $W$ is locally compact and regular (e.g. a $CW$ complex) then a function $F : U \times W \to V$ is continuous if and only if its "adjoint" $U \to \map{(W, V)}{F}$ is continuous. (The "only if" direction doesn't require any hypotheses on $W$.)

b) Show that the map $p(x) \overset{\text{def}}{=} \map{((0,1), (x, x_0))}{X}$ defined by $y \mapsto y(1)$, is a fibration, whose fiber over $x_0 \in X$ is the based loop space $\Omega_{x_0}(X) = \map{(S^1, 1)}{(x, x_0)}$.

c) Show that for any top. gp. $G$ and any universal principal $G$-bundle $EG \overset{\pi}{\to} BG$ with $EG$ contractible (not just if $\pi_* EG = 0$), there is a weak homotopy equivalence $\Omega BG \overset{f_*}{\to} G$, i.e. $f_* : \pi_* \Omega BG \to \pi_* G$ is an isom. [Hint: compare the fibration $BG \overset{\pi}{\to} BG$ to the one in part b).]
6. Show that the Gram-Schmidt orthogonalization process gives a deformation retraction \( V_n(C^{n+k}) \to V_n(C^{n+k}) \) (and similarly \( V_n(R^{n+k}) \to V_n(R^{n+k}) \)). In other words, check that the GS map is homotopic to \( \text{Id}_{V_n} \) through a htpy that fixes \( V_n \) pointwise at all times.

Note that when \( k=0 \), this gives \( \text{GL}_n(C) = U(n) \), \( \text{GL}_n(R) = O(n) \).

7. a) Let \( \phi : G \to H \) be a continuous group homomorphism and let \( \pi_1, \pi_2 \) be universal principal bundles for \( G \) and for \( H \). Show that there exists a diagram \( G \xrightarrow{\pi_1} H \xrightarrow{\pi_2} \text{equi.} \)

\[
\begin{array}{ccc}
G & \xrightarrow{\phi} & H \\
\downarrow & & \downarrow \\
EG & \xrightarrow{\tilde{\phi}} & EH \\
\downarrow & & \downarrow \\
BG & \xrightarrow{p} & BH \\
\end{array}
\]

(with the maps \( G \xrightarrow{\phi} H \) equivariant isomorphisms onto fibers of \( \rho, p \)). You may assume that \( EH \) classifies principal \( H \)-bundles over \( BG \), which holds if \( BH \)

i) \( BG \) is a CW cplx; or

ii) \( BG \) is paracompact and \( BH \)
is either Milnor's model for the classifying space - with \( EH = \ast_H \)
or \( H = \text{GL}_n(R), O(n), \text{GL}_n(C), \text{U}(n) \) and \( BH = Gr_n(R^n) / Gr_n(C^n) \).

b) Conclude that if \( \phi \) is a weak htpy equiv. (e.g. see #6), then so is \( BG \xrightarrow{\phi} BH \).

c) Conclude that the isomorphism \( \beta_n BG \cong \pi_n C \) (HW1, #3) is independent of \( EG \) so long as \( BG \) is a CW cplx.
d) Given \( f: G \to H \), a principal \( G \)-bundle over \( X \) produces a principal \( H \)-bundle over \( X \) by choosing clutching funs \( \{ \psi_{ij} \} \) for \( E \to X \), which are maps

\[
\psi_{ij} : U_i \cap U_j \to G
\]

(\( U_i \) some cover of \( X \)), and forming an \( H \)-bundle via the clutching funs

\[
\phi_{ij} : U_i \cap U_j \to H.
\]

Using b), show that this process gives a well-defined bijection between isomorphism classes of \( G \)-bundles and isomorphism classes of \( H \)-bundles over any CW complex \( X \).

(Assume that nice models for \( B_G \) and \( B_H \) exist, as in the previous parts of the problem.)