Lecture 10
Cohomology of Projective Spaces

Before addressing the Projective Bdl. Thm.,
we consider the case of computing $H^*(\mathbb{RP}^n; \mathbb{Z}/2)$
and $H^*(\mathbb{CP}^n; \mathbb{Z})$.

Propn.: For each $n$, $\mathbb{RP}^n$ is a CW cplx with
one $e_k$ in each dimension $1 \leq k \leq n$. The
$(n-1)$-skeleton of $\mathbb{RP}^n$ is precisely $\mathbb{RP}^{n-1}$, and the attaching
map $\psi_n: S^{n-1} \to \mathbb{RP}^{n-1} = (\mathbb{RP}^n)^{(n-1)}$ is just the quotient map
defining $\mathbb{RP}^n$.

PF: By induction on $n$. For $\mathbb{RP}^1 = S^1$, there is nothing
to prove. Now, $\mathbb{RP}^n = S^n/\times_{x} -x$. If $D^n \hookrightarrow S^n$ is inclusion
of the upper hemisphere, then the induced map $D^n/\times_{x} -x \to S^n/\times_{x} -x = \mathbb{RP}^n$
is a homeomorphism. But the subspace $\partial D^n/\times_{x} -x \leq D^n/\times_{x} -x$ for $x \in \mathbb{D}^n$
is precisely $\mathbb{RP}^{n-1}$! So $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup \partial D^n$ and the attaching
map is as described.

□

Corollary: $H^k(\mathbb{RP}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$ for $0 \leq k \leq n$, and 0 otherwise.

Proof: Since $H^k(X; \mathbb{Z}/2) \cong \text{Hom}(H_k(X; \mathbb{Z}/2), \mathbb{Z}/2)$ (Univ. Corr. Thm.)

it suffices to compute $H_k(\mathbb{RP}^n; \mathbb{Z}/2)$. We will compute
$H_\ast(\mathbb{R}P^n; \mathbb{Z}/2)$ via cellular homology. The cellular chain complex has the form
\[
0 \to \mathbb{Z}/2 \xrightarrow{d_n} \mathbb{Z}/2 \to \mathbb{Z}/2 \to \cdots \xrightarrow{d_2} \mathbb{Z}/2 \xrightarrow{d_1} \mathbb{Z}/2 \to 0,
\]
where $d_k$ is the cellular boundary map. It is simply multiplied by the degree (mod 2) of the composite
\[
S^{k-1} \xrightarrow{q_k} \mathbb{R}P^{k-1} \xrightarrow{q_k^{-1}} \mathbb{R}P^n \xrightarrow{\partial_k} D^{k-1} \xrightarrow{\partial_k^{-1}} S^{k-1}
\]
where $q_k : D^{k-1} \to \mathbb{R}P^n$ is the characteristic map for the $(k-1)$-cell. This map restricts to homomorphisms $S_{\pm}^{k-1} \xrightarrow{\pm} D^{k-1} - D^{k-1}$ and $S_0^{k-1} \xrightarrow{\pm} D^{k-1} - 2D^{k-1}$ (the hemispheres) so the degree is the (signed) number of inverse images of any $x \in D^{k-1} - 2D^{k-1}$, which is $(\pm 1) + (\pm 1) = 0 \bmod 2 = 0$. So all the $d_k$ are 0, and we have $H_\ast(\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2$ for $0 \leq k \leq n$, and $H_\ast(\mathbb{R}P^n; \mathbb{Z}/2) = H_{n+1}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$ for $0 \leq k \leq n$.

Before computing the ring structure, we consider the complex case.

**Propn:** $\mathbb{C}P^n$ is a CW complex with one cell in each even dimension $(0, 2, 4, \ldots, 2n)$. The $(2n-2)$-skeleton of $\mathbb{C}P^n$ is $\mathbb{C}P^{n-1}$ and the $2n$ cell is attached via $S^{2n-1} \xrightarrow{\partial} \mathbb{C}P^{n-1}$, the quotient map.
Proof: $\mathbb{C}P^n = S^{2n+1}/\{vuv \text{ for } \lambda \in S^1 \}$, and we need to view this as $D_+^{2n}/\{vuv \text{ for } \lambda \in D^{2n} = S^{2n-1} \}$; the latter is precisely a $2n$-cell attached to $\mathbb{C}P^{n-1} = S^{2n-1}/\{uv\lambda v \}$ via the quotient map. The disk we will use is $D_+^{2n} = \{(z_1, \ldots, z_{n+1}) \in S^{2n+1} \subset \mathbb{C}^{n+1} \mid z_{n+1} \in \mathbb{R}_{>0} \}$, which is just a hemisphere. The map $D_+^{2n} \to S^{2n+1} \to \mathbb{C}P^n$ is onto, and it's injective on $\{(z_1, \ldots, z_{n+1}) \in D_+^{2n} \mid z_{n+1} > 0 \}$ (be under rotation by $\lambda = e^{i\theta} \in S^1$, each $z \in \mathbb{C}$ is identified with exactly one pt. on $\mathbb{R}_{>0}$). So $D_+^{2n}/\{uv\lambda v \text{ for } \lambda \in D^{2n} \} \cong \mathbb{C}P^n$. □

Corollary: $H_\ast(\mathbb{C}P^n; \mathbb{Z}) = H^\ast(\mathbb{C}P^n; \mathbb{Z}) = \{ \mathbb{Z}, \ast \text{ even} \}$. □

Proof: The cellular chain cplx has the form $0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \cdots \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0$ b/c all the cells are in even dim's. So the bdy maps are automatically zero. The same goes for cohomology. □
The cup product structure on $H^k \mathbb{RP}^n$, $H^k \mathbb{CP}^n$ can be computed via somewhat delicate geometric arguments (see Hatcher) but can also be deduced from Poincaré Duality.

Prop (Hatcher 3.38): The cup product pairing

$$H^k(M; \mathbb{Z}) \times H^{n-k}(M; \mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$\phi, \psi \longrightarrow (\phi \cup \psi)[M]$$

is non-singular for any orientable mfld $M$ with $H^*(M; \mathbb{Z})$ torsion-free.

Similarly, the corresponding pairing with $\mathbb{Z}/2\mathbb{Z}$-coeffs is non-singular for any mfld.

\textbf{Remark:} Such a bilinear pairing $A \times B \to R$ is called \textbf{non-singular} if the induced maps

$$A \to \text{Hom}(B; R), \quad B \to \text{Hom}(A; R)$$

are isomorphisms.

\textbf{Proof:} We can factor the map $H^k \to \text{Hom}(H^{n-k}\mathbb{Z})$

$\phi \mapsto (\psi \mapsto (\psi \circ \phi)[M])$

as follows (up to sign):

$$H^k(M; \mathbb{Z}) \xrightarrow{\cup} \text{Hom}(H_{n-k}(M; \mathbb{Z}), \mathbb{Z}) \xrightarrow{D^*} \text{Hom}(H^{n-k}(M; \mathbb{Z}), \mathbb{Z}),$$

$$\phi \mapsto (\sigma \mapsto \phi(\sigma)) \quad f \mapsto (\psi \mapsto f(\phi \circ \psi)[M])$$

b/c the composite is $\phi \mapsto (\psi \mapsto \phi(\psi \circ [M]) = \pm(\psi \circ \phi)[M])$. The first map is an isom. by the UCT, the 2nd by Poincaré Duality. The $\mathbb{Z}/2\mathbb{Z}$ case is the same. Switching $k$ and $n-k$ simplifies the proof. \qed
Corollary: \( H^* (\mathbb{C}P^n, \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1}) \), with \( |\alpha| = 2 \)

Ring isomorphism \( \downarrow \)

\( H^* (\mathbb{R}P^n, \mathbb{Z}/2) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1}) \), with \( |\alpha| = 1 \).

Proof: There is a ring map \( \mathbb{Z}[\alpha]/(\alpha^{n+1}) \to H^* (\mathbb{C}P^n, \mathbb{Z}) \), defined by sending \( \alpha \mapsto \alpha = c_1 (\gamma) \in H^2 (\mathbb{C}P^n, \mathbb{Z}) \) (the canonical generator). By our additive computation of \( H^* (\mathbb{C}P^n, \mathbb{Z}) \), it will suffice to show that \( \alpha^k \in H^{2k} (\mathbb{C}P^n, \mathbb{Z}) \) is a generator (for \( k = 2, \ldots, n \)). We prove this by induction on \( n \). For \( n = 1 \), there is nothing more to prove, since we know \( \alpha \) generates \( H^2 (\mathbb{C}P^1, \mathbb{Z}) \). The inclusions \( \mathbb{C}P^{n-1} \to \mathbb{C}P^n \) induce isomorphisms \( H^k \mathbb{C}P^{n-1} \cong H^k \mathbb{C}P^n \) for \( k \leq 2n-2 \), since \( \mathbb{C}P^n \) is formed from \( \mathbb{C}P^{n-1} \) by attaching a \( 2n \)-cell. (In the real case, \( H^*(\mathbb{R}P^n, \mathbb{Z}/2) \cong H^*(\mathbb{R}P^n, \mathbb{Z}/2) \) for \( k \leq n-1 \), because the cellular boundary maps in \( C^*(\mathbb{R}P^n, \mathbb{Z}/2) \) are 0, so \( C^* \cong H^* \).) So by induction we may assume that \( \alpha^k \) generates \( H^{2k} (\mathbb{C}P^n, \mathbb{Z}) \) for \( k < n \).

By the Propin, the homomorphism \( H^2 (\mathbb{C}P^n, \mathbb{Z}) \to \mathbb{Z} \)

\( \alpha \mapsto 1 \)
is given by \( \varphi \mapsto (\varphi \cup \nu_1 [\mathbb{C}P^n]) \) for some class \( \varphi \in H^{2n-2}(\mathbb{C}P^n; \mathbb{Z}) \). So

\[ 1 = f(\alpha) = (\varphi \cup \nu_2 [\mathbb{C}P^n]), \]

which means that neither \( \varphi \) nor \( \varphi \cup \nu_2 \) can be written as \( \nu_3 \cup p \), with \( p \in H^n(\mathbb{C}P^n; \mathbb{Z}) \). Then \( \varphi \) generates \( H^{2n-2}(\mathbb{C}P^n; \mathbb{Z}) \) and we have \( \varphi = \pm \alpha^{n-1} \) by induction, and similarly \( \varphi \cup \nu_2 \) generates \( H^{2n}(\mathbb{C}P^n; \mathbb{Z}) \).

Now \( \varphi \cup \nu_2 = \pm \alpha^{n-1} \nu_2 = \pm \alpha^n \), so \( \alpha^n \) generates \( H^{2n}(\mathbb{C}P^n; \mathbb{Z}) \). \( \square \)