Lecture 7

Axioms for Chern Classes and Stiefel-Whitney Classes

The Stiefel-Whitney classes \( w_i(V) \in H^i(B; \mathbb{Z}/2) \) are defined for real vector bundles \( V \to B \) (or, equivalently, \( GL_n \mathbb{R} \)-or \( OW \)-bundles).

The Chern classes \( c_i(V) \in H^{2i}(B; \mathbb{Z}) \) are defined for complex vector bundles \( V \to B \) (equivalently, \( GL_n \mathbb{C} \) or \( U(n) \)-bundles).

(In both cases, we assume \( B \) is paracompact.)

Theorem: There exist unique sequences \( c_1, c_2, \ldots \) and \( w_1, w_2, \ldots \) of characteristic classes (for \( GL_n \mathbb{R} / GL_n \mathbb{C} \)-bodies, respectively) with \( \dim(w_i) = i \), \( \dim(c_i) = 2i \), and coefficients in \( \mathbb{Z}/2 \) and \( \mathbb{Z} \) satisfying the following axioms:

1. \( w_i(V) = 0 \) for \( i > \dim(V) \), \( w_0(V) = 1 \in H^0(B; \mathbb{Z}/2) \), \( c_i(V) = 0 \) for \( i > \dim(V) \), \( c_0(V) = 1 \in H^0(B; \mathbb{Z}) \).

2. If \( V, W \) are bundles over \( B \), then

\[
\begin{align*}
\text{"Whitney Sum Formula"} & \\
W_k(V \oplus W) = \sum_{i=0}^k w_i(V) \cup w_{k-i}(W) & (V, W \text{ real}) \\
c_k(V \oplus W) = \sum_{i+j=k} c_i(V) \cup c_j(W) & (V, W \text{ cplx})
\end{align*}
\]

3. \( -w_1(V(\mathbb{C})) \neq 0 \) in \( H^1(G_r, \mathbb{R}^o, \mathbb{Z}/2) = \mathbb{Z}/2 \) (Note: \( G_r, \mathbb{R}^o = \mathbb{R}^o \))

and this bundle is the canonical bundle over \( S^1 \), and \( H^1(S^1, \mathbb{Z}/2) = \mathbb{Z}/2 \).

\( -c_1(V(\mathbb{C})) \) is \( H^1(GL_1, \mathbb{C}^o, \mathbb{Z}) = H^1(GL_1, \mathbb{R}^o, \mathbb{Z}) \neq 0 \) is the class of the canonical bundle on \( S^1 \), and \( H^1(S^1, \mathbb{Z}/2) = \mathbb{Z}/2 \).
Note that the last axiom rules out the possibility \( w_i = c_i = 0 \) for all \( i \). Also, since all line bundles over \( \mathbb{C}^2 \) split are pulled back from \( \mathbb{C}^2 \), Axiom 3 determines \( c_i(L) \) for all line bundles \( L \). (It's actually enough to just assume \( c_1(L) \) and \( w_1(L) \) are the canonical generators, as we'll see.) To understand the Whitney sum formula, we need to define the bundle \( \mathbb{B} \), whose fiber over \( b \in \mathbb{B} \) is canonically \( V_b \oplus W_b \) (the sum of the fibers).

Note that if we define \( W(V) = \bigoplus_{i=0}^{\dim V} w_i(V) \in \bigoplus_{i=0}^{\infty} H^i(B; \mathbb{Z}) \) and \( C(V) = \bigoplus_{i=0}^{\dim V} c_i(V) \in \bigoplus_{i=0}^{\infty} H^i(B; \mathbb{Z}) \), then the Whitney sum formula takes the form:

\[
W(V \oplus W) = W(V) \cdot W(W) \quad (V, W \text{ real})
\]
\[
C(V \oplus W) = C(V) \cdot C(W) \quad (V, W \text{ complex})
\]

where the multiplication takes place in the graded ring

\[
H^*(B; \mathbb{Z}) = \bigoplus_i H^i(B; \mathbb{Z}) \quad \text{or} \quad H^*(B; \mathbb{Z}) = \bigoplus_i H^i(B; \mathbb{Z}).
\]

Whitney Sums:

Given two bundles \( V = \bigoplus_b V_b \) and \( W = \bigoplus_b W_b \), we define \( V \oplus W \) to have total space \( V \oplus W = V \times W = \{(v, w) | \pi_V(v) = \pi_W(w)\} \). The fibers are vector spaces (over \( \mathbb{R} \) or \( \mathbb{C} \)) by component-wise addition and scalar multiplication. If \( U \subseteq B \) is an open set over which both \( V \) and \( W \) are trivial, \( V \mid_U \times \bigoplus_{b \in B} W_b \). 

\[ V \oplus W \]
\[ U \times \mathbb{R}^n \xrightarrow{\varphi} W|_U \]

\[ U \times \mathbb{R}^n \times \mathbb{R}^m \xrightarrow{(v, \psi)} V \times W|_U \]

is a homeomorphism.

\[ \text{Because } (V \times W)|_U = V|_U \times W|_U \xrightarrow{(\varphi|_U, \psi)} U \times \mathbb{R}^n \times \mathbb{R}^m \text{ gives the inverse.} \]

[Here \( \mathbb{R}^n, \mathbb{R}^m \) can of course be replaced by \( \mathbb{C}^n, \mathbb{C}^m \).]

There are several other ways to view \( V \oplus W \).

- There is a bundle \( V \downarrow \) associated to any \( \frac{V}{B_1, B_2} \).
  
  (The topology on \( V \times W \) is the product topology, and the bundle is trivial over \( U_1 \times U_2 \) if \( V|_{U_1} \cong U_1 \times \mathbb{R}^n, W|_{U_2} \cong U_2 \times \mathbb{R}^m \).)

  The Whitney sum is then the pullback \( \delta : (V \times W) \rightarrow V \times W \)

  \[ \begin{array}{ccc}
  \mathbb{R}^n & \to & \mathbb{R}^n \times \mathbb{R}^m \\
  \downarrow & & \downarrow \\
  \mathbb{R}^n & \to & \mathbb{R}^n \times \mathbb{R}^m \\
  \end{array} \]

  \[ \text{If } \{ U_{ij}, \varphi_{ij} ; U_i \cap U_j \rightarrow \mathbb{C}^n \mathbb{R}^j \}_{i,j} \text{ and } \{ \psi_{ij} ; U_i \cap U_j \rightarrow \mathbb{C}^m \mathbb{R}^j \}_{i,j} \]

  give clutching data for \( V \) and \( W \) (respectively),

  then \( \varphi_{ij} \circ \psi_{ij} : U_i \cap U_j \rightarrow \mathbb{G}^{n+m}(\mathbb{R}) \) gives

  clutching data for \( V \oplus W \). Here \( \varphi_{ij} \circ \psi_{ij} \) is

  defined via the block-sum maps \( \mathbb{G}^n \mathbb{R} \times \mathbb{G}^m \mathbb{R} \rightarrow \mathbb{G}^{n+m} \mathbb{R} \).

  \[ [A], [B] \rightarrow [A \circ B] \]

  To see this, we just need to look at the trivializations

  given above: the transitions for \( V \oplus W \) have the form

  \[ (\varphi_{ij}, \psi_{ij})^{-1} \circ (\varphi_{i'}, \psi_{i'}) : U_i \cap U_{i'} \times \mathbb{R}^n \rightarrow U_i \cap U_{i'} \times \mathbb{R}^m \]

  and the matrix for this transformation at \( u \in U_i \cap U_{i'} \) is exactly \( \begin{bmatrix} v_{ij} & 0 \\ 0 & v_{ij} \end{bmatrix} \).
In MS 53, a general construction is given, which works for other "continuous" functors such as $\otimes$, $\text{Hom}$, etc.

It is easy to see that their description of $V\otimes W$ agrees with the first one given above. We'll return to the general construction later.

In order to prove the existence of Stiefel-Whitney and Chern classes, we'll study the cohomology of projective bundles.

**Defn:** Let $\frac{E}{B}$ be a (real or cpx) vector bundle. The projective bundle associated to $E$ is the space $\mathbb{P}(E) = \frac{(E-E_0)/\text{Xuex}}{\text{Xuex}}$, where $E_0$, the zero section of $E$, consists of all the zero vectors, $c(1(0))$.

**Lemma:** The natural projection $\frac{\mathbb{P}(E)\times x}{\text{Xuex}}$ is a locally trivial fiber bundle whose fiber is the projective space $\mathbb{P}^{n-1}$ or $\mathbb{C}P^{n-1}$ (when $V$ is a $\mathbb{R}^n$ or a cpx bundle, respectively).

**Proof:** It suffices to check that $\frac{(\mathbb{R}^n-E_0)\times U}{\text{Xuex}}$ is homeomorphic to $\mathbb{P}^{n-1}\times U$, but this is immediate. \(\square\)

Our goal will be to understand the cohomology of projective space bundles (with $\mathbb{Z}$ coeffs in the cpx case, and with $\mathbb{Z}_2$ coeffs in the real case).

These cohomology groups will be described in terms of the Chern/Stiefel-Whitney classes of the topological line bundle.
Definition: If \( \mathcal{E} \) is a vector bundle, the tonological line bundle \( \gamma = \gamma_\mathcal{E} \) over \( \mathcal{P}(\mathcal{E}) \) is defined by \( \gamma = \{ (L, v) \in \mathcal{P}(\mathcal{E}) \times \mathcal{E} \mid v \in L \} \subseteq \mathcal{P}(\mathcal{E}) \times \mathcal{E} \). (Here we think of points in \( \mathcal{P}(\mathcal{E}) \) as lines through the origin in the fibers of \( \mathcal{E} \).)

Note that by definition, the restriction of \( \gamma_\mathcal{E} \) to any fiber of \( \mathcal{P}(\mathcal{E}) \) is precisely the tonological line bundle on that fiber.

Lemma: \( \gamma_\mathcal{E} \) is a locally trivial line bundle over \( \mathcal{P}(\mathcal{E}) \).

Proof: Say \( \mathcal{P}(\mathcal{E})|_U = U \times \mathbb{R}^{n-1} \) for some \( U \). If \( \gamma_\mathcal{E}|_{\mathbb{R}^{n-1}} \) is trivial over \( W \subseteq \mathbb{R}^{n-1} \), then \( \gamma_\mathcal{E}|_{U \times W} \) is trivial as well.

The Projective Bundle Theorem:

Let \( \mathcal{P}(\mathcal{E}) \) be the projective bundle associated to a cplx n-plane bundle \( \mathcal{E} \).

Then the map \( \pi^*: H^*(B; \mathbb{Z}) \rightarrow H^*(\mathcal{P}(\mathcal{E}); \mathbb{Z}) \) is injective, and there is an isomorphism of graded \( H^*(B; \mathbb{Z}) \)-modules

\[
H^*(B; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[x]/(x^{n+1}) \xrightarrow{\alpha} H^*(\mathcal{P}(\mathcal{E}); \mathbb{Z}),
\]

where \( \text{deg}(x) = 2g \) and for each \( i \), \( \alpha(i^*x^i) = c_i(\mathcal{L}_E) \), the \( i \)th cup-power of \( c_i(\mathcal{L}_E) \). In particular, \( H^*(\mathcal{P}(\mathcal{E}); \mathbb{Z}) \) is free as an \( H^*(B; \mathbb{Z}) \)-module.

If \( \mathcal{E} \) is a real n-plane bundle, we have

\[
H^*(B; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[x]/(x^{n+1}) \xrightarrow{\alpha} H^*(\mathcal{P}(\mathcal{E}); \mathbb{Z}) \quad \text{for} \ i = 0, \ldots, n-1.
\]
Note: This theorem does not describe the full ring structure of $H^*(PE)$.

To make sense of the first Chern/Stiefel-Whitney classes of $\frac{LE}{PE}$, we need to know that bundles over $PE$ are pulled back from $\frac{V_1(\mathbb{C}^\infty)}{Gr_1(\mathbb{C}^\infty)}$. This is somewhat subtle. The argument in MS 5.6, which shows that all bundles over paracompact spaces are pulled back from $\frac{V_m^{\mathbb{R}^\infty}}{Gr_m^{\mathbb{R}^\infty}}$, can be modified to work in the complete case, and there is also a "projective" version which shows that $LE$ has a classifying map.

This is explained in Hatcher's Vector Bundles notes. In the real case, we'll see that there is a simple general definition of $w$ that can be used.

**Grothendieck's Definition of Chern/Stiefel-Whitney Classes:**

Given a complex $n$-plane bundle $\frac{E}{B}$, the class $c_i(LE) \in H^*(PE; \mathbb{Z})$ must be expressible in terms of the basis $1, c_1(LE), \ldots, c_i(LE)^{\cdot n}$ for $H^*(PE; \mathbb{Z})$ as an $H^*(B; \mathbb{Z})$-module. In other words, there are unique elements $c_1(E), \ldots, c_n(E) \in H^*(B; \mathbb{Z})$ s.t.

$$c_i(LE) = (-1)^i c_n(E) \cdot 1 + \sum_{j=1}^{i-1} c_{i-j}(E) \cdot c_j(LE) + \cdots + c_1(E) \cdot c_i(LE)^{n-1}.$$  

**Rank:** For Stiefel-Whitney classes, the signs have no effect because we work with $\mathbb{Z}_2$ coefs.

**Def'n:** The class $c_i(E) \in H^*(B; \mathbb{Z})$ appearing in (A) is the $i$th Chern class of the bundle $LE$, and the Stiefel-Whitney classes of real $n$-plane bundles are defined analogously.
Here is a straightforward general definition of the class $W_1\left(\frac{E}{X}\right)$, where $L$ is a real line bundle. (One can define $W_1\left(\frac{E}{X}\right) \subseteq H'(X; \mathbb{Z}/2)$ similarly for any real bundle $E$.)

First, we reinterpret the group $H'(X; \mathbb{Z}/2)$.

By the Univ. Coeff. Thm., we have

$$H'(X; \mathbb{Z}/2) \cong \text{Hom}(H_1(X; \mathbb{Z}), \mathbb{Z}/2) \otimes \text{Ext}(H_0(X; \mathbb{Z}), \mathbb{Z}/2)$$

$$\cong \text{Hom}(H_1(X; \mathbb{Z}), \mathbb{Z}/2)$$

But $H_0(X; \mathbb{Z}) = \mathbb{Z}$ is free. Since $H_1(X; \mathbb{Z}) = \pi_1(X)$, we find that

$$H'(X; \mathbb{Z}/2) \cong \text{Hom}(\pi_1X, \mathbb{Z}/2).$$

Claim: The first Stiefel-Whitney class $W_1\left(\frac{E}{X}\right)$ corresponds to the function $\pi_1X \xrightarrow{w_1} \mathbb{Z}/2$ defined by

$$w_1([x]) = \begin{cases} 1, & \text{if } x^*L \text{ is non-trivial} \\ 0, & \text{else} \end{cases}$$

Proof: The function is well-defined by the Bundle Htpy Thm. First, let's check the formula on the universal line bundle $\overline{\gamma} \xrightarrow{\overline{\gamma}} \mathbb{R}P^2$. We know that $\pi_1(\mathbb{R}P^2) \cong \pi_1\mathbb{R}P^2 = \mathbb{Z}/2$, so both $H'(\mathbb{R}P^2; \mathbb{Z})$ and $\text{Hom}(\pi_1\mathbb{R}P^2; \mathbb{Z}/2)$ have a single non-zero element, which by definition is $W_1([\gamma])$. On the other hand, the pullback of $\overline{\gamma}$ along the generator $\alpha: S^1 = \mathbb{R}P^1 \to \mathbb{R}P^2$
Proof of Lemma 2.2. (Real Case)

Here we prove the commutativity for any $f: X \to \mathbb{R}^m$.

Colonel of map $TV = Z_{1,2}$, this means it is constant.

Choose a metric on $V$ giving an isomorphism $V \cong \mathbb{R}^s$. Make the $x$-coordinates.

$\mathbb{R}^s \times V \cong \mathbb{R}^n$.

The unique non-trivial $M: \mathbb{R}^{m} \to \mathbb{R}^1$.

To complete the proof, we just note that the new defn is natural under pullbacks and the diagram.

is precisely the topological ball over $\mathbb{R}^1$ which we have shown is non-trivial. So our new defn also gives us

$Z_1$. 

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**Rmk:** We could take this as our defin
of the First Stiefel-Whitney class of line bundles. In fact, it makes sense for any line bundle \( \xi \) over \( X \), even if \( X \) is not paracompact (and then \( L \) need not be
pulled back from \( \xi \)).

So in the real case, we don't need to
use Milnor's result (MS §5) that bundles
over paracompact spaces are pulled back from
the universal bundle. (Although we have
just shown that this new defn agrees
with the old when whenever Milnor's result applies.)