Instructions: Write your name at the top. You have ten minutes to complete the following four questions. Show your work as clearly as possible. No calculators, books, or notes are allowed.

1. Fill in the blanks in the statements of the following theorem.

Theorem: If \( f(x) \) is a \underline{continuous} function on the interval \([a, b]\) and \( M \) is any number between \( f(a) \) and \( f(b) \), then there exists a number \( \alpha \) in the interval \( (a, b) \) satisfying \( f(\alpha) = M \).

2. Show that the polynomial \( p(x) = x^5 - x^4 - x - 1 \) has a root (i.e. show that for some real number \( x \), \( p(x) = 0 \)).

We use the Intermediate Value Theorem. This means we need to find points \( x = a \) and \( x = b \) where \( p(a) < 0 \), \( p(b) > 0 \).

\[
p(1) = -1, \quad p(2) = 2^5 - 2^4 - 2 - 1 = 32 - 16 - 2 - 1 = 13 > 0.
\]

So by the IVT, there is some \( x \in (-1, 2) \) with \( p(x) = 0 \).

3. a) Use the Squeeze Theorem to calculate \( \lim_{x \to 0} x^2 \cos(1/x) \).

Since \( -1 \leq \cos(\frac{1}{x}) \leq 1 \), we have \( -x^2 \leq x^2 \cos(\frac{1}{x}) \leq x^2 \).

b) Compute \( \lim_{x \to 0} \frac{1 - \cos(3x)}{\sin(6x)} \).

See both sides for another sol'n, not involving absolute values.

\[
\lim_{x \to 0} \frac{1 - \cos(3x)}{\sin(6x)} = \lim_{x \to 0} \frac{1 - \cos(3x)}{3x} \cdot \lim_{x \to 0} \frac{3x}{\sin(6x)} = \lim_{x \to 0} \frac{3}{2} \lim_{x \to 0} \frac{\sin(3x)}{6x} = \frac{3}{2} \lim_{x \to 0} \frac{3x}{6x} = \frac{3}{2} \lim_{x \to 0} \frac{\sin(3x)}{3x} = \frac{3}{2} \lim_{x \to 0} 1 = \frac{3}{2}.
\]

c) Compute \( \lim_{x \to 0} \frac{e^{\sin x}}{x} \).

\[
\lim_{x \to 0} \frac{e^{\sin x}}{x} = e \lim_{x \to 0} e^{\frac{\sin x}{x}} = e \lim_{x \to 0} \frac{\sin x}{x} = e \cdot 1 = e.
\]
Another way to solve 3a):

First we compute \( \lim_{x \to 0^+} x^3 2 \cos(\frac{1}{x^2}) \).

For \( x > 0 \), we found that
\[
\frac{1}{2} x^3 \leq x^3 \cos(\frac{1}{x^2}) \leq 2 x^3.
\]
So letting \( U^+(x) = 2 x^3 \), \( L^+(x) = \frac{1}{2} x^3 \), we have
\[
\lim_{x \to 0^+} U^+(x) = \lim_{x \to 0^+} L^+(x) = 0,
\]
and by the Squeeze Theorem, \( \lim_{x \to 0^+} x^3 2 \cos(\frac{1}{x^2}) = 0 \).

Next we compute \( \lim_{x \to 0^-} x^3 2 \cos(\frac{1}{x^2}) \).

We found that for \( x < 0 \),
\[
2 x^3 \leq x^3 2 \cos(\frac{1}{x^2}) \leq \frac{1}{2} x^3,
\]
so let \( U^-(x) = \frac{1}{2} x^3 \), \( L^-(x) = 2 x^3 \). Then
\[
\lim_{x \to 0^-} U^-(x) = \lim_{x \to 0^-} L^-(x) = 0,
\]
so by the Squeeze Theorem, we have
\[
\lim_{x \to 0^-} x^3 2 \cos(\frac{1}{x^2}) = 0.
\]
Since the left- and right-handed limits are both 0, we conclude that \( \lim_{x \to 0} x^3 2 \cos(\frac{1}{x^2}) = 0 \) too.