In this note we prove some of the standard results of commutative ring theory that lead up to proofs of the main theorem of dimension theory and of the Nullstellensatz. An alternative approach to dimension theory is given in Chapter 11 of Atiyah and Macdonald [AM], who base their approach on the analysis of the Hilbert function of a local ring.

We need a couple of facts about localization. The proofs are fairly straightforward, and they can be found in Chapter 5 of [AM].

**Lemma 1.** Let $A \subseteq B$ be rings with $B$ integral over $A$.

1. If $b$ is an ideal of $B$ with $a = b \cap A$, then $B/b$ is integral over $A/a$.

2. If $S$ is a multiplicatively closed subset of $A$, then $S^{-1}B$ is integral over $S^{-1}A$.

3. If $q \in \text{spec}(B)$ and $p = q \cap A$, then $q$ is a maximal ideal of $B$ if and only if $p$ is a maximal ideal of $A$.

**Theorem 2 (Incomparability).** Let $A \subseteq B$ be rings with $B$ integral over $A$. If $q \subseteq q'$ are prime ideals of $B$ with $q \cap A = q' \cap A$, then $q' = q$.

**Proof.** If $p \in \text{spec}(A)$, we write $B_p$ for the localization $S^{-1}B$, where $S = A \setminus p$. Let $p = q \cap A$. Then $B_p$ is integral over $A_p$ by Lemma 1. Moreover, $qB_p \subseteq q'B_p$ are both prime ideals of $B_p$, and each intersect with $A_p$ in $pA_p$. Therefore, by Lemma 1, $qB_p$ is a maximal ideal, so $qB_p = q'B_p$. Intersecting down to $B$ gives $q = q'$.

**Theorem 3.** Let $A \subseteq B$ be rings with $B$ integral over $A$.

1. (Lying over) Let $p \in \text{spec}(A)$. Then there is a $q \in \text{spec}(B)$ with $q \cap A = p$.

2. (Going up) Let $p_1 \subseteq p_2 \subseteq \cdots \subseteq p_n$ be prime ideals of $A$, and let $q_1 \subseteq q_2 \subseteq \cdots \subseteq q_m$ be primes of $B$ with $m < n$ and $q_i \cap A = p_i$ for each $i$. Then there are primes $q_{m+1}, \ldots, q_n$ of $B$ with $q_m \subseteq q_{m+1} \subseteq \cdots \subseteq q_n$ and $q_i \cap A = p_i$ for each $i$. 

Proof. The extension $B_p/A_p$ is integral by Lemma 1. Let $m$ be a maximal ideal of $B_p$. Then $m \cap A_p$ is maximal in $A_p$ by Lemma 1, so $m \cap A_p = pA_p$. However, $pA_p \cap A = p$. Consequently, if $q = m \cap B$, then $q \cap A = p$.

For (2), by induction we can reduce to the case $m = 1$ and $n = 2$. Then we have a chain $p_1 \subseteq p_2$ and a prime ideal $q_1$ with $q_1 \cap A = p_1$. Since $B/q_1$ is integral over $A/p_1$ by Lemma 1, by (1) there is a $Q \in \text{spec}(B/q_1)$ with $q_2/q_1 \cap A/p_1 = p_2/p_1$. If $q_2 = \{x \in B : x + q_1 \in Q\}$, then $q_2 \in \text{spec}(B)$ with $q_1 \subseteq q_2$ and $q_2 \cap A = p_2$.

To prove the following lemma we use ideas from infinite Galois theory. We can avoid this by a Zorn’s lemma argument, or by using ideas of integrality over ideals of a subring (see Chapter 5 of [AM]).

**Corollary 4.** Let $A \subseteq B$ be rings with $B$ integral over $A$. Let $q \in \text{spec}(B)$ with $p = q \cap A$.

1. $\text{ht}(q) \leq \text{ht}(p)$.
2. If $\text{ht}(p) = n$, then there is a $q' \in \text{spec}(B)$ with $\text{ht}(q') = n$ and $q' \cap A = p$.
3. $\dim(B) = \dim(A)$.

**Proof.** For (1), suppose that $\text{ht}(q) = n$, and let $q_0 \subseteq \cdots \subseteq q_n = q$ be a chain of primes in $B$. If $p_i = q_i \cap A$, then there is a chain $p_0 \subseteq \cdots \subseteq p_n = p$. Moreover, if the inclusions in the chain for $q$ are all proper, then the same holds for the chain for $p$ by incomparability. Therefore, $\text{ht}(p) \geq n$.

To prove (2), suppose that $\text{ht}(p) = n$, and suppose that $p_0 \subseteq \cdots \subseteq p_n = p$ is a chain of length $n$. By the going up theorem, there is a chain $q_0 \subseteq \cdots \subseteq q_n$ with $q_i \cap A = p_i$ for each $i$. Let $q = q_n$. Then $\text{ht}(q) \geq n$ and $q \cap A = p$. However, by (1) we know that $\text{ht}(q) \leq n$, so $\text{ht}(q) = n$.

Finally, for (3) we know that

$$
\dim(B) = \sup \{\text{ht}(q) : q \in \text{spec}(B)\} \leq \sup \{\text{ht}(q \cap A) : q \in \text{spec}(A)\} \leq \dim(A)
$$

by (1). However, $\dim(B) \geq \dim(A)$ by (2), so the two inequalities together proves that $\dim(B) = \dim(A)$.

We point out that our proofs appear to assume that all heights in question are finite. By being careful with our arguments, we see that our arguments will work even when heights are infinite.

**Lemma 5 (Prime avoidance lemma).** Let $A$ be a ring and let $p_1, \ldots, p_n$ be prime ideals of $A$ and let $a$ be an ideal of $A$ such that $a \subseteq \bigcup p_i$. Then $a \subseteq p_i$ for some $i$. 

Proof. We prove the contrapositive by induction on \( n \). The case \( n = 1 \) is clear. Suppose that \( n > 1 \) and that \( a \not\subseteq \bigcup_{i=1}^{n} p_i \). Then \( a \not\subseteq \bigcup_{i \neq j} p_i \) for each \( j \). By induction, for each \( j \) there is an \( x_j \in a \) such that \( x_j \not\in p_i \) for each \( i \neq j \). If for some \( j \) we have \( x_j \not\in p_j \) then we are done. If not, then \( x_j \in p_j \) for all \( j \). Consider the element

\[
y = \sum_{i=1}^{n} x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n.
\]

We have \( y \in a \) and \( y \not\in p_i \) for each \( i \). Hence, \( a \not\subseteq \bigcup_{i=1}^{n} p_i \). \( \square \)

Lemma 6. Let \( K/F \) be a normal field extension, let \( A \) be an integrally closed domain with quotient field \( F \) and let \( B \) the integral closure of \( A \) in \( K \). If \( q_1, q_2 \in \mathrm{spec}(B) \) with \( q_1 \cap A = q_2 \cap A \), then there is a \( \sigma \in \mathrm{Gal}(K/F) \) with \( \sigma(q_1) = q_2 \).

Proof. If \( x \in K \) with \( x \) integral over \( A \), then \( \sigma(x) \) is also integral over \( A \) for any \( \sigma \in \mathrm{Gal}(K/F) \), so \( \sigma(B) = B \). It is then easy to see that if \( q \in \mathrm{spec}(B) \), then \( \sigma(q) \in \mathrm{spec}(B) \) for any \( \sigma \). Also, \( B \cap F = A \) since \( A \) is integrally closed in \( F \).

Let \( L \) be the fixed field of \( G = \mathrm{Gal}(K/F) \). Then \( K/L \) is Galois and \( L/F \) is purely inseparable. If \( L \neq F \), then \( \text{char}(F) = p > 0 \). Let \( B' = B \cap L \), the integral closure of \( A \) in \( L \). If \( p = q_i \cap A \), then there is a unique \( p' \in \mathrm{spec}(B') \) with \( p' \cap A = p \), namely

\[
p' = \{ x \in B' \mid x^{p^n} \in p \text{ for some } n \}.
\]

Since \( q_i \cap A = p \), this uniqueness gives \( q_i \cap B' = p' \). Therefore, we may assume that \( L = F \), and so \( K/F \) is Galois.

First suppose that \([K:F] < \infty\). Then \(|G| < \infty\). If \( x \in q_2 \), then

\[
N_{K/F}(x) = \prod_{\sigma \in G} \sigma(x) \in q_2 \cap F = p,
\]

so \( N_{K/F}(x) \in q_1 \), hence \( \sigma(x) \in q_1 \) for some \( \sigma \) since \( q_1 \) is prime. Thus, \( x \in \sigma^{-1}(q_1) \). Therefore, \( q_2 \subseteq \bigcup_{\sigma \in G} \sigma(q_1) \). By the prime avoidance lemma, \( q_2 \subseteq \sigma(q_1) \) for some \( \sigma \). But since

\[
q_2 \cap p = q_1 \cap A = \sigma(q_1 \cap A) = \sigma(q_1) \cap A,
\]

incomparability gives \( q_2 = \sigma(q_1) \).

We now consider the case where \([K:F] \) is infinite. If \( L \) is a Galois subextension of \( K \) with \([L:F] < \infty\), set

\[
G_L = \{ \sigma \in G \mid \sigma(q_1 \cap L) = q_2 \cap L \}.
\]

The argument above shows that \( G_L \neq \emptyset \). Note that if \( N = \mathrm{Gal}(K/L) \), then \(|G : N| = [L:F] < \infty\), and \( \sigma \in G_L \) implies that \( \sigma N \subseteq G_L \). If \( \tau_1, \ldots, \tau_m \) are coset representatives of \( N \) in \( G \) with \( \tau_i \in G_L \), then \( G_L = \bigcup \tau_i N \). Thus, \( G_L \) is closed in the Krull topology on \( G \). Also note if \( L \subseteq L' \), then \( G_{L'} \subseteq G_L \). If \( L_1, \ldots, L_r \) are finite Galois subextensions of \( K/F \),
then the composite $M$ of the $L_i$ is also a finite Galois extension of $F$. Then, by previous remarks, we get $\emptyset \neq G_M \subseteq \bigcap_i G_{L_i}$. Therefore, the collection of closed sets \{G_L\} has the finite intersection property. Since $G$ is compact in the Krull topology (see Theorem 21 of §11 in [Mc] or Theorem 17.6 of [Mo]), we see $\bigcap G_L \neq \emptyset$, where the intersection is taken over all subfields $L$ with $[L : F] < \infty$. If $\sigma$ is in this intersection, then $\sigma(q_1) = q_2$ since $K$ is the union of all these subfields. \hfill $\square$

**Theorem 7 (Going down).** Let $A \subseteq B$ be domains such that $B$ is integral over $A$ and $A$ is integrally closed. If $p_1 \subseteq p_2 \subseteq \cdots \subseteq p_n$ are prime ideals of $A$ and if $q_m \subseteq q_{m+1} \subseteq \cdots \subseteq q_n$ are prime ideals in $B$ with $q_i \cap A = p_i$ for each $i$, then there are primes $q_1 \subseteq \cdots \subseteq q_{m-1} \subseteq q_m$ such that $q_i \cap A = p_i$ for each $i$.

**Proof.** By induction we can reduce to the case $m = 1$ and $n = 2$. We then have a chain $p_1 \subseteq p_2$ in $\text{spec}(A)$ and a $q_2 \in \text{spec}(B)$ with $q_2 \cap A = p_2$. Let $F$ be the quotient field of $A$ and let $L$ be the quotient field of $B$. Also, let $K$ be the normal closure of $L/F$, and let $C$ be the integral closure of $B$ in $K$. By going up, there are $r_1 \subseteq r_2$ in $\text{spec}(C)$ with $r_i \cap A = p_i$ for $i = 1, 2$. Also, there is an $r \in \text{spec}(C)$ with $r \cap B = q_2$. Then $r \cap A = p_2 = r_2 \cap A$, so there is a $\sigma \in \text{Gal}(K/F)$ with $\sigma(r_2) = r$ by the previous lemma. Set $q_1 = \sigma(r_1) \cap B$. Then $q_1 \subseteq q_2 = \sigma(r_2) \cap B$ and $q_1 \cap A = \sigma(r_1) \cap A = r_1 \cap A = p_1$. \hfill $\square$

We now have the tools to approach dimension theory. In order to prove the main theorem we need a lemma, the Noether normalization lemma, and Zariski’s theorem.

**Lemma 8.** Let $A$ be a $k$-algebra and a domain, and let $p$ be a nonzero prime ideal of $A$. Let $t_1, \ldots, t_n \in A$ and let $c \in p$ with $c \neq 0$. Suppose that the $t_i$ are algebraically independent over $k$ in $A/p$. Then $t_1, \ldots, t_n, c$ are algebraically independent over $k$ in $A$.

**Proof.** Suppose this is false. Then there is a nonzero polynomial $f \in k[x_1, \ldots, x_n, y]$ with $f(t_1, \ldots, t_n, c) = 0$. Choose $f$ with the degree in $y$ minimal. Write $f = \sum_{i=0}^d g_i y^i$ with $g_i \in k[x_1, \ldots, x_n]$, and $g_d \neq 0$. If $d = 0$, then $f = g_0 \in k[x_1, \ldots, x_n]$ is nonzero with $f(t_1, \ldots, t_n, c) = g_0(t_1, \ldots, t_n) = 0$, so $g(\overline{t_1}, \ldots, \overline{t_n}) = 0$ in $A/p$. This is false since the $t_i$ are algebraically independent in $A/p$. Therefore, $d > 0$. Now in $A/p$,

$$0 = f(t_1, \ldots, t_n, c) = f(\overline{t_1}, \ldots, \overline{t_n}, c) = g_0(\overline{t_1}, \ldots, \overline{t_n})$$

as $c \in p$. Thus, $g_0 = 0$. So $f = y \cdot \sum_{i=0}^{d-1} g_{i+1} y^i$. Since $c \neq 0$ and $A$ is a domain, $f/y$ evaluated at $(t_1, \ldots, t_n, c)$ is 0, contradicting minimality of $d$. Thus, $t_1, \ldots, t_n, c$ are algebraically independent over $k$. \hfill $\square$

If $A$ is a $k$-algebra and a domain with quotient field $F$, let $\text{trdeg}_k A = \text{trdeg}_k F$, the transcendence degree of $F/k$. 

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Corollary 9. Let $A$ be an $k$–algebra and a domain, and let $F$ be the quotient field of $A$. Let $p \in \text{spec}(A)$. Then $\text{trdeg}_k A \geq \text{trdeg}_k(A/p) + \text{ht}(p)$.

Proof. Take $t_1, \ldots, t_n \in A$ such that $\overline{t_1}, \ldots, \overline{t_n}$ are algebraically independent in $A/p$ over $k$. Say $p = p_0 \supset p_1 \supset \cdots \supset p_m = (0)$ is a chain in $\text{spec}(A)$. Thus, $\text{trdeg}_k(A/p) \geq n$ and $\text{ht}(p) \geq m$. Choose $c_1, \ldots, c_m$ with $c_i \in p_i-1 - p_i$. Assume that $t_1 + p_i, \ldots, t_n + p_i, c_1 + p_i, \ldots, c_i + p_i \in A/p_i$ are algebraically independent over $k$. Then as $p_i/p_{i+1}$ is prime in $A/p_{i+1}$ and $(A/p_{i+1})/(p_i/p_{i+1}) = A/p_i$, by the lemma $t_1 + p_i, \ldots, c_i + p_i$ are algebraically independent over $k$. Thus, by induction $t_1 + p_m, \ldots, c_m + p_m$ are algebraically independent over $k$. But as $p_m = (0)$, this means $t_1, \ldots, c_m$ are algebraically independent over $k$. Thus $\text{trdeg}_k A \geq n + m$. Hence, $\text{trdeg}_k A \geq \text{trdeg}_k(A/p) + \text{ht}(p)$. □

Recall that a $k$–algebra $A$ is an affine $k$–algebra if $A$ can be generated by a finite number of elements as a ring extension of $k$. That is, $A$ is affine if $A = k[a_1, \ldots, a_n]$ for some $a_i \in A$. If $A$ is an affine $k$–algebra and is an integral domain, we say that $A$ is an affine $k$–domain.

Theorem 10 (Noether Normalization). Let $A$ be an affine $k$–domain. Then there are $t_1, \ldots, t_d \in A$ algebraically independent over $k$ with $A$ integral over $k[t_1, \ldots, t_d]$.

Proof. We prove this in the case that $|k| = \infty$. The proof for finite fields is slightly different, and we indicate the changes required for $k$ finite after the proof. Say $A = k[a_1, \ldots, a_n]$. We use induction on $n$. If $n = 1$, then either $a_1$ is transcendental over $k$, in which case we take $t_1 = a_1$ (and $d = 1$), or $a_1$ is algebraic over $k$, in which case we take $d = 0$, since $A$ is then integral over $k$. So, assume that $n > 1$. If $a_1, \ldots, a_n$ are algebraically independent over $k$ we can take $t_i = a_i$ for each $i$. Thus, suppose that they are not independent over $k$. Then there is a nonzero polynomial $f \in k[x_1, \ldots, x_n]$ with $f(a_1, \ldots, a_n) = 0$. Say $f = \sum_{i=0}^l f_i$, where $f_i$ is homogeneous of total degree $i$ and $f_l \neq 0$. Let $y_i = x_i - c_i x_n$ (for $i < n$), where the $c_i \in k$ are to be determined. Then $f = \sum_i f_i(y_1 + c_1 x_n, \ldots, x_n) = \sum_i f_i(y_1, \ldots, y_{n-1}, x_n)$. Each $f'_i$ then has degree $i$ in $x_n$. Expanding $f'_i$ we see that the coefficient of $x_n^i$ is $f_i(c_1, \ldots, c_{n-1}, 1)$. Thus, we can write

$$f = f_i(c_1, \ldots, c_{n-1}, 1) x_n^i + \sum_{i=0}^{l-1} g_i(y_1, \ldots, y_{n-1}) x_n^i.$$  

Choose the $c_i$ so $f_i(c_1, \ldots, c_{n-1}, 1) \neq 0$, possible as $|k| = \infty$ and $f_i$ is a nonzero homogeneous polynomial. Let $b_i = a_i - c_i a_n$ for $i < n$. Then under the map $\phi : k[x_1, \ldots, x_n] \to A$ with $\phi(x_i) = a_i$, we have $\phi(y_i) = b_i$, so

$$0 = f(a_1, \ldots, a_n) = f_i(c_1, \ldots, c_{n-1}) a_n^i + \sum_{i=0}^{l-1} g_i(b_1, \ldots, b_{n-1}) a_n^i.$$  

Dividing by $f_i(c_1, \ldots, c_{n-1})$ shows that $a_n$ is integral over the subring $k[b_1, \ldots, b_{n-1}]$ of $A$. By induction there are $t_1, \ldots, t_d \in k[b_1, \ldots, b_{n-1}]$ algebraically independent over $k$.
with \(k[b_1, \ldots, b_n]\) integral over \(k[t_1, \ldots, t_d]\). Thus, \(A = k[b_1, \ldots, b_{n-1}][a_n]\) is integral over \(k[t_1, \ldots, t_d]\).

For the case that \(k\) is finite we can use the same argument except we set \(y_i = x_i - x_n^{r_i}\) for suitably chosen large integers \(r_i\).

**Theorem 11 (Zariski).** Let \(k\) be a field, and let \(A\) be an affine \(k\)-algebra that is a field. Then \(A\) is an algebraic extension of \(k\).

**Proof.** Suppose that \(A\) is an affine \(k\)-algebra that is a field. By Noether normalization, there are \(t_1, \ldots, t_r \in A\) such that \(t_1, \ldots, t_r\) are algebraically independent over \(k\) and \(A\) is integral over \(k[t_1, \ldots, t_r]\). The zero ideal of \(A\) is maximal since \(A\) is a field. Thus, by Lemma 1, the zero ideal of \(k[t_1, \ldots, t_r]\) is maximal. This forces \(k[t_1, \ldots, t_r]\) to be a field. However, by the algebraic independence of the \(t_i, k[t_1, \ldots, t_r]\) is isomorphic to a polynomial ring in \(r\) variables. The only way it can be a field is if \(r = 0\). Thus, \(A\) is integral over \(k\), and so the field \(A\) is algebraic over \(k\).

**Theorem 12 (Dimension Theorem).** Let \(A = k[a_1, \ldots, a_n]\) be an affine \(k\)-algebra and a domain. Then

1. \(\dim(A) = \text{trdeg}_k A \leq n\).

2. If \(p \in \text{spec}(A)\), then \(\text{trdeg}_k A = \text{ht}(p) + \text{trdeg}_k (A/p)\), and so \(\dim(A) = \text{ht}(p) + \dim(A/p)\).

3. Every maximal chain of primes in \(A\) has length \(\dim(A)\).

**Proof.** Let \(F\) be the quotient field of \(A\). Then \(F = k(a_1, \ldots, a_n)\), so \(\{a_1, \ldots, a_n\}\) contains a transcendence basis of \(F/k\). Thus, \(\text{trdeg}_k F \leq n\). If \(p \in \text{spec}(A)\) then \(\text{ht}(p) + \text{trdeg}_k A/p \leq \text{trdeg}_k F \leq n\) by the corollary. Say \(\text{trdeg}_k F = d\). Then \(\text{ht}(p) \leq d\) for any prime \(p\), so \(\dim(A) \leq d\). Let us refer to the first part of (2) by (2'). We first prove (2'), and do this by induction on \(d\). If \(d = 0\), then the inequalities above show that all terms in (2') are 0, so (2') holds. Thus, assume that \(d > 0\) and that (2') holds for \(d - 1\). By Noether Normalization, \(A\) is integral over a subring \(A_0 = k[t_1, \ldots, t_d]\), where the \(t_i\) are algebraically independent over \(k\). This is the same \(d\) as above since \(F\) is algebraic over the quotient field of \(A_0\). Let \(B = k[t_1, \ldots, t_{d-1}] \subseteq A_0\). Then \(A_0 \cong B[x]\). Since \(\text{trdeg}_k B = d - 1\), by induction (2') holds for \(B\). Let \(q \in \text{spec}(B[x])\) and \(p = q \cap B\). Then \(q \supseteq pB[x]\), a prime of \(B[x]\) (as \(B[x]/pB[x] \cong (B/p)[x]\)). We consider two cases.

Case 1: \(q = pB[x]\). Then \(\text{ht}(q) \geq \text{ht}(p)\), since given a chain \(p_0 \subset p_1 \subset \cdots \subset p_r = p\), we have \(p_0B[x] \subset \cdots \subset p_rB[x] = q\). Now \(B[x]/q \cong (B/p)[x]\), so by the corollary and induction,

\[
\begin{align*}
d &= \text{trdeg}_k B[x] \geq \text{ht}(q) + \text{trdeg}_k (B[x]/q) \geq \text{ht}(p) + \text{trdeg}_k (B/p) + 1 \\
&= \text{trdeg}_k B + 1 = d
\end{align*}
\]
so \( \text{trdeg}_k B[x] = \text{ht}(q) + \text{trdeg}_k(B[x]/q) \).

Case 2: \( q \supset pB[x] \). Then \( \text{ht}(q) > \text{ht}(pB[x]) \geq \text{ht}(p) \) by the argument above. Thus \( \text{ht}(q) \geq 1 + \text{ht}(pB[x]) \). So by case 1,

\[
\text{ht}(q) \geq 1 + \text{ht}(pB[x]) \geq 1 + \text{ht}(p),
\]

so

\[
\text{ht}(q) + \text{trdeg}_k(B[x]/q) \geq \text{ht}(p) + 1 + \text{trdeg}_k(B[x]/q) \\
\geq \text{ht}(p) + 1 + \text{trdeg}_k(B/p)
\]
as \( B/p \subseteq B[x]/q \). So

\[
\text{ht}(q) + \text{trdeg}_k(B[x]/q) \geq \text{ht}(p) + \text{trdeg}_k(B/p) + 1 \\
= \text{trdeg}_k B + 1 \\
= d
\]

by induction. By the corollary,

\[
d = \text{trdeg}_k B[x] \geq \text{ht}(q) + \text{trdeg}_k(B[x]/q),
\]

so \( \text{trdeg}_k B[x] = \text{ht}(q) + \text{trdeg}_k(B[x]/q) \). Thus, we have \( (2') \) for \( A_0 \).

Let \( \mathfrak{p} \in \text{spec}(A) \). Since \( A/A_0 \) is an integral extension of integral domains and \( A_0 \cong k[x_1, \ldots, x_{d-1}] \) is integrally closed, the going down theorem shows that \( \text{ht}(\mathfrak{p}) = \text{ht}(\mathfrak{p} \cap A_0) \).

Also \( \text{trdeg}_k A/\mathfrak{p} = \text{trdeg}_k(A_0/\mathfrak{p} \cap A_0) \) since \( A/\mathfrak{p} \) is integral over \( A_0/\mathfrak{p} \cap A_0 \), so the quotient field of \( A/\mathfrak{p} \) is algebraic over the quotient field of \( A_0/\mathfrak{p} \cap A_0 \). Thus,

\[
\text{ht}(\mathfrak{p}) + \text{trdeg}_k(A/\mathfrak{p}) = \text{ht}(\mathfrak{p} \cap A_0) + \text{trdeg}_k(A_0/\mathfrak{p} \cap A_0) \\
= \text{trdeg}_k A_0 = d = \text{trdeg}_k A
\]

by what has been proven. Therefore, we have proved \( (2') \). To prove \( (1) \), note that we have seen that \( \dim(A) \leq d = \text{trdeg}_k A \). Let \( M \) be a maximal ideal of \( A \). By Zariski’s theorem, \( A/M \) is algebraic over \( k \), so \( (2) \) gives \( \text{ht}(M) = d \). But \( \text{ht}(M) \leq \dim(A) \), so we get \( \dim(A) = \text{trdeg}_k(A) = \text{ht}(M) \) for each \( M \). This proves \( (1) \). As \( A/\mathfrak{p} \) is also an affine \( k \)-domain for \( \mathfrak{p} \in \text{spec}(A) \), we see \( \dim(A/\mathfrak{p}) = \text{trdeg}_k(A/\mathfrak{p}) \) by \( (1) \), so \( \dim(A) = \text{ht}(\mathfrak{p}) + \dim(A/\mathfrak{p}) \). This finishes \( (2) \).

For \( (3) \), let \( \mathfrak{p}_0 \subseteq \cdots \subseteq \mathfrak{p}_r \) be a maximal chain. Therefore, \( \mathfrak{p}_r \) is a maximal ideal of \( A \). We have \( r \leq d = \text{dim}(A) \). We prove \( r = d \) by induction on \( d \). If \( d = 0 \) this is trivial. Assume that \( d > 0 \). Then \( A \) is not a field since \( (0) \) is not maximal, so \( r \geq 1 \). Thus \( \mathfrak{p}_1 \neq (0) \). Now

\[
(0) = \mathfrak{p}_1/\mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_r/\mathfrak{p}_1
\]
is a maximal chain of length \( r - 1 \) in \( A/p_1 \). Since the original chain is maximal, \( \text{ht}(p_1) = 1 \) as there is no \( p \) with \((0) \subset p \subset p_1 \). So

\[
\dim(A/p_1) = \dim(A) - \text{ht}(p_1) = d - 1.
\]

Induction gives \( r - 1 = d - 1 \), so \( r = d \). This finishes the proof of the dimension theorem. \( \square \)

We can use some of the results we have done to prove the Nullstellensatz. We start off with a lemma, which is the heart of the Nullstellensatz.

**Lemma 13.** Let \( k \) be an algebraically closed field and let \( I \) be a proper ideal of \( k[x_1, \ldots, x_n] \). Then the zero set \( Z(I) \) is nonempty. Moreover, any maximal ideal of \( k[x_1, \ldots, x_n] \) is of the form \( (x_1 - a_1, \ldots, x_n - a_n) \) for some \( a_i \in k \).

**Proof.** Let \( M \) be a maximal ideal of \( k[x_1, \ldots, x_n] \). We write \( A = k[x_1, \ldots, x_n] \) for convenience. Then \( A/M \) is a field, and the natural inclusion \( k \rightarrow A \) induces a field injection \( k \rightarrow A/M \). We view \( A/M \) as a field extension of \( k \). However, \( A/M \) is an affine \( k \)-algebra since \( A/M \) is generated by \( x_1 + M, \ldots, x_n + M \). Thus, by Zariski’s theorem, \( A/M \) is algebraic over \( k \). Since \( k \) is algebraically closed, this forces \( A/M = k \). From this we can characterize \( M \). Let \( \varphi : A \rightarrow k \) be the ring homomorphism that sends \( A \) to \( A/M \) followed by an isomorphism \( A/M \rightarrow k \). Let \( a_i = \varphi(x_i) \). Notice that \( \varphi \) is the identity on \( k \). Then \( \varphi(x_i - a_i) = 0 \), so \( x_i - a_i \in \ker(\varphi) = M \). Consequently, \( M \) contains the ideal \( (x_1 - a_1, \ldots, x_n - a_n) \). However, this is a maximal ideal, so \( M = (x_1 - a_1, \ldots, x_n - a_n) \). \( \square \)

We can now prove the first statement of the lemma. Let \( I \) be a proper ideal of \( k[x_1, \ldots, x_n] \), and let \( M \) be a maximal ideal that contains \( I \). Then \( M = (x_1 - a_1, \ldots, x_n - a_n) \) for some \( a_i \in k \) by what we have proved. If \( P = (a_1, \ldots, a_n) \), then \( P \in Z(M) \subseteq Z(I) \), so \( Z(I) \) is nonempty.

**Theorem 14 (Nullstellensatz).** Let \( J \) be an ideal of \( k[x_1, \ldots, x_n] \). Then \( I(Z(J)) = \sqrt{J} \).

**Proof.** Let \( J \) be an ideal of \( k[x_1, \ldots, x_n] \). The inclusion \( \sqrt{J} \subseteq I(Z(J)) \) is clear. Also, since the result is trivial if \( J = k[x_1, \ldots, x_n] \), we assume that \( J \) is a proper ideal. Let \( f \in I(Z(J)) \), and let \( z \) be a new variable. Consider the ideal \( J' \) generated by \( J \) and \( 1 - zf \) in \( k[x_1, \ldots, x_n, z] \). An arbitrary element of this ideal is of the form \( g(1 - zf) + \sum h_i g_i \), where \( g, g_i \in k[x_1, \ldots, x_n, z] \) and each \( h_i \in J \). Suppose that \( Q \in Z(J') \). If \( Q = (a_1, \ldots, a_{n+1}) \) and \( P = (a_1, \ldots, a_n) \), then \( h(Q) = h(P) = 0 \) for all \( h \in J \), so \( P \in Z(J) \). But then \( 0 = (1 - zf)(Q) = 1 - a_{n+1}f(P) = 1 \) since \( f \in I(Z(J)) \). This is impossible, so \( Z(J') \) is empty. By the previous lemma, \( J' = k[x_1, \ldots, x_n, z] \). Thus, we have an equation \( g(1 - zf) + \sum h_i g_i = 1 \), where \( h_i \in J \) and \( g, g_i \in k[x_1, \ldots, x_n, z] \). By writing each \( g_i \) as a polynomial in \( z \) with coefficients in \( k[x_1, \ldots, x_n] \) and by doing the same for \( \sum h_i g_i \), we see that \( \sum_i h_i g_i = \sum_{i=1}^m h'_i z^i \) for some \( h'_i \in J \). By setting \( z = f \), we get an equation \( \sum_{i=1}^m h'_i(x_1, \ldots, x_n)(1/f)^i = 1 \) in \( k[x_1, \ldots, x_n][1/f] \). Multiplying
by \( f^m \) gives

\[
f^m = \sum_{i=1}^{m} h_i'(x_1, \ldots, x_n) f^{m-i} \in J\]

since each \( h_i' \in J \). Therefore, \( f \in \sqrt{J} \) as desired.

\[\square\]

References

