WEIGHTED NORM INEQUALITIES FOR
PSEUDO-PSEUDODIFFERENTIAL OPERATORS DEFINED BY
AMPLITUDES

NICHOLAS MICHALOWSKI, DAVID J. RULE, AND WOLFGANG STAUBACH

Abstract. We prove weighted norm inequalities for pseudodifferential operators with
amplitudes which are only measurable in the spatial variables. The result is sharp, even
for smooth amplitudes. Nevertheless, in the case when the amplitude contains the oscil-
latory factor $\xi \mapsto e^{i|\xi|^{1-\rho}}$, the result can be substantially improved. We extend the $L^p$-
boundedness of pseudo-pseudodifferential operators to certain weights. End-point results
are obtained when the amplitude is either smooth or satisfies a homogeneity condition
in the frequency variable. Our weighted norm inequalities also yield the boundedness
of commutators of these pseudodifferential operators with functions of bounded mean
oscillation.

1. Introduction

Weighted norm inequalities have a long history in harmonic analysis and partial differential
equations. Since the introduction of $A_p$ weights by B. Muckenhoupt [21] (see Definition 2.4),
there has been a large amount of activity establishing weighted $L^p$ estimates with weights
belonging to $A_p$. Such estimates have concerned singular integral operators of Calderón-
Zygmund type, strongly singular integral operators (or singular integrals of Hirschman-
Wainger type), maximal functions, standard pseudodifferential operators, and mildly regular
pseudodifferential operators. However, there are still some gaps in the knowledge of weighted
norm inequalities for pseudodifferential operators, and the aim of this paper is to fill in
some of these. We do this first in the context of symbols introduced by C.E. Kenig and
W. Staubach in [18] which are only measurable in the spatial variable. Our methods also
extend to amplitudes and we obtain some sharp results in this direction. Most of these
weighted boundedness results are new even in the smooth case. However, in the smooth
case we can go on to consider end-point cases by using an interpolation technique. The
results of this paper extend the applications derived in [18] and as a separate application
we prove new boundedness results for the commutators of pseudodifferential operators with
functions of bounded mean oscillation (written BMO, see Definition 4.1).

Recall that for a function $u \in C^\infty_0(\mathbb{R}^n)$ a pseudodifferential operator is an operator given
formally by

\begin{equation}
T_a u(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x, y, \xi) e^{i(x-y, \xi)} u(y) \, dy \, d\xi,
\end{equation}

whose amplitude $(x, y, \xi) \mapsto a(x, y, \xi)$ is assumed to satisfies certain growth conditions. The
most common class of amplitudes are those introduced by L. Hörmander in [15] and we refer
the reader to Definitions 2.1, 2.2 and 2.3 below for the specific amplitudes we will consider
here and the standard notation we will use.
By the \textit{weighted boundedness} of a pseudodifferential operator \( T_a \) or a \textit{weighted norm inequality} for a pseudodifferential operator we mean the existence of an estimate of the form
\[
\|T_a u\|_{L^p_w} \leq C\|u\|_{L^p_w},
\]
for some weight \( w \) and \( 1 \leq p \leq \infty \). As usual \( L^p_w \) denotes the weighted \( L^p \) space with weight \( w \):
\[
\|u\|_{L^p_w} = \left( \int_{\mathbb{R}^n} |u(x)|^p w(x) dx \right)^{\frac{1}{p}}.
\]
All our results will concern \( w \in A_q \) (see Definition 2.4) for some \( q \) which may not always be equal to \( p \).

![Figure 1. Regions of \( L^p \)-Boundedness](image)

In the remainder of this section we summarize the results to follow. Section 2 sets out some definitions, fixes some notation and records some well-known results we will need later. In Section 3 our first main result is Theorem 3.3. It is a pointwise bound of operators in \( OPL^\infty S^m_{\rho,0} \) by a maximal function and corresponds to the \( L^p \)-boundedness of \( OPL^\infty S^m_{\rho,0} \) in the region \( C \cup D \) of Figure 1 obtained in [18]. Such an estimate immediately leads to weighted boundedness results of the form (1.2).

In Theorem 3.4 we go on to consider symbols of the form \((x,\xi) \mapsto e^{i|\xi|^{1-\rho}}\sigma(x,\xi)\), with \( \sigma \in L^\infty S^m_{\rho,0} \) and \( m < n(\rho - 1)/2 \). We build on the methods developed by S. Chanillo and A. Torchinsky [7] to prove that operators corresponding to these symbols can be bounded pointwise by certain maximal functions and consequently are bounded on \( L^p_w \) for \( w \in A_p \) and \( 1 < p < \infty \). This estimate is obtained when \( m < n(\rho - 1)/2 \) and corresponds to the region \( B \cup C \cup D \) of Figure 1. In Corollary 3.8 we observe the result can be generalized to amplitudes \((x, y, \xi) \mapsto e^{i|\xi|^{1-\rho}}\sigma(x, y, \xi)\), with \( \sigma \in L^\infty A^m_{\rho,0} \). In the case of symbols of the aforementioned form, if we make the additional assumption that the symbol is homogeneous in the \( \xi \) variables, we can extend the weighted boundedness result to the end-point value \( m = n(\rho - 1)/2 \). This is formulated as Theorem 3.6. These results are particularly interesting in connection to weighted norm inequalities for maximal functions associated to strongly singular integrals. In the case of \( \rho = 1 \), Theorem 3.6 yields the weighted version of an \( L^p \)-boundedness result due to R. Coifman and Y. Meyer [10].

The weighted boundedness of operators corresponding to symbols can be used to prove pointwise bounds by a maximal function for operators corresponding to amplitudes. This idea is first used to prove Theorem 3.7, where weighted boundedness results are shown for \( OPL^\infty A^m_{\rho,0} \) in region \( D \) of Figure 1. This is shown to be sharp. The amplitudes, and in
particular the rough amplitudes, considered here are interesting because of the connection
to the Weyl quantization of rough symbols, and the potential applications of the latter in
semiclassical microlocal analysis. The Weyl quantization is when the amplitude takes the
form \((x, y, \xi) \mapsto a((x + y)/2, \xi)\).

We can then use a fairly general method of interpolation, based on complex interpolation, to
obtain end-point results for smooth amplitudes. The idea is to use properties of \(A_p\)-weights
to enable us to interpolate between the weighted boundedness for values of \(m\) less than the
end-point and the unweighted boundedness which is known for values of \(m\) greater than the
end-point. An example of this is Theorem 3.10, where we show weighted boundedness for
operators in \(OPA_n^{1(p-1)}\) with \(0 < \rho \leq 1\) and \(0 \leq \delta < 1\). The interpolation technique can
also, for example, be used to prove weighted boundedness results for smooth symbols in the
particular the rough amplitudes, considered here are interesting because of the connection
to the Weyl quantization of rough symbols, and the potential applications of the latter in
semiclassical microlocal analysis. The Weyl quantization is when the amplitude takes the
form \((x, y, \xi) \mapsto a((x + y)/2, \xi)\).

We can then use a fairly general method of interpolation, based on complex interpolation, to
obtain end-point results for smooth amplitudes. The idea is to use properties of \(A_p\)-weights
to enable us to interpolate between the weighted boundedness for values of \(m\) less than the
end-point and the unweighted boundedness which is known for values of \(m\) greater than the
end-point. An example of this is Theorem 3.10, where we show weighted boundedness for
operators in \(OPA_n^{1(p-1)}\) with \(0 < \rho \leq 1\) and \(0 \leq \delta < 1\). The interpolation technique can
also, for example, be used to prove weighted boundedness results for smooth symbols in the

In Section 4 we use the weighted norm inequalities to prove a variety of boundedness results
for the commutators of pseudodifferential operators with BMO functions. When we have the
\(L^p\)-boundedness of pseudodifferential operators for all weights \(w \in A_p\), we are actually able
to show weighted boundedness of \(k\)-th commutators as a direct consequence of a general
result due to J. Álvarez, R.J. Bagby, D.S. Kurtz and C. Pérez [1], see Section 4 for the
definition of these commutators. For these results see Theorem 4.5. However, in Theorem
4.4, we also obtain unweighted \(L^p\)-boundedness of a commutator knowing only the weighted
\(L^p\)-boundedness of the original operator for \(w\) in a much smaller class of weights.

Theorems 4.4 and 4.5 extend the \(L^p\) boundedness of BMO commutators with \(OPA_n^{1}\) due
to R. Coifman, R. Rochberg and G. Weiss [11], and the \(L^p\) boundedness of commutators of
bmo functions with rough pseudodifferential operators arising from homogeneous symbols
of order zero (the symbol class \(L^\infty S^{0}_{cl}\)), proved by F. Chiarenza, M. Frasca and P. Longo [8].
The space of bmo functions is a localized version of the class BMO. To our knowledge, there
are no results in the existing literature concerning the boundedness of BMO commutators
of operators with rough amplitudes, even in the case of the Weyl quantization.

We would like to thank Andrew Hassell for prompting us to consider the problem of weighted
norm inequalities for pseudodifferential operators arising from amplitudes.

2. Definitions, Notation and Preliminaries

First we introduce a standard Littlewood-Paley partition of unity \(\{\varphi_k\}_{k \geq 0}\). Let \(\varphi_0 : \mathbb{R}^n \to \mathbb{R}\)
be a smooth radial function which is equal to one on the unit ball centred at the origin and
supported on its concentric double. Set \(\varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi)\) and \(\varphi_k(\xi) = \varphi(2^{-k}\xi)\). Then

\[
\varphi_0(\xi) + \sum_{k=1}^{\infty} \varphi_k(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R},
\]

and \(\text{supp}(\varphi_0) \subset \{\xi \mid 2^{k-1} \leq |\xi| \leq 2^{k+1}\} \text{ for } k \geq 1\). One also has, for all multi-indices \(\alpha\) and
\(N \geq 0\),

\[
|\partial_\xi^\alpha \varphi_0(\xi)| \leq c_{\alpha,N}(\xi)^{-N},
\]

where \(\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}\), and

\[
(2.1) \quad |\partial_\xi^\alpha \varphi_k(\xi)| \leq c_\alpha 2^{-k|\alpha|} \quad \text{for some } c_\alpha > 0 \text{ and all } k \geq 1.
\]

We now fix some common notation and terminology.

**Definition 2.1.** A function \(\alpha : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\) is called an amplitude when it belongs
to any one of the following sets. Let \(m \in \mathbb{R}, \rho \in [0,1]\) and \(\delta \in [0,1]\).
We say \( a \in A_{\rho,\delta}^m \) when for each triple of multi-indices \( \alpha, \beta \) and \( \gamma \) there exists a constant \( C_{\alpha,\beta,\gamma} \) such that
\[
|\partial_\xi^\alpha \partial_\eta^\beta a(x,y,\xi)| \leq C_{\alpha,\beta,\gamma} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|+\gamma},
\]

(b) We say \( a \in L^\infty A_{\rho}^m \) when for each multi-index \( \alpha \) there exists a constant \( C_{\alpha} \) such that
\[
\|\partial_\xi^\alpha a(\cdot,\cdot,\xi)\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq C_{\alpha} \langle \xi \rangle^{m-\rho|\alpha|}.
\]

Therefore, here we are only assuming measurability in the \((x,y)\)-variables.

Given an amplitude \( a \) we define the pseudodifferential operator \( T_a \) associated to \( a \) formally by (1.1), although it is not clear that the integral in (1.1) is well-defined. This can be made rigorous by considering the partial sums of the Littlewood-Paley pieces as follows. We define \( a_k = a \varphi_k \). The integral in (1.1) converges absolutely when \( a \) is replaced with \( a_k \), so in this way we obtain a family of operators \( \{T_{a_k}\}_k \). The methods of proof we use will show in each case that the partial sums
\[
\sum_{k=0}^N T_{a_k}
\]
converge in operator norm as \( N \to \infty \). Consequently, we obtain our desired operator as
\[
T_a := \lim_{N \to \infty} \sum_{k=0}^N T_{a_k}.
\]

An important special case of these operators is when there is no dependency on the \( y \)-variable. It is convenient to have a slightly different terminology in this case.

**Definition 2.2.** A function \( a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) is called a symbol when it belongs to any one of the following sets. Let \( m \in \mathbb{R} \), \( \rho \in [0,1] \) and \( \delta \in [0,1] \).

(a) We say \( a \in S_{\rho,\delta}^m \) when for each pair of multi-indices \( \alpha \) and \( \beta \) there exists a constant \( C_{\alpha,\beta} \) such that
\[
|\partial_\xi^\alpha \partial_\eta^\beta a(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|},
\]

(b) We say \( a \in L^\infty S_{\rho}^m \) when for each multi-index \( \alpha \) there exists a constant \( C_{\alpha} \) such that
\[
\|\partial_\xi^\alpha a(\cdot,\xi)\|_{L^\infty} \leq C_{\alpha} \langle \xi \rangle^{m-\rho|\alpha|}.
\]

Therefore, here we are only assuming measurability in the \( x \)-variable.

Obviously, we have \( S_{\rho,\delta}^m \subset A_{\rho,\delta}^m \), \( L^\infty S_{\rho}^m \subset L^\infty A_{\rho}^m \), \( S_{\rho}^m \subset L^\infty A_{\rho}^m \) and \( A_{\rho,\delta}^m \subset L^\infty A_{\rho}^m \). The amplitudes of (a) in both Definitions 2.1 and 2.2 were first introduced in \([14, 15]\). If the amplitude is smooth and \( \delta < \rho \), then operators arising from amplitudes in \( A_{\rho,\delta}^m \) can be rewritten as a sum of operators arising from symbols in \( S_{\rho,\delta}^m \), see, for example, \([23]\).

However, if \( \delta \geq \rho \), then the boundedness results are known to be different \([15, 19]\), as we will see played out in this paper. The symbols of (b) in Definition 2.2 were introduced in \([18]\), and Definition 2.2 (b) is the natural generalization of this to amplitudes. They are, for example, much rougher than those considered by S. Nishigaki \([22]\) and K. Yabuta \([26]\) in their investigations of weighted norm inequalities for pseudodifferential operators.

**Definition 2.3.** Given a class \( X \) of symbols or amplitudes, operators which arise from elements in \( X \) are denoted by \( OPX \), that is, we say \( T \in OPX \) when there exists \( a \in X \) such that \( T = T_a \), defined as in (1.1).

When the amplitude of an operator is only measurable in the spatial variables, that is the operator belongs to \( OPL^\infty S_{\rho}^m \) or \( OPL^\infty A_{\rho}^m \), then following \([18]\), we say it is a pseudo-pseudodifferential operator. It is well-known that for standard pseudodifferential operators the singular support of the Schwartz kernel of the operator is contained in the diagonal
— this is called the pseudo-local property. Singularities in the Schwartz kernel of pseudo-pseudodifferential operators will in general go beyond the diagonal, so they do not have the pseudo-local property.

Given \( u \in L^p_{\text{loc}} \), the \( L^p \) maximal function \( M_p(u) \) is defined by

\[
M_p(u)(x) = \sup_{B \ni x} \left( \frac{1}{|B|} \int_B |u(y)|^p \, dy \right)^{1/p}
\]

where the supremum is taken over balls \( B \) in \( \mathbb{R}^n \) containing \( x \). Clearly then, the Hardy-Littlewood maximal function is given by

\[
M_1(u) := \sup_{B \ni x} \left( \frac{1}{|B|} \int_B |u(y)| \, dy \right).
\]

An immediate consequence of Hölder’s inequality is that

\[
M_p(u)(x) \leq M_1(u)(x) \quad \text{for } p \geq 1.
\]

We shall use the notation

\[
u_B := \frac{1}{|B|} \int_B |u(y)| \, dy
\]

for the average of the function \( u \) over \( B \). One can then define the class of Muckenhoupt \( A_p \) weights as follows.

**Definition 2.4.** Let \( w \in L^1_{\text{loc}} \) be a positive function. One says that \( w \in A_1 \) if there exists a constant \( C > 0 \) such that

\[
Mw(x) \leq Cw(x)
\]

for almost all \( x \in \mathbb{R}^n \).

One says that \( w \in A_p \) for \( p \in (1, \infty) \) if

\[
[\lambda]_{A_p} := \sup_{B \text{ balls in } \mathbb{R}^n} \lambda |B|^{-1/p} < \infty.
\]

The \( A_p \) constants of a weight \( w \in A_p \) are defined by

\[
[w]_{A_p} := \sup_{B \text{ balls in } \mathbb{R}^n} w_B \|w^{-1}\|_{L^\infty(B)},
\]

and

\[
[w]_{A_p} := \sup_{B \text{ balls in } \mathbb{R}^n} w_B \left( \frac{1}{p} \right)^{p-1} |B|^{-1}.
\]

The following results are well-known and can be found in, for example, [17, 23].

**Theorem 2.5.** Suppose \( p > 1 \) and \( w \in A_p \). There exists an exponent \( q < p \), which depends only on \( p \) and \( [w]_{A_p} \), such that \( w \in A_q \). There exists \( \varepsilon > 0 \), which depends only on \( p \) and \( [w]_{A_p} \), such that \( w^{1+\varepsilon} \in A_p \).

**Theorem 2.6.** For \( 1 < q < \infty \), the Hardy-Littlewood maximal operator is bounded on \( L^q_w \) if and only if \( w \in A_q \). Consequently, for \( 1 \leq p < \infty \), \( M_p \) is bounded on \( L^p_w \) if and only if \( w \in A_{q/p} \).

**Theorem 2.7.** Suppose that \( \phi : \mathbb{R}^n \to \mathbb{R} \) is integrable non-increasing and radial. Then, for \( u \in L^1 \), we have

\[
\int \phi(y)u(x-y) \, dy \leq \|\phi\|_{L^1} M(u)(x)
\]

for all \( x \in \mathbb{R}^n \).

It is useful to record here results of J. Álvarez and J. Hounie [2] see also [15]. These will be used in Section 3.

**Theorem 2.8.** Let \( 1 < q < \infty \), \( 0 < \rho \leq 1 \), \( 0 \leq \delta < 1 \) and suppose either:

(a) \( a \in A^m_{p,\delta} \) and \( m \leq n(\rho - 1) \left( \frac{1}{q} - \frac{1}{2} \right) + \min\{0, n(\rho - \delta)\} \); or
Fix $a \in S^m_{\rho, \delta}$ and $m \leq n\rho + \min\{0, n(\rho - \delta)/2\}$. Then $T_a$ is bounded on $L^q$.

Part (a) is a consequence of remark (d) on page 11 of [2], together with straightforward adjustments of Theorem 2.2, Lemma 3.1 and Theorem 3.4 therein to the case of amplitudes. Part (b) is explicitly stated in [2] as Theorem 3.4 on page 13. As is common practice, we will denote constants which can be determined by know parameters in a given situation, but whose value is not crucial to the problem at hand, by $C$. Such parameters in this paper would be, for example, $m$, $\rho$, $\delta$, $p$, $n$, $|w|_{A_n}$, and the constants $C_{\alpha, \beta, \gamma}$, $C_{\alpha, \beta}$ and $C_\alpha$ in Definitions 2.1 and 2.2. The value of $C$ may differ from line to line, but in each instance could be estimated if necessary. We sometimes write $a \lesssim b$ as shorthand for $a \leqCb$.

3. Pseudodifferential operators and their weighted $L^p$ boundedness

The first fairly simple lemma yields a classical kernel estimate, which in particular implies the rapid decrease off the diagonal of the Schwartz kernel of pseudodifferential operators with certain amplitudes.

**Lemma 3.1.** Let $a \in L^\infty A^m_\rho$ with $m \in \mathbb{R}$ and $\rho \in [0, 1]$. Let $a_k(x, y, \xi) = a(x, y, \xi)\varphi_k(\xi)$, for $k \geq 0$ with $\varphi_k$ as in the above Littlewood-Paley decomposition. Then, for each $l \geq 0$,

$$|z|^l \left| \int a_k(x, y, \xi) e^{i(z, \xi) d\xi} \right| \lesssim 2^{k(n+m-\rho l)},$$

for all $x, y, z \in \mathbb{R}^n$.

**Proof.** Using the definition of $L^\infty A^m_\rho$, inequality (2.1) and the Leibniz rule we see that

$$|\partial^\alpha_k a_k(x, \xi)| \leq c_\alpha 2^{k(m-\rho |\alpha|)}, \text{ for some } c_\alpha > 0 \text{ and } k = 1, 2, \ldots,$$

where we have also used the assumption $\rho \leq 1$. First suppose $l$ is an integer. For a multi-index $\alpha$ with $|\alpha| = l$, integration by parts then yields

$$|z^\alpha \int a_k(x, y, \xi) e^{i(z, \xi) d\xi}| = \left| \int a_k(x, y, \xi) \partial^\alpha \phi_k e^{i(z, \xi) d\xi} \right|$$

$$= \left| \int \partial^\alpha k a_k(x, y, \xi) e^{i(z, \xi) d\xi} \right| \lesssim 2^{k(n+m-\rho l)}.$$

Summing over all $\alpha$ with $|\alpha| = l$ proves this special case of the lemma. The general result of non-integer values of $l$ follows by interpolation of the inequality for $k$ and $k+1$, where $k < l < k+1$. \hfill \Box

The kernel of operators of the form (1.1) is $K(x, x-y) = \frac{1}{(2\pi)^n} \int a(x, y, \xi) e^{i(x-y, \xi)} d\xi$. From Lemma 3.1 it is easy to conclude

$$|K(x, x-y)| \lesssim |x-y|^{-N} \text{ for } N > 0, \text{ } |x-y| \geq 1,$$

provided either $\rho > 0$ and $m \in \mathbb{R}$, or $\rho = 0$ and $m < -n$.

It was shown in [18] that under certain conditions on the order $m$ pseudo-pseudodifferential operators are bounded on $L^p$ spaces. We record that result here.

**Theorem 3.2.** Fix $p \in [1, 2]$ and let $a \in L^\infty S^m_\rho$ with $0 \leq \rho < 1$ and $m < \frac{n}{p}(\rho - 1)$. Then $T_a$ is a bounded operator on $L^q$ for each $q \geq p$. 

Now, our goal is to show that under the same assumptions, $OPL^\infty S^m_\rho$ are also bounded on weighted $L^p$ spaces. More precisely, we prove the following theorem. The proof uses a similar method to [18].

**Theorem 3.3.** Fix $p \in [1,2]$ and let $a \in L^\infty S^m_\rho$ with $0 \leq \rho \leq 1$ and $m < \frac{n}{p} (\rho - 1)$. Then there exists a constant $C$, depending only on $n$, $p$, $m$, $\rho$ and a finite number of the constants $C_a$ in Definition 2.1, such that

$$|T_a(u)(x)| \leq CM_p u(x),$$

for all $x \in \mathbb{R}^n$. Consequently, $T_a$ is a bounded operator on $L^q_p$ for each $q > p$ and $w \in A_{q/p}$.

**Proof.** The $L^q_p$-boundedness follows immediately from the pointwise estimate, by Theorem 2.6.

To prove the pointwise estimate we use the Littlewood-Paley partition of unity introduced in Section 2, we decompose the symbol as

$$a(x,\xi) = a_0(x,\xi) + \sum_{k=1}^{\infty} a_k(x,\xi)$$

with $a_k(x,\xi) = a(x,\xi) \varphi_k(\xi)$, $k \geq 0$.

First we consider the operator $T_{a_0}$. We have

$$T_{a_0}(u)(x) = \frac{1}{(2\pi)^n} \int \int a_0(x,\xi) e^{i(x-y,\xi)} u(y) dy d\xi = \int K_0(x,y) u(x-y) dy,$$

with

$$K_0(x,y) = \frac{1}{(2\pi)^n} \int a_0(x,\xi) e^{i(y,\xi)} d\xi.$$

Lemma 3.1 gives us the estimate

$$|K_0(x,y)| \lesssim |y|^{-M},$$

for each $M > n$. Theorem 2.7 yields

$$|T_{a_0}(u)(x)| \lesssim \langle y \rangle^{-M} |u(x-y)| dy \lesssim M u(x) \lesssim M_p u(x),$$

for all $1 \leq p \leq 2$.

Now let us analyse $T_{a_k}(u)(x) = \frac{1}{(2\pi)^n} \int a_k(x,\xi) \hat{u}(\xi) e^{i(x,\xi)} d\xi$ for $k \geq 1$. We note, just as before, that $T_{a_k}(u)(x)$ can be written as

$$T_{a_k}(u)(x) = \int K_k(x,y) u(x-y) dy$$

with

$$K_k(x,y) = \frac{1}{(2\pi)^n} \int a_k(x,\xi) e^{i(y,\xi)} d\xi = \hat{a}_k(x,y),$$

where $\hat{a}_k$ here denotes the inverse Fourier transform of $a_k(x,\xi)$ with respect to $\xi$. One observes that

$$|T_{a_k}(u)(x)|^p = \left| \int K_k(x,y) u(x-y) dy \right|^p = \left| \int K_k(x,y) \sigma_k(y) \frac{1}{\sigma_k(y)} u(x-y) dy \right|^p,$$

with weight functions $\sigma_k(y)$ which will be chosen momentarily. Therefore, Hölder’s inequality yields

$$|T_{a_k}(u)(x)|^p \leq \left\{ \int |K_k(x,y)|^p \sigma_k(y)^p dy \right\} \frac{p}{p'} \left\{ \int \frac{|u(x-y)|^p}{\sigma_k(y)^p} dy \right\},$$

for each $p$. Finally, we observe that

$$|T_{a_k}(u)(x)|^p \leq \left\{ \int |K_k(x,y)|^p \sigma_k(y)^p dy \right\} \frac{p}{p'} \left\{ \int \frac{|u(x-y)|^p}{\sigma_k(y)^p} dy \right\},$$

for each $p$. Consequently, we have shown that $T_a$ is a bounded operator on $L^q_p$ for each $q > p$ and $w \in A_{q/p}$.
where $\frac{1}{p} + \frac{1}{p'} = 1$. Now for an $l > \frac{n}{p}$, we define $\sigma_k$ by
\[
\sigma_k(y) = \begin{cases} 
2^{-k\rho p}, & |y| \leq 2^{-kp}\rho; \\
2^{-k\rho (\frac{p}{p'} - 1)|y|}, & |y| > 2^{-kp}\rho.
\end{cases}
\]
By Hausdorff-Young’s theorem and the estimate (3.1), first for $\alpha = 0$ and then for $|\alpha| = l$, we have
\[
\int 2^{-kp\rho p} |K_k(x, y)|^{p'} \, dy \leq 2^{-kp\rho p} \left\{ \int |a_k(x, \xi)|^{p} \, d\xi \right\}^{\frac{p'}{p}} 
\lesssim 2^{-kp\rho p} \left\{ \int_{|\xi|<2^k} 2^{pm_k} \, d\xi \right\}^{\frac{p'}{p}} \lesssim 2^{kp(p - \frac{n}{2}(\rho - 1))},
\]
and
\[
\int 2^{-kp\rho (\frac{p}{p'} - 1)} |K_k(x, y)|^{p'} |y|^{p'} \, dy \leq 2^{-kp\rho (\frac{p}{p'} - 1)} \left\{ \int |\nabla_{\xi} a_k(x, \xi)|^{p} \, d\xi \right\}^{\frac{p'}{p}} 
\lesssim 2^{-kp\rho (\frac{p}{p'} - 1)} \left\{ \int_{|\xi|<2^k} 2^{kp(m - \rho l)} \, d\xi \right\}^{\frac{p'}{p}} \lesssim 2^{kp(m - \frac{n}{2}(\rho - 1))}.
\]
Hence, splitting the integral into $|y| \leq 2^{-kp}$ and $|y| > 2^{-kp}$ yields
\[
\left\{ \int |K_k(x, y)|^{p'} |\sigma_k(y)|^{p'} \, dy \right\}^{\frac{p}{p'}} \lesssim \left\{ 2^{kp(m - \frac{n}{2}(\rho - 1))} \right\}^{\frac{p}{p'}} = 2^{kp(m - \frac{n}{2}(\rho - 1))}.
\]
Furthermore, once again using Theorem 2.7, we have
\[
\int \frac{|u(x - y)|^{p} \, dy}{|\sigma_k(y)|^{p}} \lesssim (M_{p} u(x))^{p}
\]
with a constant that only depends on the dimension $n$. Thus (3.4) yields
\[
|T_{a_{0}} u(x)|^{p} \lesssim 2^{k(m - \frac{n}{2}(\rho - 1))} (M_{p} u(x))^{p}
\]
Summing in $k$ using (3.3) and (3.5), we obtain
\[
|T_{a} u(x)|^{p} \lesssim |a_{0}(x, D)u(x)|^{p} + \sum_{k=1}^{\infty} |a_{k}(x, D)u(x)|^{p}
\]
\[
\lesssim (M_{p} u(x))^{p} \left( 1 + \sum_{k=1}^{\infty} 2^{k(m - \frac{n}{2}(\rho - 1))} \right)
\]
We observe that the series above converges if $m < \frac{n}{2}(\rho - 1)$. This ends the proof of the theorem. \hfill \square

For symbols in $L^\infty S_{p}^{m} \alpha$ which contain the oscillatory factor $\xi \mapsto e^{i|\xi|^{1-\rho}}$, one can improve the $L^p_{\alpha}$-boundedness result of Theorem 3.3 to all weights $w \in A_p$ for $m < \frac{n}{2}(\rho - 1)$.

**Theorem 3.4.** Let $\sigma \in L^\infty S_{1}^{m}$, with $m < \frac{n}{2}(\rho - 1)$ and set $a(x, \xi) = e^{i|\xi|^{1-\rho}} \sigma(x, \xi)$, with $0 < \rho < 1$. For each $1 < p < \infty$, one has the pointwise estimate
\[
|T_{a} u(x)| \lesssim M(Mu)(x) + M_{p} u(x),
\]
with a constant depending only on $n, p, m, \rho$ and a finite number of the constants $C_{\alpha}$ in Definition 2.1. Consequently
\[
\|T_{a} u\|_{L^q_{\alpha}} \lesssim \|u\|_{L^q_{\alpha}},
\]
for all $1 < q < \infty$ and $w \in A_q$. 


Proof. It is clear from Theorems 2.5 and 2.6 that the weighted \( L^p \) boundedness will follow from (3.6). To prove this we will use the Littlewood-Paley partition of unity \( \{ \varphi_k \}_k \) of Section 2 just as we did above. Observe that we may choose \( \{ \varphi_k \}_k \) so that
\[
\varphi_k(\xi) = \varphi_k(\xi)(\varphi_{k-1}(\xi) + \varphi_k(\xi) + \varphi_{k+1}(\xi)),
\]
for all \( \xi \in \mathbb{R}^n \) and each \( k \geq 1 \). Now, again for \( k \geq 0 \), we set \( a_k(x, \xi) = a(x, \xi)\varphi_k(\xi) \). As in the proof of Theorem 3.3 we can deal with the piece of the operator containing \( \varphi_0(\xi) \) using integration by parts, which yields \( |T_{a_k}(u)(x)| \leq CM(u)(x) \). For \( k \geq 1 \), we compute
\[
T_{a_k}(u)(x) = \left( \int a_k(x, \xi)e^{iy \xi} \, d\xi \right) u(x-y) \, dy = \int K_k(x,y)u(x-y) \, dy.
\]
Using (3.7), we can see
\[
K_k(x,y) = \int a_k(x, \xi)e^{iy \xi} \, d\xi = \int a(x, \xi)\varphi_k(\xi)e^{iy \xi} \, d\xi
\]
\[
= \int a(x, \xi)\varphi_k(\xi)(\varphi_{k-1}(\xi) + \varphi_k(\xi) + \varphi_{k+1}(\xi))e^{iy \xi} \, d\xi
\]
\[
= T_{\varphi_k}(\psi_k(y),)
\]
where \( T_{\varphi_k}(\psi) \) is the operator with the symbol \( a_k(x, \xi) \) with \( x \) fixed (and hence is a multiplier) and
\[
(\varphi_{k-1} + \varphi_k + \varphi_{k+1})^* := 2^{nk}\psi(2^k.) := \psi_k,
\]
for some function \( \psi \) in the Schwartz class.

Now the fact that \( T_{\varphi_k}(\psi) \) is a multiplier operator, that multipliers are translation invariant, and that \( T_{\varphi_k}(\psi) = T_{\varphi_k}(\psi) \) enables us to write
\[
\int K_k(x,y)u(x-y) \, dy = \int T_{\varphi_k}(\psi_k(y))u(x-y) \, dy = \int \psi_k(y)T_{\varphi_k}(\psi_k(y)u(x-y) \, dy.
\]
Combining this with (3.8), we have
\[
|T_{a_k}(u)(x)| = \left| \int \psi_k(y)T_{\varphi_k}(\psi_k(y)u(x-y) \, dy \right|
\]
\[
\leq \int |\psi_k(y)||T_{\varphi_k}(\psi_k(y)u(x-y) | \, dy.
\]
We observe that \( |\psi_k| \) can be majorized by, say
\[
\sum_{j=1}^{\infty} \alpha_j(2^{-k}d_j)^{-n} \chi_{Q_j^k},
\]
where, for each \( k \) and \( j \), \( \chi_{Q_j^k} \) is the characteristic function of a cube \( Q_j^k \) centred at the origin with diameter \( 2^{-k}d_j \), and \( \sum_j |\alpha_j| < \infty \). We are also free to assume \( d_j \geq 1 \) for all \( j \). Therefore,
\[
|T_{a_k}(u)(x)| \leq \sum_{j=1}^{\infty} \alpha_j(2^{-k}d_j)^{-n} \int_{Q_j^k} |T_{\varphi_k}(\psi_k(y)u(x-y) | \, dy.
\]
In what follows we omit the bar over \( a_k \) for notational simplicity. Since \( a_k \) and \( \varphi_k \) satisfy the same symbol estimates, and since the proof is identical for \( \varphi_k \) replaced by \( a_k \), this shouldn’t cause any confusion for the reader.

Now we consider two cases. First, suppose \( d := 2^{1-k}d_j < 1/4 \). For such \( j \) and \( k \) we decompose \( u = u_1^{k,j} + u_2^{k,j} + u_3^{k,j} \), where \( u_1^{k,j} = u\chi_{Q_j^k} \) and \( \chi_{Q_j^k} \) is the characteristic function of \( \{ y | |x-y| \leq 2^{-k}d_j \} \), \( \chi_{Q_j^k}^{k,j} \) is the characteristic function of \( \{ y | 2^{2^{-k}d_j} < |x-y| \leq (2^{1-k}d_j)^{\nu} \} \), and \( \chi_{Q_j^k}^{k,j} \) is the characteristic function of \( \{ y | |x-y| > (2^{1-k}d_j)^{\nu} \} \). The number \( \nu \) will be
We can estimate $J_1$ using the $L^1$ boundedness of multipliers of this form: Indeed, since $|\partial_x^\alpha a_k(\cdot, \xi)| \lesssim 2^{k(m-\rho|\alpha|)}$, following [18], if $K_k(x,y)$ denotes the kernel of $T_{ak(x,\cdot)}$ (recall again that $x$ is fixed here) then

$$\int |K_k(x,y)|dy = \int_{|y| \leq 2^{-k\rho}} |K_k(x,y)|dy + \int_{|y| > 2^{-k\rho}} |K_k(x,y)|dy := J_{1,1} + J_{1,2}$$

To estimate $J_{1,1}$ we use the Cauchy-Schwarz inequality and (3) for the case $p = p' = 2$. Therefore

$$J_{1,1} \leq \left\{ \int_{|y| \leq 2^{-k\rho}} |y|^{-2}dy \right\}^{\frac{1}{2}} \left\{ \int |K_k(x,y)|^2dy \right\}^{\frac{1}{2}} \lesssim 2^{k(m-\frac{2}{2}(\rho-1))}.$$ 

To estimate $J_{1,2}$ we use again the Cauchy-Schwarz inequality and (3) for the case $p = p' = 2$.

This yields

$$J_{1,2} \leq \left\{ \int_{|y| > 2^{-k\rho}} |y|^{-2}dy \right\}^{\frac{1}{2}} \left\{ \int |K_k(x,y)|^2|y|^{2}dy \right\}^{\frac{1}{2}} \lesssim 2^{k(m-\frac{2}{2}(\rho-1))}.$$ 

Thus, by Young’s inequality, the $L^1$ norm of the multiplier $T_{ak(x,\cdot)}$ has the size $2^{-\varepsilon k}$ where $\varepsilon = \frac{\rho}{2}(\rho - 1) - m$, which in turn yields

$$J_1 \leq (2^{-k\rho})^{-n} \int |T_{ak(x,\cdot)}(u^{k,j})(x-y)|dy$$

$$\leq C 2^{-\varepsilon k}(2^{-k\rho})^{-n} \int |u^{k,j}(x-y)|dy$$

$$\leq C 2^{-\varepsilon k}Mu(x).$$

To estimate $J_3$ we use Lemma 3.1 with $l$ so large that $\varepsilon = \frac{m}{2}(1-\rho) \geq (1-\nu) + l(\nu - \rho)$ to obtain the estimate

$$|K_k(x,z)| \leq C 2^{k(n+m-\rho l)}|z|^{-l}$$

and so conclude, for any $y \in Q_n^\rho$,

$$|T_{ak(x,\cdot)}(u^{k,j})(x-y)| = \left| \int K_k(x,z)u^{k,j}_1(x-y-z)dz \right|$$

$$\leq C 2^{k(n+m-\rho l)} \int_{|y+z| > (2^{-k\rho})^\nu} |z|^{-l}|u(x-y-z)|dz$$

$$\leq C 2^{k(n+m-\rho l)} \int_{|z| > (2^{\nu - 1})2^{-k\rho\rho}^\nu} |z|^{-l}|u(x-y-z)|dz$$

$$\leq C 2^{k(n+m-\rho l)}((2^{\nu} - 1)2^{-k\nu\rho}^\nu(n-1)Mu(x-y)$$

$$\leq C 2^{k(n+m-\rho l)}(2^{-k\nu\rho}(n-1)Mu(x-y)$$

$$\leq C 2^{-\varepsilon k}Mu(x-y)$$
since \(d_j \geq 1\) and \(2^{1-k}d_j < 4/4\), so \(|z| > |y + z| - |y| > (2^{1-k}d_j)^{2} - 2^{-k}d_j \geq (2^{v} - 1)2^{-kv}d_j\). Consequently,

\[
J_3 \leq C2^{-ek}M(Mu)(x).
\]

For the case \(d < 1\), it remains only to estimate \(J_2\). We will use the arguments in Theorem 1.2 of [7]. We observe that we may write \(a_k(x, \xi) = 2^{-\epsilon k}b_k(x, \xi)\) where \(b_k(x, \xi) = e^{i|\xi|^{2} - \epsilon k(x, \xi)}\) and \(\tau_k \in L^{\infty}S^{\frac{p}{p-1}}\), with constants \(C_\alpha\) independent of \(k\). Suppressing the indices \(j\) and \(k\), we further partition the function \(u_{2, j}^{k}\) in \(y\), we may write \(|\gamma| = 2\) for \(\xi \neq 0\) and \(2^{\alpha}d \sim d^{\nu}\), where \(d = 2^{1-k}d_j\). We obtain, then, the estimate analogous to (3.1) in [7];

\[
J_2 \leq C2^{-ek}Mu(x) + 2^{-ek} \sum_{\gamma=2}^{\infty} d^{-n} \int_{Q_{y}^{\gamma}} |S_{\gamma}(g_{\gamma})(y)| \, dy
\]

where \(S_{\gamma}\) is defined as follows. Let \(\delta = n(1 + \rho)(1 - \frac{1}{p} - \frac{1}{2})\) and set

\[
\eta_{\gamma}(\xi) = \eta([(1 - \rho)/(2^\gamma d)]^{1/\rho})\xi
\]

where \(\eta\) is radial, smooth and such that

\[
\eta(\xi) = \begin{cases} 
0, & \text{if } |\xi| < (1/8)\frac{1}{\rho}; \\
1, & \text{if } (1/7)^{1/\rho} < |\xi| < 30^{1/\rho}; \\
0, & \text{if } |\xi| > 40^{1/\rho}.
\end{cases}
\]

Let \(\theta\) be a smooth cut-off function which is equal to one near infinity and zero near the origin, then define \(S_{\gamma}\) as

\[
S_{\gamma}(u)(y) = \int_{\mathbb{R}^{n}} e^{i(y + \xi + |\xi|^{-1} - n)} \frac{\theta(\xi)\tau_k(x, \xi)}{|\xi|^\delta} |\xi|^{n(1-\rho)/2} \eta_{\gamma}(\xi)|\xi|^{\delta + n(\rho - 1)/2} u(\xi) \, d\xi.
\]

Following the reasoning of [7], we obtain for fixed \(x\) that

\[
\|S_{\gamma}(g_{\gamma})\|_{L^{p'}} \leq C(2^\gamma d)^{(\delta - \frac{2}{2}(\rho - 1))/\rho}\|g_{\gamma}\|_{L^{p'}} \leq C(2^\gamma d)^{(\delta - \frac{2}{2}(\rho - 1))/\rho + n/p} M_p u(x).
\]

Therefore, observing that \(\delta - \frac{n}{2}(\rho - 1)/\rho + n/p = n/(\rho p')\), we have

\[
\sum_{\gamma=2}^{\infty} d^{-n} \int_{Q_{y}^{\gamma}} |S_{\gamma}(g_{\gamma})(y)| \, dy \leq C d^{-n/p} M_p u(x) \sum_{\gamma=2}^{\infty} (2^\gamma d)^{n/(\rho p')}
\]

\[
= C d^{n}(\frac{\xi}{(\rho - 1)/p}) M_p u(x).
\]

So we choose \(\nu\) so that \(n(1 - \frac{2}{p})/p' = \epsilon/2\), and then

\[
2^{-ek}d^{n(\frac{\xi}{(\rho - 1)/p})} = 2^{-ek/2}d_{\xi/2}^{n/2} \leq 2^{-ek/2}
\]

and

\[
J_2 \leq C2^{-ek/2}M_p u(x).
\]

Now consider the case \(d \geq 1/4\). We decompose \(u = u_{1,j} + (u - u_{1,j})\), where \(u_{1,j}\) is defined as before. We have

\[
(2^{-k}d)_{\nu}^{-n} \int_{Q_{y}^{\gamma}} |T_{ak(x, \cdot)}(u)(x - y)| \, dy
\]

\[
= (2^{-k}d)_{\nu}^{-n} \int_{Q_{y}^{\gamma}} |T_{ak(x, \cdot)}(u_{1,j})(x - y)| \, dy
\]

\[
+ (2^{-k}d)_{\nu}^{-n} \int_{Q_{y}^{\gamma}} |T_{ak(x, \cdot)}(u - u_{1,j})(x - y)| \, dy
\]

\[
= L_1 + L_2.
\]
The term $L_1$ is identical to $J_1$, and in fact the same analysis works for $d \geq 1/4$ as did for $d < 1/4$. For $L_2$, by (3.10) with $l$ sufficiently large, we see that, for $y \in Q^j_{\rho}$,

$$|T_{a_k}(x)(u - u_{1})^{k,j}(x - y)| = \left| \int K_k(x, z)(u - u_{1})^{k,j}(x - y - z) \, dz \right| \leq C2^{k(n + \rho l)} \int_{|y + z| > 2^k \rho_j} |z|^{-l} |u(x - y - z)| \, dz \leq C2^{k(n + \rho l)} \int_{|y + z| > 2^k \rho_j} |y + z|^{-l} |u(x - y - z)| \, dz \leq C2^{k(n + \rho l)} M u(x) \leq 2^{-e_k} M u(x),$$

so, $L_2 \leq C2^{-e_k} M u(x)$.

Collecting all these estimates together, we find

$$|T_a(u)(x)| \leq \sum_{k=0}^{\infty} |T_{a_k}(u)(x)| \leq M(u)(x) + \sum_{d<1/4} M(J_1 + J_2 + J_3) + \sum_{d \geq 1/4} \alpha_j(L_1 + L_2) \leq M u(x) + \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j (2^{-e_k} (M u)(x) + 2^{-e_k/2} M_p u(x)) \leq (M(M u)(x) + M_p u(x)),$$

which proves (3.6). \hfill \Box

By considering a homogeneous version of the symbols in the previous theorem, we are able to prove the weighted $L^p$ boundedness of the corresponding operators at the end-point $m = n(p - 1)/2$. We now define this class precisely.

**Definition 3.5.** The class $L^\infty S^m_{cl}$ consists of symbols which are bounded and measurable in the spatial variable and satisfy

1. $\|\partial^\alpha a(\cdot, \xi)\|_{L^\infty} \leq c_\alpha (\xi)^{m-|\alpha|}$, for each multi-index $\alpha$.
2. $a(x, t \xi) = t^m a(x, \xi), \ t \geq 1, \ |\xi| \geq 1$.

Now we are ready to state and prove our boundedness result for a sub-class of operators in $L^\infty S^m_{cl}$.

**Theorem 3.6.** Let $\sigma \in L^\infty S^n_{cl}$ and set $a(x, \xi) = e^{i|\xi|^{1-\rho}} \sigma(x, \xi)$, with $0 < \rho \leq 1$. Then for each $1 < p < \infty$ and each $w \in A_p$, the operator $T_a$ is bounded from $L^p_w$ to itself.

**Proof.** Let us introduce a radial cut-off function $\theta(\xi)$ which is smooth and equal to one for $|\xi| \geq 1$ and zero for $|\xi| \leq 1/2$. Then it is obvious that the operator

$$T_{a_1} u(x) = \frac{1}{(2\pi)^n} \int e^{i|\xi|^{1-\rho}} \sigma(x, \xi) e^{i(x, \xi)} (1 - \theta(\xi)) \hat{u}(\xi) \, d\xi$$

has a rapidly decreasing Schwartz kernel and is therefore bounded on $L^p_w$. So, it is enough to only consider the operator

$$T_{a_2} u(x) = \frac{1}{(2\pi)^n} \int e^{i|\xi|^{1-\rho}} \sigma(x, \xi) e^{i(x, \xi)} \theta(\xi) \hat{u}(\xi) \, d\xi.$$
Suppose the weighted norm inequality follows from the pointwise estimate by Theorems 2.5 and 2.6. Now since
\[ \int |\xi|^{-p} \hat{f}(\xi) \hat{g}(\xi) \chi(\xi) \leq |\hat{f}(\xi)||\hat{g}(\xi)| \chi(\xi), \]
we have polynomial bounds in \( \chi \in C_0^\infty \). Namely let
\[ \chi(\xi) = 1 \quad \text{for} \quad |\xi| \leq \frac{1}{2}, \]
and \( \sigma_0(x, \xi) \) supported in \( |\xi| \leq \frac{1}{2} \). Furthermore
\[ \|f_k\|_{L^\infty} \leq C_N(k)^{-N}, \]
for all \( N > 0 \). For \( \sigma_0(x, \xi) \) one has \( \sigma_0(x, \xi)e^{i(|\xi|^{-p-1})} = 0 \) because of the disjoint supports of \( \sigma_0 \) and \( \theta \), therefore we have
\[ T_{a_k}u(x) = \sum_k f_k(x) \int |\xi|^{-p} e^{i(|\xi|^{-p-1})} \hat{\sigma}_k(\xi) \hat{\theta}(\xi) d\xi = T(D)u(x). \]
Now it is clear that \( U_k(D)u(x) = (S_k(D) \circ T(D)u)(x) \) with
\[ S_k(D)u(x) := \int w_k \left( \frac{\xi}{|\xi|} \right) (1 - \chi) e^{i(\xi \cdot x)} \hat{u}(\xi) d\xi, \]
and
\[ T(D)u(x) := \int |\xi|^{-p} e^{i(|\xi|^{-p-1})} \hat{\theta}(\xi) e^{i(\xi \cdot x)} \hat{u}(\xi) d\xi. \]
Now since \( w_k \left( \frac{\xi}{|\xi|} \right) (1 - \chi) \) is a result of N. Miller [20] concerning weighted \( L^p \) boundedness of operators with symbols in \( S_0^m \), yields
\[ \|U_k(D)u\|_{L_p^\infty} \lesssim \|T(D)u\|_{L_p^\infty}, \]
where the constants of the inequality only depend on finite number of derivatives of \( w_k \left( \frac{\xi}{|\xi|} \right) (1 - \chi) \), and are of polynomial growth in \( k \). Furthermore, a result of S. Chanillo concerning weighted \( L^p \) boundedness of strongly singular integral operators of Hirschman-Wainger type [5], yields \( \|T(D)u\|_{L_p^\infty} \lesssim \|u\|_{L_p^\infty} \) from which it follows that
\[ \|U_k(D)u\|_{L_p^\infty} \lesssim \|u\|_{L_p^\infty} \] with polynomial bounds in \( k \) and hence summing in \( k \) and using the fact that \( \|f_k\|_{L^\infty} \leq C_N(k)^{-N} \), for all \( N > 0 \) yields
\[ \|T_{a_k}u\|_{L_p^\infty} \lesssim \|u\|_{L_p^\infty}, \]
which concludes the proof of the theorem. \( \square \)

Now we turn to the problem of weighted boundedness of operators arising from amplitudes. To this end we have the following result, which is sharp modulo the end-point \( m = n(\rho - 1) \).

**Theorem 3.7.** Suppose \( 0 \leq \rho \leq 1, \) \( m < n(\rho - 1) \) and \( a \in L^\infty A_m^\rho \), then, for each \( p > 1 \), we have
\[ |T_a u(x)| \lesssim M_p u(x), \]
and consequently
\[ \|T_a u\|_{L_p^\infty} \lesssim \|u\|_{L_p^\infty}. \]

**Proof.** The weighted norm inequality follows from the pointwise estimate by Theorems 2.5 and 2.6.

Let \( K(x, y, z) := \frac{1}{(2\pi)^n} \int a(x, y, \xi) e^{i(z \cdot \xi)} d\xi \), then we have
\[ T_a u(x) = \int_{|x| \leq 1} K(x, y, x-y) u(y) dy + \int_{|x| > 1} K(x, y, x-y) u(y) dy = I + II. \]
However since \(m < n(p - 1)\), we also know from estimate (3.2) that \(|K(x, y, x - y)| \lesssim C|x - y|^{-N}\) for sufficiently large \(N\) and \(|x - y| \geq 1\). So \(II\) can be easily majorized by \(M(u)(x)\).

As mentioned earlier we will again use the Littlewood-Paley partition of unity \(\{\varphi_k\}_k\) just as we did in Theorem 3.4. Using that partition of unity and setting

\[
K_k(x, y, z) := \frac{1}{(2\pi)^n} \int a_k(x, y, \xi) e^{i(y \cdot \xi)} d\xi
\]

yields

\[
I = \sum_{k=0}^{\infty} \int_{|x-y| \leq 1} K_k(x, y, x - y)u(y) dy = \sum_{k=0}^{\infty} I_k,
\]

Now once again for \(k = 0\) we observe that \(|K_0(x, y, x - y)| \lesssim (x - y)^{-N}\) for all \(N > 0\), hence \(|I_0| \lesssim Mu(x)\).

If we consider an individual term with \(k \geq 1\), we have

\[
|I_k| = \left| \int_{|x-y| \leq 1} K_k(x, y, x - y)u(y) dy \right|
\]

\[
= \left| \int_{|x-y| \leq 1} K_k(x, y, x - y)b(x - y)|r| \frac{1}{|b(x - y)|^r} u(y) dy \right|
\]

where \(b\) and \(r\) are parameters to be chosen later. Therefore, Hölder’s inequality yields

\[
|I_k| \leq \left\{ \int_{|x-y| \leq 1} |K_k(x, y, x - y)|^{p'} |b(x - y)|^{r p'} dy \right\}^{\frac{1}{p'}} \times \left\{ \int_{|x-y| \leq 1} \frac{|u(y)|^p}{|b(x - y)|^{r p}} dy \right\}^{\frac{1}{p'}}.
\]

By Theorem 2.7, for \(r < \frac{n}{p}\), we have

\[
\left\{ \int_{|x-y| \leq 1} \frac{|u(y)|^p}{|b(x - y)|^{r p}} dy \right\}^{\frac{1}{p'}} \leq C b^{-r} M_p u(x),
\]

therefore

\[
(3.11) \quad |I_k| \leq C \left\{ \int |K_k(x, x - z, z)|^{p'} |b z|^{r p'} dz \right\}^{\frac{1}{p'}} b^{-r} M_p u(x).
\]

Considering the remaining integral, setting \(\sigma_k^r(z, \xi) := a_k(x, x - z, \xi)\) and using (3.9) we have

\[
K_k(x, x - z, z) = \int a_k(x, x - z, \xi) e^{iz \cdot \xi} d\xi
\]

\[
= \int \sigma_k^r(z, \xi) e^{iz \cdot \xi} d\xi
\]

\[
= \int \sigma_k^r(z, \xi) (\phi_{k-1}(\xi) + \phi_k(\xi) + \phi_{k+1}(\xi)) e^{iz \cdot \xi} d\xi
\]

\[
= \int \sigma_k^r(z, \xi) \tilde{\psi}_k(\xi) e^{iz \cdot \xi} d\xi = T_{\sigma_k^r}(\tilde{\psi}_k)(z),
\]

Therefore, taking \(b = 2^k\),

\[
\left\{ \int |K_k(x, x - z, z)|^{p'} |b z|^{r p'} dy \right\}^{\frac{1}{p'}} = \left\{ \int |T_{\sigma_k^r}(\tilde{\psi}_k)(z)|^{p'} |2^k z|^{r p'} dz \right\}^{\frac{1}{p'}}
\]

Now we observe that since \(x\) is fixed, \(\sigma_k^r\) belongs to the symbol class \(L^\infty S^m_n\) with seminorms that are uniform in \(x\). Furthermore, the weight \(z \mapsto |z|^{r p'}\) is in \(A_{p'}\) if and only if
Let operators in $OPL$ also be bounded on $L^p$, enabling us to prove an analogue of (3.12). We leave the details to the interested reader. □

We now prove an interpolation result that will allow us to extend some of our previous results when we also assume the amplitudes are smooth. We also have a generalization of the result in Theorem 3.4 to the settings of amplitudes.

Corollary 3.8. Let $a(x,y,\xi) = e^{i\xi^1 + p} \sigma(x,y,\xi)$ with $0 < \rho \leq 1$ and assume that $\sigma(x,y,\xi) \in L^\infty_{\alpha}A^m_p$ with $m = \frac{m_1 + 1}{2}(\rho - 1)$, then $T_\alpha$ is bounded on $L^p_w$, for all $p \in (1, \infty)$ and $w \in A_p$.

Proof. In the case $\rho = 1$, we are exactly in the situation of Theorem 3.7. For the case $0 < \rho < 1$ we first observe that in the proof of Theorem 3.4 we have, in fact, proved for the Littlewood-Paley pieces that

$$|T_\alpha(u)(x)| \lesssim 2^{-zh}M(Mu)(x) + 2^{-zh/2}Mu(x),$$

so Theorem 2.6 gives us the estimate

$$\|T_\alpha(u)\|_{L^p_w} \lesssim 2^{-zh/2}\|u\|_{L^p_w}$$

for $1 < p < \infty$ and $w \in A_p$. Now we may repeat the proof of Theorem 3.7, the bound (3.13) enabling us to prove an analogue of (3.12). We leave the details to the interested reader. □

We now prove an interpolation result that will allow us to extend some of our previous results when we also assume the amplitudes are smooth.

Lemma 3.9. Let $0 \leq \rho \leq 1$, $0 \leq \delta \leq 1$, $1 < p < \infty$ and $m_1 < m_2$. Suppose that

(a) Operators in $OPA_{\rho,\delta}^{m_1}$ (or $OPL^\infty A_{\rho}^{m_1}$) are bounded on $L^p_w$ for a fixed $w \in A_p$, and
(b) Operators in $OPA_{\rho,\delta}^{m_2}$ (or $OPL^\infty A_{\rho}^{m_2}$) are bounded on $L^p_w$,

where the bounds depend only on a finite number of $C_{\alpha,\beta,\gamma}$ (or $C_{\alpha}$) in Definition 2.1. Then, for each $m \in (m_1, m_2)$, operators in $OPA_{\rho,\delta}^{m}$ (or $OPL^\infty A_{\rho}^{m}$) are bounded on $L^p_w$, where $\mu = u^\nu$ and

$$\nu = \frac{m_2 - m}{m_2 - m_1}.$$
Proof. For $a \in A^m_{p,\delta}$ (or $a \in L^\infty_A^m$) we introduce a family of symbols $a_z(x, y, \xi) := (\xi)^z a(x, y, \xi)$, where $z \in \Omega := \{ z \in \mathbb{C}; m_1 - m \leq Re z \leq m_2 - m \}$. It is easy to see that, for $|a + \beta| \leq C_1$ with $C_1$ large enough and $z \in \Omega$,

$$|\partial_z^\alpha \partial_{\xi}^\beta a_z(x, y, \xi)| \lesssim (1 + |\text{Im } z|)^C z^{-|\alpha|+|\beta|+\gamma},$$

for some $C_2$. (We only require $\beta = \gamma = 0$ if $a \in L^\infty_A^m$.)

We introduce the operator

$$T_z u := w^{m_2 - m - z} T_{a_z}(w^{-m_2 - m} u).$$

First we consider the case of $p \in [1, 2]$. In this case, $A_p \subset A_2$ which in turn implies that both $w^z$ and $w^{-\frac{z}{2}}$ belong to $L^p_{\text{log}}$ and therefore for $z \in \Omega$, $T_z$ is an analytic family of operators in the sense of Stein-Weiss [25].

Now we claim that for $z_1 \in \mathbb{C}$ with $Re z_1 = m_1 - m$, the operator $(1 + |\text{Im } z_1|)^{-C_2} T_{a_z}$ is bounded on $L^p$ with bounds uniform in $z_1$. Indeed the amplitude of this operator is $(1 + |\text{Im } z_1|)^{-C_2} a_{z_1}(x, y, \xi)$ which belongs to $A^m_{p,\delta}$ (or $L^\infty_A^m$) with constants uniform in $z_1$. Thus, the claim follows from assumption (a).

Consequently, we have

$$\|T_{z_1} u\|^p_{L^p} = (1 + |\text{Im } z_1|)^{pC_2} \| (1 + |\text{Im } z_1|)^{-C_2} w^{m_2 - m - z_1} T_{a_{z_1}}(w^{-m_2} u) \|^p_{L^p} \lesssim (1 + |\text{Im } z_1|)^{pC_2} \| w^{-m_2} u \|^p_{L^p} = (1 + |\text{Im } z_1|)^{pC_2} \| u \|^p_{L^p},$$

where we have used the fact that $|w^{-m_2} u| = w$.

Similarly if $z_2 \in \mathbb{C}$ with $Re z_2 = m_2 - m$, then $|w^{m_2 - m - z_2} u| = 1$ and the amplitude of the operator $(1 + |\text{Im } z_2|)^{-C_2} T_{a_{z_2}}$ belongs to $A^m_{p,\delta}$ (or $L^\infty_A^m$) with constants uniform in $z_2$. Assumption (b) therefore implies that

$$\|T_{z_2} u\|^p_{L^p} \lesssim (1 + |\text{Im } z_2|)^{pC_2} \| u \|^p_{L^p}.$$

Therefore the complex interpolation of operators in [4] implies that for $z = 0$ we have

$$\|T_{00} u\|^p_{L^p} = \| w^{m_2 - m} T_0(u^{-m_2 - m_1} u) \|^p_{L^p} \leq C \| u \|^p_{L^p}.$$

Hence, setting $v = w^{-\frac{m_2 - m}{m_2 - m_1}} u$ this reads

$$\|T_a(v)\|^p_{L^p} \leq C \| u \|^p_{L^p},$$

where $\mu = w^\nu$ and $\nu = (m_2 - m)/(m_2 - m_1)$. This ends the proof in the case $1 \leq p \leq 2$.

At this point we recall the fact that if a linear operator $T$ is bounded on $L^p_w$, then its adjoint $T^*$ is bounded on $L^p_{w^{-1}}$. Therefore, in the case $p > 2$, we apply the above proof to $T_{a^*}$, with $p' \in [1, 2)$ and $v = w^{1-p}$, which yields that $T_{a^*}$ is bounded on $L^p_{w^{1-p}}$ and since $w \in A_p$, we have $v \in A_{p'}$ and so $T_{a^*}$ is bounded on $L^p_{w^{1-p} v} = L^p_{w^{1-p}(1-p)} v = L^p_{\mu}$, which concludes the proof of the theorem. \hfill \square

For smooth amplitudes we can use Lemma 3.9 to show the following end-point versions of Theorems 3.7 and 3.3.
Theorem 3.10. If \( a \in A^{m(\rho - 1)}_{\rho, \delta} \) with \( 0 < \rho \leq 1, 0 \leq \delta < 1 \), then for all \( 1 < p < \infty \) and all \( w \in A_p \), the corresponding pseudodifferential operator \( T_a \) is bounded on \( L^p_w \). If \( a \in S^{m(\rho - 1)/2}_{\rho, \delta} \) with \( 0 < \rho < 1, 0 \leq \delta < 1 \), then for all \( 2 \leq p < \infty \) and all \( w \in A_p^{1/2} \), the corresponding pseudodifferential operator \( T_a \) is bounded on \( L^p_w \).

Proof. We begin by proving the first statement for \( 0 < \rho < 1 \). By the Extrapolation Theorem of J. Rubio De Francia (see [12]) it is sufficient to show the boundedness of \( T_a \) on \( L^2_w \) spaces for each \( w \in A_2 \), with constants depending only on \([w]_{A_2}\). Fix \( m_2 \) such that \( n(\rho - 1) < m_2 < \min\{0, n(\rho - \delta)\} \). By Theorem 2.5, given \( w \in A_2 \) we can find \( \varepsilon > 0 \) so that \( w^{1+\varepsilon} \in A_2 \). For this \( \varepsilon \) take \( m_1 < n(\rho - 1) \) in such a way that the straight line \( L \) that joins points with coordinates \((m_1, 1+\varepsilon)\) and \((m_2, 0)\), passes through the point \((n(\rho - 1), 1)\). Clearly this is possible due to the fact that we can choose the point \( m_1 \) as close as we like to \( n(\rho - 1) \).

By Theorem 3.7, \( OPA^{m_1}_{\rho, \delta} \) are bounded operators on \( L^2_{w^{1+\varepsilon}} \) for \( w \in A_2 \) and, by Theorem 2.8 (a), \( OPA^{m_2}_{\rho, \delta} \) are bounded on \( L^2 \). Therefore, by Lemma 3.9, \( OPA^{m(\rho - 1)}_{\rho, \delta} \) are bounded operators on \( L^2_w \).

In the case \( a \in A^0_{\rho, \delta} \), one just uses the fact that \( T_a = T_{\alpha} + T_{\alpha_1} \), where \( T_{\alpha_j}, j = 0, 1 \), are pseudodifferential operators belonging to \( OPS^{-1}_{1, \delta}(1-\delta) \). Now a straightforward modification of the proof of the weighted boundedness of operators in \( OPS^{-1}_{1, \delta}(1) \) in [20], yields that \( T_{\alpha_j} \) are both bounded on \( L^p_w \) for \( w \in A_p \), and therefore the same is true for \( T_a \).

The proof of the second statement is similar. By the extrapolation theorem of P. Auscher and J. Martell (see [3, Thm 4.9]) it is sufficient to show the boundedness of \( T_a \) on \( L^2_w \) spaces for each \( w \in A_1 \), with constants depending only on \([w]_{A_1}\). We now repeat the argument above with \( n(\rho - 1)/2 \) instead of \( n(\rho - 1) \), \( n(\rho - \delta)/2 \) instead of \( n(\rho - \delta) \), and Theorem 3.3 and Theorem 2.8 replacing Theorem 3.7 and Theorem 2.8 (a), respectively. \( \square \)

In connection to Theorem 3.10, it should be mentioned that the second part is the extension of the result in [7] to the case of \( \delta \geq \rho \). The first part of the theorem extends the weighted \( L^p \) boundedness of \( T_a \in OPS^{m(\rho - 1)}_{\rho, \delta} \) proved in [2] for the range \( 0 < \rho \leq \frac{1}{2} \) to the range \( 0 < \rho < 1 \). However, for symbols we can extend the first part of the theorem to \( \rho = 0 \). Indeed, Theorem 3.7 yields the \( L^p_{w^{1+\varepsilon}} \) boundedness of operators with symbols in \( S^{m_1}_{0, \delta} \) with \( m < -n \). Furthermore by Theorem 3.2, operators with symbols in \( S^{m_2}_{0, \delta} \) with \( m < -\frac{n}{2} \) are bounded on \( L^2 \). Hence, an interpolation procedure as above yields the boundedness in \( L^p_w \) of operators in \( OPS^{m(\rho - 1)}_{\rho, \delta} \) for \( 0 \leq \rho < 1, 0 \leq \delta < 1 \).

Also using Lemma 3.9, we can extend the range of \( m \) at the price of obtaining boundedness for fewer weights.

Theorem 3.11. Let \( 1 < q < \infty \), \( 0 < \rho < 1 \), and \( 0 \leq \delta < 1 \) and suppose either:

(a) \( a \in A^m_{\rho, \delta} \) and \( m < n(\rho - 1) \left( \frac{1}{q} - \frac{1}{2} \right) + \min\{0, n(\rho - \delta)\} \); or

(b) \( a \in S^m_{\rho, \delta} \) and \( m < n(\rho - 1) \left( \frac{1}{q} - \frac{1}{2} \right) + \min\{0, n(\rho - \delta)/2\} \).

Then, for all \( w \in A_q \), there exists \( \alpha \in (0,1) \), depending on \( m, \rho, \delta, q \) and \([w]_{A_w} \), such that, for all \( \varepsilon \in [0,\alpha] \), \( T_a \) is bounded on \( L^{q, \varepsilon}_w \).

Proof. If \( m < n(\rho - 1) \), then, by Theorem 3.7, there is nothing to prove, so assume \( m > n(\rho - 1) \). First assuming (a) we fix \( m_2 \) such that \( m < m_2 < n(\rho - 1) \left( \frac{1}{q} - \frac{1}{2} \right) + \min\{0, n(\rho - \delta)\} \) and \( m_1 < n(\rho - 1) \). Then since \( OPA^{m_1}_{\rho, \delta} \) are a bounded operators on \( L^p_w \) for all \( w \in A_p \).
by Theorem 3.7, and $OPA^{m_2}_{p,d}$ are bounded on $L^p$, by Theorem 2.8 (a). We see that all the assumptions of Lemma 3.9 are fulfilled and therefore we obtain the desired result.

The proof under assumption (b) is the same, except we replace Theorem 2.8 (a) by Theorem 2.8 (b).

4. Some applications in harmonic analysis

In this section we show how our weighted norm inequalities can be used to derive the $L^p$ boundedness of commutators of functions of bounded mean oscillation with a wide range of pseudodifferential operators. We start with the precise definition of a function of bounded mean oscillation.

**Definition 4.1.** A locally integrable function $u$ is of bounded mean oscillation if

$$\|f\|_{\text{BMO}} := \sup_B \frac{1}{|B|} \int_B |f(x) - f_B| \, dx < \infty,$$

where the supremum is taken over all balls in $\mathbb{R}^n$. We denote the set of such functions by $\text{BMO}$.

For $u \in \text{BMO}$ it is well-known that $e^{r|u(x)|}$ is locally integrable for $r < 1$. This is a consequence of the John-Nirenberg theorem, see, for example, [13, p. 524]. Furthermore, for all $\gamma < \frac{1}{2e}$, there exists a constant $C_{n, \gamma}$ so that for $u \in \text{BMO}$ and all balls $B$,

$$\frac{1}{|B|} \int_B e^{\gamma |u(x) - u_B|/\|u\|_{\text{BMO}}} \, dx \leq C_{n, \gamma}. \tag{4.2}$$

For this see [13, p. 528].

The following abstract lemma will enable us to prove the $L^p$ boundedness of the BMO commutators of pseudodifferential operators.

**Lemma 4.2.** For $1 < p < \infty$, let $T$ be a linear operator which is bounded on $L^p_{\text{wt}}$ for all $w \in A_p$ for some fixed $\alpha \in (0,1]$. Then given a function $f \in \text{BMO}$, if $\Phi(z) := \int e^{zf(x)}T(e^{-zf(x)}u)(x)u(x) \, dx$ is holomorphic in a disc $|z| < \lambda$, then the commutator $[f,T]$ is bounded on $L^p$.

**Proof.** Without loss of generality we can assume that $\|f\|_{\text{BMO}} = 1$. We take $u$ and $v$ in $C_c^\infty$ with $\|u\|_{L^p} \leq 1$ and $\|v\|_{L^{p'}} \leq 1$, and an application of Hölder’s inequality to the holomorphic function $\Phi(z)$ together with the assumption on $v$ yield

$$|\Phi(z)|^p \leq \int e^{p\Re z f(x)}|T(e^{-zf(x)}u)|^p \, dx.$$

Our first goal is to show that the function $\Phi(z)$ defined above is bounded on a disc with centre at the origin and sufficiently small radius. At this point we recall a lemma due to Chanillo [6] which states that if $\|f\|_{\text{BMO}} = 1$, then for $2 < s < \infty$, there is an $r_0$ depending on $s$ such that for all $r \in [-r_0, r_0]$, $e^{sf(x)} \in A_p$.

Taking $s = 2p$ in Chanillo’s lemma, we see that there is some $r_1$ depending on $p$ such that for $|r| < r_1$, $e^{sf(x)} \in A_p$. Then we claim that if $R := \min(\lambda, \frac{\alpha r_1}{p})$ and $|z| < R$ then $|\Phi(z)| \lesssim 1$. Indeed since $R < \frac{\alpha r_1}{p}$ we have $|\Re z| < \frac{\alpha r_1}{p}$ and therefore $|\frac{p\Re z}{\alpha}| < r_1$. Therefore Chanillo’s lemma yields that for $|z| < R$, $w := e^{\frac{p\Re z}{\alpha}f(x)} \in A_p$, and since $e^{p\Re z f(x)} = w^n$, we have $w^n \in A_p$. Therefore $\Phi(z) < \lambda^n$ and hence $|\Phi(z)| \lesssim 1$. This completes the proof of Lemma 4.2.

□
the assumption of weighted boundedness of $T$ and the $L^p$ bound on $u$, imply that for $|z| < R$
\[ |\Phi(z)|^p \leq \int e^{pRe z f(x)} |T(e^{-z f(x)} u)|^p dx \]
\[ = \int w^p |T(e^{-z f(x)} u)|^p dx \]
\[ \leq \int w^p e^{-z f(x)} u^p dx = \int w^p w^{-\alpha} |u|^p dx \lesssim 1, \]
and therefore $|\Phi(z)| \lesssim 1$ for $|z| < R$.

Finally, using the holomorphicity of $\Phi(z)$ in the disc $|z| < R$, Cauchy’s integral formula applied to the circle $|z| = R' < R$, and the estimate $|\Phi(z)| \lesssim 1$, we conclude that
\[ |\Phi'(0)| \leq \frac{1}{2\pi} \int_{|z|=R'} |\Phi(z)| |dz| \lesssim 1. \]

By construction of $\Phi(z)$, we actually have that $\Phi'(0) = \int v(x)[f, T]u(x) dx$ and the definition of the $L^p$ norm of the operator $[f, T]$ together with the assumptions on $u$ and $v$ yield at once that $[f, T]$ is a bounded operator from $L^p$ to itself for $p$.

The following lemma guarantees the holomorphicity of $\Phi(z) := \int e^{zf(x)} T(e^{-zf(x)} u(x)) v(x) dx$, when $T$ is a bounded pseudo-pseudodifferential operator.

**Lemma 4.3.** Let $T_\alpha \in \text{OpL}^\infty A_p^{m,\delta}$ be an $L^2$ bounded operator. Given $f \in \text{BMO}$ with $\|f\|_{\text{BMO}} = 1$ and $u$ and $v$ in $C_0^\infty$, there exists $\lambda > 0$ such that the function $\Phi(z) := \int e^{zf(x)} T_\alpha(e^{-zf(x)} u(x)) v(x) dx$ is holomorphic in the disc $|z| < \lambda$.

**Proof.** By the definition of pseudo-pseudodifferential operators above Definition 2.2 as the operator limit of the partial sums of the Littlewood-Paley pieces, $\sum_{k=0}^N T_{\alpha_k} := T_N$, it follows that for all $g \in L^2$
\begin{equation}
\lim_{N \to \infty} \|T_N g - T_\alpha g\|_{L^2} = 0.
\end{equation}

Now recall the fact mentioned in the beginning of this section concerning functions $f \in \text{BMO}$, namely the local integrability of $e^{r|f(x)|}$ for $r < 1$. This means $e^{r|f|} \in L^2_{\text{loc}}$ if $r < \frac{1}{2}$, and hence for $u \in C_0^\infty$ and $|z| < \frac{1}{2}$, one has $ue^{\pm zf} \in L^2$. Now, if we define $\Phi_N(z) = \int v(x)e^{zf(x)} T_N(e^{-zf(x)} u(x)) v(x) dx$, then since the integral defining $\Phi_N$ is absolutely convergent and its integrand is holomorphic in $z$ for $|z| < 1$, it follows that $\Phi_N$ is a holomorphic function in $|z| < 1$. Now we claim that for $\gamma$ as in (4.2),
\[ \lim_{N \to \infty} \sup_{|z| < \frac{\gamma}{2}} |\Phi_N(z) - \Phi(z)| = 0. \]

Indeed, since $\frac{\gamma}{2} < \frac{1}{2}$, for $|z| < \frac{\gamma}{2}$,
\[ |\Phi_N(z) - \Phi(z)| = \left| \int v(x)e^{zf(x)} (T_N - T_\alpha)(e^{-zf} u) dx \right| \]
\[ \leq \|v e^{zf}\|_{L^2} \|(T_N - T_\alpha)(u e^{-zf})\|_{L^2} \]
\[ \leq \|v\|_{L^\infty} \left\{ \int_{\text{supp } v} e^{2Re z f(x)} dx \right\}^{\frac{1}{2}} \|(T_N - T_\alpha)(u e^{-zf})\|_{L^2}. \]

Using the assumption $\|f\|_{\text{BMO}} = 1$, and (4.2), it follows that for any compact set $K$, $\int_K e^{\pm 2Re z f(x)} dx \leq C_\gamma(K)$, for $|z| < \frac{\gamma}{2}$. Therefore, (4.3) yields
\[ \lim_{N \to \infty} \sup_{|z| < \frac{\gamma}{2}} |\Phi_N(z) - \Phi(z)| = 0 \]
and hence $\Phi(z)$ is a holomorphic function in $|z| < \frac{\gamma}{2}$. \qed
Lemmas 4.2 and 4.3 yield our main result concerning commutators with BMO functions, namely

**Theorem 4.4.** Suppose either:

(a) \( a \in A^m_{p,\delta}, \ 0 < p \leq 1, \ 0 \leq \delta < 1 \) and \( m < n(p-1)(\frac{1}{2} - \frac{1}{p} + \min\{0, n(p-\delta)\}); \) or

(b) \( a \in S^m_{p,\delta}, \ 0 < p \leq 1, \ 0 \leq \delta < 1 \) and \( m < n(p-1)(\frac{1}{2} - \frac{1}{p} + \min\{0, n(p-\delta)/2\}). \)

Then, for \( f \in \text{BMO} \), the commutator \([f, T_a]\) is bounded on \( L^q \).

If

(c) \( a \in L^\infty S^m_{p,\delta} \) with \( 0 \leq p \leq 1 \) and \( m < \frac{n}{p}(p-1) \) with \( p \in [1,2] \),

then, for \( f \in \text{BMO} \), the commutator \([f, T_a]\) is bounded on \( L^q \) for all \( q \in (p,\infty) \).

**Proof.** (a) By Theorem 3.11 part (a), there is an \( \alpha \in (0,1) \) for which \( T_a \in \text{OPA}^m_{p,\delta} \) with \( m < n(p-1)(\frac{1}{2} - \frac{1}{p} + \min\{0, n(p-\delta)\}) \) is \( L^q_{w,a} \)-bounded. Furthermore if we define the function

\[
\Phi(z) := \int e^{z f(x)} T_a(e^{-z f(x)} u)(x)v(x) dx
\]

then Lemma 4.3 yields that \( \Phi(z) \) is a holomorphic function in a disc around the origin. So an application of Lemma 4.2 with \( T = T_a \) yields that the commutator \([f, T_a]\) is a bounded operator from \( L^p \) to itself.

(b) We repeat the argument above, but with (a) from Theorem 3.11 replaced by (b).

(c) If \( a \in L^\infty S^m_{p,\delta} \) with \( 0 \leq p \leq 1 \), \( m < \frac{n}{p}(p-1) \), then by Theorem 3.3 we know \( T_a \) is bounded on \( L^q_w \) for \( w \in A^e_{q/p} \). But for \( \alpha > 0 \) sufficiently small, it follows from Definition 2.4 that \( w^\alpha \in A^e_{q/p} \) when \( w \in A^e_q \) and \( 0 \leq e \leq \alpha \). Therefore, from Lemma 4.2, we conclude that \( ||[f, a(x, D)] u ||_{L^p} \lesssim ||u||_{L^p} \). \( \square \)

**Theorem 4.5.** Suppose either:

(a) \( a \in L^\infty A^m_{p,\delta} \) with \( m < n(p-1) \) and \( 0 \leq p \leq 1 \); or

(b) \( a(x, y, \xi) = e^{|\xi|^{\gamma-\delta}} \sigma(x, y, \xi) \) and \( \sigma \in L^\infty A^m_{p,\delta} \) with \( 0 < p \leq 1 \) and \( m < \frac{n}{p}(p-1) \); or

(c) \( a \in A^{n(p-1)}_{p,\delta} \) with \( 0 \leq \delta < 1 \) and \( 0 < p \leq 1 \); or

(d) \( a(x, \xi) = e^{|\xi|^{\gamma-\delta}} \sigma(x, \xi) \) and \( \sigma \in L^\infty S^m_{\delta}(p-1) \) with \( 0 < p \leq 1 \).

Then, for \( f \in \text{BMO} \) and \( k \) a positive integer, the \( k \)-th commutator defined by

\[
T_{a,f,k}u(x) := T_a((f(x) - f(\cdot))^k u)(x)
\]

is bounded on \( L^q_w \) for each \( w \in A^e_q \) and \( q \in (1,\infty) \).

**Proof.** The claims follow from Theorem 2.13 in [1], and Theorem 3.7, Corollary 3.8, Theorem 3.10 and Theorem 3.6, respectively. \( \square \)

**References**


