1. A k-chromatic graph \( G \), (i.e. \( \chi(G) = k \)) is called critically k-chromatic or just critical, if \( \chi(G - v) < k \) for all \( v \). Show that every k-chromatic graph has a critical k-chromatic induced subgraph, and that any such subgraph has minimum degree \( \delta \geq k - 1 \).

**Proof:** Let \( G \) be a k-chromatic graph. Then
\[
\chi(G) = k.
\]
[We use the proof of Thm 3.22]

Let \( H \) be any smallest induced subgraph such that \( \chi(H) = k \). Since \( H \) is smallest w/ this property, then \( H - v \) is \((k - 1)\)-chromatic \( \forall v \in V(H) \), i.e.
\[
\chi(H - v) = k - 1.
\]
Hence \( H \) is a critical k-chromatic induced subgraph.

We claim that \( \deg(v) \geq k - 1 \). Suppose not. Then \( \deg(v) < k - 1 \).

Let \( V_1, \ldots, V_{k-1} \) be the coloring classes of \( H - v \).
If \( \deg(v) < k - 1 \), then there exists \( V_i \) s.t. \( \forall u \in V_i \)
\[
v \notin E(H - v).
\]
Then \( V_1, \ldots, V_i \cup \{v\}, \ldots, V_{k-1} \) are then coloring classes for \( H - v \). Hence \( \chi(H - v) \leq k - 1 \).
Thus \( \deg(v) \geq k - 1 \).

2. Determine the critical 3-chromatic graphs.

**Solution:** A graph is critical 3-chromatic if and only if it is an odd cycle of length \( n \geq 3 \).

**Proof:** \( \Rightarrow \) Suppose \( G \) is a critical 3-chromatic graph. Hence \( \chi(G) = 3 \) and \( \chi(G - v) < 3 \) \( \forall v \in V(G) \).

Since bipartite graphs have \( \chi(H) = 2 \) \( \forall H \) bipartite.
2. Let graph $G$ be not bipartite. Hence any cycle in $G$ is not even. If $G$ has no cycle, then it is a forest and hence bipartite. Therefore $G$ must contain an odd cycle, say $C$. If $v \in V(G)$ st $v \notin V(C)$, then $C$ is a subgraph of $G-v$ and thus $G-v$ is not bipartite. Therefore $\chi(G-v) \geq 3$, which is a contradiction, since $G$ is critical 3-chromatic. Thus \( \forall v \in V(G) \ v \notin V(C) \), i.e., $G=C$ is an odd cycle.

"\( \leq \)" Suppose $G$ is an odd cycle of length $n \geq 3$. It is straightforward to see that $\chi(G) = 3$. Let $v \in V(G)$. Since $G$ is a cycle, there exist $v_1, v_2 \in V(G)$ st $v_1, v_2 \in E(G)$. Then $G-v$ is a path from $v_1$ to $v_2$. Hence $G-v$ is bipartite and thus $\chi(G-v) = 2$. Thus $G$ is critical.

3. Let $G$ be a simple graph of order $n$. Let $e = e(G)$ be the number of edges. Prove that $\chi(G) \geq \frac{n^2}{n^2 - 2e}$.

Proof: Let $G$ be a graph of order $n$. Let $e = e(G)$ and let $k = \chi(G)$. Then there exists a partition $V_1, \ldots, V_k$ of the vertices into coloring classes. Notice that $G$ is then $k$-partite.

Since $n = |V(G)|$ then $|V_i| \geq \frac{n}{k}$ for each $i$.

Then the upper bound on $e$ would be obtained if $G$ is a complete $k$-partite graph. Thus $e \leq \frac{n}{k} \cdot \left(\frac{(k-1)}{k} \cdot \frac{n}{k}\right) + \frac{n}{k} \cdot \left(\frac{(k-2)}{k} \cdot \frac{n}{k}\right) + \cdots + \frac{n}{k} \cdot \frac{1}{k} \cdot \frac{n}{k}$.
Therefore \[ e \leq \left(\frac{n}{k}\right)^2 \left[\frac{(k-1)+(k-2)+\ldots+1}{2}\right] = \left(\frac{n}{k}\right)^2 \cdot \frac{(k-1)(k)}{2} = \frac{n^2(k-1)}{2k} \]

Hence \[ 2ek \leq n^2k - n^2 \Rightarrow n^2 \leq n^2k - 2ek \]

\[ \Rightarrow n^2 \leq k \left(\frac{n^2-2e}{n^2}\right) \Rightarrow n^2 \leq k \]

\[ \Rightarrow 0 \leq \frac{n^2}{n^2-2e} \]

i.e. \[ \chi'(G) \geq \frac{n^2}{n^2-2e} \]

4. Find the edge chromatic number of \( K_n \) and prove your answer.

Proof: Claim: \[ \chi'(K_n) = \begin{cases} n & \text{if } n \text{ is odd} \\ n-1 & \text{if } n \text{ is even}. \end{cases} \]

Proof of claim:
First suppose \( n \) is odd. We may arrange the vertices of \( K_n \) in an arrangement of a regular \( n \)-polygon.

Then we may color each of the edges that form the polygon a different color. Then each of the "internal" edges are parallel to one of the edges of the "perimeter" 

![Picture](https://via.placeholder.com/150)

So these could be colored the same color as the edge they are parallel to.

Hence \[ \chi'(K_n) \leq n. \]

Notice that \( K_n \) is regular of degree \( n-1 \).

Still need to establish why \( \chi'(K_n) = n \).
4.
Now suppose $n$ is even.
Let $v \in V(K_n)$. Then $K_{n-v} \cong K_{n-1}$ and $n-1$ is odd.

Therefore $\chi'(K_{n-1}) = n-1$. by the previous part.

Then $K_{n-1} + v$ has $n-1$ new edges. Each one has an edge from $K_{n-1}$ that is not adjacent to.

Then color $v$ the same color. Hence we can use the $n-1$ colors of $K_{n-1}$ to color the remaining edges and therefore $\chi'(K_n) \leq n-1$. Notice that this cannot be lowered since then $\chi'(K_{n-1}) \neq n-1$.

5. Let $G_1, G_2$ be two graphs. Consider their join $G_1 \cup G_2$. Prove that (a) $\chi(G_1) + \chi(G_2) = \chi(G_1 \cup G_2)$ and (b) $G_1$ & $G_2$ are critical if and only if $G_1 \cup G_2$ is.

Proof:
(a) Let $\chi(G_1) = n$ and $\chi(G_2) = m$.

We can use the colors $1, \ldots, n$ to color the vertices of $G_1$ and $n+1, \ldots, n+m$ for the $m$ vertices of $G_2$.

Since $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$. This would color all the vertices of $G_1 \cup G_2$ and no two adjacent vertices will have the same color.

Hence $\chi(G_1 \cup G_2) \leq n+m$.

For each vertex $v \in V(G_1)$, $v \in E(G_1 \cup G_2)$ $\forall v \in V(G_2)$.

Hence the vertices of $G_1$ can not have the same color as any in $G_2$. Therefore $\chi(G_1 \cup G_2) \geq n+m$.
(b) \(\Rightarrow\) Suppose \(G_1\) and \(G_2\) are critical graphs and 
\(\chi(G_1) = n\) and \(\chi(G_2) = m\).

By part (a), \(\chi(G_1 + G_2) = m + n\).

Let \(v \in V(G_1 + G_2)\). Since \(V(G_1 + G_2) = V(G_1) \cup V(G_2)\)
then \(v \in V(G_1)\) or \(v \in V(G_2)\). Wlog spse \(v \in V(G_1)\).

Then \(\chi(G_1 + G_2 - v) = (\chi(G_1 - v) + \chi(G_2))\)
\(\leq n + m\) since \(G_1\) is critical.

Similarly, if \(v \in V(G_2)\), \(\chi(G_1 + G_2 - v) \leq n + m\).

Therefore \(G_1 + G_2\) is critical.

\(\Leftarrow\) Spse \(G_1 + G_2\) is critical. Again by part (a)
\(\chi(G_1 + G_2) = \chi(G_1) + \chi(G_2) = n + m\)

Let \(v \in V(G_1)\). Spse \(\chi(G_1 - v) = n\), i.e. \(G_1\) is
not critical. Then \(\chi(G_1 + G_2 - v) = \chi((G_1 - v) + G_2)\)
\(= n + m\) and thus \(G_1 + G_2\) is not critical.

Hence \(G_1\) is critical. Similarly, \(G_2\) is critical.

6. Give an example of a graph \(G\) that satisfies
\(\chi(G) = \rho(G)\) and \(\omega(G) < \chi(G)\). Why does this not contradict the Perfect Graph Thm?

Solution: Notice that if \(\omega(G) < \chi(G)\) then \(G\) is not perfect. If \(\chi(G) = \rho(G)\) then \(\omega(G) = \chi(G)\).

In order to not contradict the Perfect Graph Thm
6. $\overline{G}$ must have an induced subgraph $H$ s.t $w(H) < \chi(H)$, i.e. $\overline{G}$ is not perfect.

Example:

Notice that since $\overline{G}$ contains a triangle $\{2\}$, then $w(\overline{G}) \geq 3$.

Also since $\overline{G}$ is not bipartite then $\chi(\overline{G}) \geq 3$. Notice that we can color the vertices with 3 colors. The coloring classes are

$\{1, 3, 4, 7\}, \{2, 5, 3\}, \{6\}$. 

Thus $\chi(\overline{G}) = 3$.

Therefore $\alpha(G) = \beta_c(G)$.

Let $H$:  

is an induced graph of $\overline{G}$ and $w(H) = 2$, but $\chi(H) \geq 3$ $[H$ not bipartite].
7. Spse that $G$ satisfies $\alpha(G) = k(G)$. Let $k$ be the clique cover of $G$, where $1 \leq k = k(G)$, and let $A$ be the collection of all independent sets of cardinality $\alpha(G)$. Show that $|\{A \cap K | K \in k \} | = 1 \iff A \neq \emptyset$ and $K \in k$.

Proof: Let $A \in k$ and let $K \in k$. Then $K$ is a clique and every vertex in $K$ is adjacent to all the other vertices of $K$. Let $n = \alpha(G) = k(G)$.

Let $v \in A$. Then $v \notin K$ for some $K \in k$.

Thus $\{A \cap K \} \neq \emptyset$. Since $K$ is a clique, then if $v \in A \cap K$, then $v, v \notin A$ but $v \in E(G)$.

Thus $|\{ A \cap K \} | = 1$.

We still need to establish that $|\{ A \cap K \} | = 1 \iff K \in k$.

Let $K' \in k$ s.t. $K \neq K'$.

Since $|A| = n$ then by above $|\{A \cap K \} | = 1$ and

Since $|A| = n$, let $A = \{V_1, \ldots, V_n\}$.

Since $|K| = n$, let $K = \{K_1, \ldots, K_n\}$.

For each $v \in A$ s.t. $K_i \cap K_j \neq \emptyset$, let $v \notin \{V_i \cup V_j\}$ as seen above. Let $v, v' \in A$ s.t. $i \neq j$. We claim that $K_i \neq K_j$.

If $K_i = K_j$ then $v, v' \in K_i = K_j$ and thus $v, v \in E(G)$ since $K_i$ is a clique. But adjacent vertices can not be in $A$. Therefore $K_i \neq K_j$.

Hence there is a 1-1 correspondence $A \rightarrow k$

$Vi \rightarrow K_i$

Thus $\{A \cap K \} \neq \emptyset \iff A$ and $|\{A \cap K \} | = 1$. 

\[\rightarrow\]
The dual statement for graphs satisfying \( \omega(G) = \chi(G) \) is:

Let \( C \) be the collection of the vertices all maximal cliques of \( G \). Each such set has cardinality \( \omega(G) \).

Let \( B \) be the collection of coloring classes of \( G \). Then \( 1 \subseteq CB \) \( \forall C \in C \) and \( B \subseteq B \).

Proof: Since \( \omega(G) = \alpha(G) \) \& \( \chi(G) = k(G) \)
Hence \( \alpha(G) = k(G) \).
Also \( \forall C \subseteq C \Rightarrow C = \alpha(G) \) \& \( B = \chi(G) \) and the result follows immediately from the first part.