Every graph is assumed to be simple, unless otherwise stated.

1. A $k$-chromatic graph $G$ (i.e. $\chi(G) = k$) is called critically $k$-chromatic or just critical, if $\chi(G - v) < k$ for every $v \in V(G)$. Show that every $k$-chromatic graph has a critical $k$-chromatic induced subgraph, and that any such subgraph has minimum degree $\delta \geq k - 1$.

2. Determine the critical 3-chromatic graphs.

3. Let $G$ be a simple graph of order $n$. Let $e = e(G)$ be the number of edges. Prove that $\chi(G) \geq \frac{n^2}{n^2 - 2e}$.

4. Find the edge chromatic number of $K_n$ and prove your answer.

5. Let $G_1, G_2$ be two graphs. Consider their join $G_1 + G_2$. Prove that
   
   (a) $\chi(G_1 + G_2) = \chi(G_1) + \chi(G_2)$
   
   (b) $G_1$ and $G_2$ are critical if and only if $G_1 + G_2$ is.

6. Give an example of a graph $G$ for which $\alpha(G) = k(G)$ and $\omega(G) < \chi(G)$. Why does this not contradict the Perfect Graph theorem?

7. Suppose that $G$ satisfies $\alpha(G) = k(G)$. Let $\mathcal{K}$ be the clique cover of $G$ where $|\mathcal{K}| = k(G)$, and let $\mathcal{A}$ be the collection of all independent sets of cardinality $\alpha(G)$. Show that
   
   $|A \cap K| = 1$ for all $A \in \mathcal{A}$ and $K \in \mathcal{K}$.

   Give a dual statement for a graph satisfying $\omega(G) = \chi(G)$.

Extra Problems for Graduate Students:

8. Show that if any two odd cycles of a graph $G$ have a vertex in common then $\chi(G) \leq 5$.

9. Prove that the only regular graph of degree $n \geq 3$ which is $(n + 1)$-chromatic is $K_{n+1}$.