1. Find $\omega(K_4)$ and $\omega(K_5)$. Prove your answer.

**Solution:** $\omega(K_4) = 3$

**Proof:** Let $S = \{x_1, x_2, x_3, x_4\}$ be a set and let $J = \{S_1, S_2, S_3, S_4\}$ such that $S_1 = \{x_1, x_2, x_3\}$, $S_2 = \{x_1, x_2\}$, $S_3 = \{x_1, x_3\}$ and $S_4 = \{x_1\}$. Then $S_i \cap S_j \neq \emptyset \quad \forall \ i \neq j$. Thus $\Omega(J) \cong K_4$ and hence $\omega(K_4) \leq 3$.

If $\omega(K_4) < 3$ then either $\omega(K_4) = 1$ or $\omega(K_4) = 2$.

If $\omega(K_4) = 1$ then there exists a set $S = \{x_1\}$ with $J = \{S_1, S_2, S_3, S_4\}$ and $S_i \subseteq S$ , $S_i \neq \emptyset$ , and $S_i \neq S_j \quad \forall \ i \neq j$.

This is impossible, since there is only one nonempty subset of $S$, namely $S$ itself. So $\omega(K_4) \neq 1$.

If $\omega(K_4) = 2$ then there exists a set $S = \{x_1, x_2\}$ with $J = \{S_1, S_2, S_3, S_4\}$ and $S_i \subseteq S$ , $S_i \neq \emptyset$ , $S_i \neq S_j \quad \forall \ i \neq j$.

The subsets of $S$ are: $\emptyset$, $\{x_1\}$, $\{x_2\}$, $\{x_1, x_2\}$

Notice that there do not exist 4 nonempty distinct subsets. Thus $\omega(K_4) \neq 2$.

- $\omega(K_5) = 4$

**Proof:** Let $S = \{x_1, x_2, x_3, x_4, x_5\}$ be a set and let $J = \{S_1, S_2, S_3, S_4, S_5\}$ such that $S_1 = \{x_1, x_2, x_3, x_4\}$, $S_2 = \{x_1, x_2\}$, $S_3 = \{x_1, x_3\}$, $S_4 = \{x_1, x_4\}$, $S_5 = \{x_1\}$. Then $S_i \cap S_j \neq \emptyset \quad \forall \ i \neq j$.

Thus $\Omega(J) \cong K_5$ and hence $\omega(K_5) \leq 4$.

If $\omega(K_5) < 4$ then either $\omega(K_5) = 1$, $\omega(K_5) = 2$ or $\omega(K_5) = 3$. As above $\omega(K_5) \neq 1, 2$. 

As above $\omega(K_5) \neq 1, 2$. 

2. If \( \omega(K_5)=3 \) then there exists \( S=\{x_1, x_2, x_3\} \) and \( \mathcal{F}=\{S_1, S_2, S_3, S_4, S_5\} \) such that \( \mathcal{I}(\mathcal{F})=K_5 \). Then \( S_i \neq \emptyset, S_i \subseteq S \forall i \) and \( S_i \cap S_j \neq \emptyset \ \forall i \neq j \). Consider all the possible subsets of \( S \):
\[ \emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\} \]
Notice that there are 7 nonempty ones.
If \( S_i=\{x_i\} \) for some \( i \) then \( S_j \neq \{x_1, x_2, x_3\}, \{x_2, x_3\}, \{x_1, x_3\}, \{x_1, x_2\} \)
But then there are only 3 more subsets that intersect \( \{x_i\} \), and we need a total of 5 subsets. Similarly,
\( S_i \neq \{x_2\}, S_i \neq \{x_3\} \ \forall i \). Then again there are only 4 sets left and we need 5. Hence \( \omega(K_5) \neq 3 \).

2. Prove that the least number of vertices in a cubic graph with a bridge is 10.

Proof: Let \( G \) be a graph and let \( uv \) be a bridge in \( G \). Then the smallest cubic graph must be connected.
Since otherwise each connected component would be cubic of smaller order.
Since \( uv \) is a bridge then \( G-uv \) is disconnected.
Hence \( G-uv \) has at least 2 connected components.
Let \( C_1 \) be one of the connected components. Wlog suppose \( uv \in \mathcal{V}(C_1) \). Since \( uv \) is removed then \( \deg(u)=2 \) in \( G-uv \). Thus there exists \( u_1, u_2 \in \mathcal{V}(G-uv) \) such that \( u_1, u_2, u \in \mathcal{E}(G-uv) \).
Since \( uv \in \mathcal{V}(G) \) and \( G \) is connected then \( u_1, u_2 \in \mathcal{V}(C_1) \). Note that \( v \notin \mathcal{V}(C_1) \), since otherwise \( G-uv \) would not be disconnected.
Now \( \deg(u_1)=\deg(u_2)=3 \) and thus each one of \( u_1, u_2 \)
are connected to 1 more vertices in \( C_1 \). Hence \( u_1, u_2 \in C_1 \).
Since \( v \notin V(G_1) \).

Picture:

\[
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{example.png}}
\end{array}
\]

Notice that then there exists \( u_3, u_4 \) such that \( u_1, u_3, u_2, u_4 \in E(G) \). Then if \( u_1 u_2 \in E(G) \) then since \( \deg u_3 = \deg u_4 = 3 \) then there exists a \( u_5 \) such that \( u_3 u_5 \in E(G) \).

Hence \( G_1 \) has at least 5 vertices.

Similarly, \( v \) belongs in some other connected component \( G_2 \) of \( G - uv \). and \( G_2 \) has at least 5 vertices.

So overall \( G \) must contain at least 10 vertices.

3. Let \( G \) be a block with \( k \geq 3 \). Prove that there exists a vertex \( v \) such that \( G - v \) is also a block.

Proof: Omitted

4. Prove or disprove: The number of cliques of a graph \( G \) does not exceed \( w(G) \) (\( w(G) \): intersection number).

Proof: Wlog suppose \( G \) is connected.

Let \( C_1, \ldots, C_n \) be the cliques of a graph \( G \).

Let \( k = |V(G)| \) and let \( S \) be a set s.t. \( F = \{ S_1, \ldots, S_k \} \) is a family of subsets of \( S \) s.t. \( \bigcup_{i=1}^{k} S_i = S \), \( \bigcap_{i=1}^{k} S_i = \emptyset \), \( |S| = w(G) \). and \( G \cong \tau(F) \).

By the definition of what a clique is we conclude that either \( C_i \cap C_j = \emptyset \) or if \( C_i \cap C_j \neq \emptyset \) then there exists a vertex \( vi \in V(C_i) \) and \( vj \in V(C_j) \) s.t. \( vi \notin V(C_j) \) & \( vj \notin V(C_i) \).
4. Wlog let's assume that $S_1, \ldots, S_{n_1}$ correspond to vertices of $C_1$, $S_{n_1+1}, \ldots, S_{n_2}$ to the vertices of $C_2$, etc.

Notice that $C_1, \ldots, C_n$ forms a clique cover for $G$.

Suppose (wlog) $C_{i_1}, \ldots, C_{i_m}$ forms a minimal clique cover. Then for each $C_i$ there is a vertex that does not belong to any of the other $C_j$. Otherwise, this would not be a minimal clique cover.

Then let $v_i \in V(C_{i_j})$ s.t. $v_i \not\in V(C_{i_j})$ $\forall j \neq i, 1 \leq i, j \leq m$.

Therefore for each $C_i$ there is a $C_j$ s.t. $C \cap C_j \neq \emptyset$ ($G$ is connected) and thus a $v_i \in V(C_{i_j})$ s.t. $v_i \not\in V(C_{i_j})$.

Thus $v_i \in S_{i_1}, \ldots, S_{i_m}$ & $v_i \not\in S_{i_1+1}, \ldots, S_{i_j}$. Therefore $|S| \geq n = \# cliques$. Thus $n \leq \omega(G)$. \( \square \)

5. Characterize the adjacency matrix of a bipartite graph.

**Proof:** Let $G$ be a bipartite graph. Then let $V(G) = V_1 \cup V_2$ s.t. there are no edges b/w the vertices in $V_1$ and no edges b/w the vertices in $V_2$. Suppose $k=|V_1|$ and $m=|V_2|$. Then let $\{V_1, \ldots, V_k\} = V_1$ and $\{V_2, \ldots, V_{k+m}\}$.

Hence the adjacency matrix is of the form

$$A = \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{k \times m} \times \mathbb{R}^{m \times k}$$