Solve all of the following problems.

1. Let $R = \mathbb{Z}[x]/I$ be the quotient ring of $\mathbb{Z}[x]$ by the ideal $I$ generated by $10(x^3 - 1)$.
   (i) Find all the prime ideals of $R$.
   (ii) Find all the maximal ideals of $R$.

2. Show that the ring $\mathbb{Z}[i]/(1 + 3i)$ is isomorphic to $\mathbb{Z}/(10)$.

3. Show that $R = \mathbb{Z}[\omega]$, where $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is a Euclidean domain.

4. Let $f(x) = x^4 - x^2 + 1$.
   (i) Determine whether $f(x)$ is irreducible or not in $\mathbb{Z}[x]$.
   (ii) Determine whether $f(x)$ is irreducible or not in $\mathbb{Q}[x]$.
   (iii) Determine whether $f(x)$ is irreducible or not in $\mathbb{Z}[i][x]$.

5. Let $R$ be a commutative ring with $1 \neq 0$. Let $S$ be a a commutative ring with $1 \neq 0$ and let $R \rightarrow S$ be a ring homomorphism.
   For every ring $S$ we define $\text{ann}_R(S) = 0 :_R S = \{ r \in R \mid \phi(r)s = 0_S \text{ for every } s \in S \}$.
   Let $I$ be an ideal in $R$. Prove that $\sqrt{\text{ann}_R(R/I)} = \sqrt{\text{ann}_R(R)} + I$, where $\sqrt{M} = \text{rad}M$ is the radical of a ring $M$.
   [Hint: When computing $\text{ann}_R(R/I)$ consider $\phi : R \rightarrow R/I$ the natural projection map and consider the identity map on $R$ when computing the $\text{ann}_R(R)$.]