Math 392  
Exam 1  
Fall 2010

Name: Solutions

Instructions:

1. Make sure you have all 10 pages of the test (including this cover page).

2. No books, no notes, and no calculators are allowed.

3. You must show sufficient work to receive credit.
   Correct answers without sufficient work will receive no credit.

4. The point value of each problem occurs to the left of each problem.

<table>
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1. (10 pts) Find the general solution to the differential equation
\[ \frac{dy}{dt} = \frac{1}{y^2 - 2ty} \frac{du}{dt} = \frac{1}{-t(y^2 - 2y)} \Rightarrow \int (y^2 - 2y) \, dy = \int \frac{1}{-t} \, dt \]

\[ \Rightarrow \frac{y^3}{3} - \frac{y^2}{2} = \ln |t| + C_1 \quad \Rightarrow \quad y^3 - 3y^2 = \frac{3}{2} \ln |t| + C_2 \]

2. (8 pts) Perform Euler’s method with the given step size \( \Delta t \) on the given initial-value problem over the time interval given. Your answer should be given in the form of a table. Recall the formula for Euler’s method is: \( y_{k+1} = y_k + f(t_k, y_k)\Delta t \).

\[ \frac{dy}{dt} = 3t - y, \, y(0) = 0, \, 0 \leq t \leq 1, \, \text{and} \, \Delta t = 0.5. \]

<table>
<thead>
<tr>
<th>( k )</th>
<th>( t_k )</th>
<th>( y_k )</th>
<th>( f(t_k, y_k) = 3t_k - y_k )</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( 3(0) - 0 = 0 )</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>0</td>
<td>( 3(0.5) - 0 = 3/2 )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.75</td>
<td>( 3(1) - \frac{3}{4} = 2.75 = 2.25 )</td>
</tr>
</tbody>
</table>

\[ y_1 = y_0 + f(t_0, y_0)\Delta t = 0 + 0 \cdot (0.5) = 0 \]

\[ y_2 = y_1 + f(t_1, y_1)\Delta t = 0 + \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4} \]
3. (10 pts) Find the bifurcation values for \( \frac{dy}{dt} = y^2 - 4y + a \). Show your work.

**Equilibrium points:** \( \frac{dy}{dt} = 0 \)

\[
y^2 - 4y + a = 0 \Rightarrow y = \frac{4 \pm \sqrt{16 - 4a}}{2}
\]

\[
\Rightarrow y = 2 \pm \sqrt{4 - a}
\]

If \( 4 - a > 0 \Rightarrow 4 > a \) then we have two real roots.

If \( 4 - a = 0 \Rightarrow a = 4 \) then one real root.

If \( 4 - a < 0 \Rightarrow 4 < a \) no real roots.

For \( a < 4 \) there are two equilibrium points.

For \( a = 4 \) there is only one equilibrium point.

For \( a > 4 \) we have no equilibrium points.

Hence the only bifurcation value is \( a = 4 \).
4. (10 pts) Find a particular solution to the differential equation

\[ \frac{dy}{dt} + 2y = 5\cos t + t. \]

\[ y_p(t) = a\cos t + b\sin t + ct + d \]

\[ \frac{dy_p}{dt} = -a\sin t + b\cos t + c \]

So

\[ -a\sin t + b\cos t + c + 2a\cos t + 2b\sin t + 2ct + 2d = 5\cos t + t \]

\[ \Rightarrow (2b-a)\sin t + (b+2a)\cos t + 2ct + 2d + c = 5\cos t + t \]

\[ \Rightarrow 2b - a = 0 \]
\[ b + 2a = 5 \]
\[ \Rightarrow 2b = a \]
\[ \Rightarrow b + 2(2b) = 5 \]
\[ \Rightarrow 5b = 5 \]
\[ \Rightarrow b = 1 \]
\[ \Rightarrow a = 2 \]

\[ 2c = 1 \]
\[ \Rightarrow c = \frac{1}{2} \]

\[ 2d + c = 0 \]
\[ \Rightarrow 2d = -\frac{1}{2} \]
\[ \Rightarrow d = -\frac{1}{4} \]

\[ \Rightarrow y_p(t) = 2\cos t + \sin t + \frac{1}{2}t - \frac{1}{4} \]
5. (10 pts) Find the solution to the initial value problem.

\[ \frac{dy}{dt} = 3y + e^{7t}, \quad y(0) = 1. \]

\[ \mu(t) = e^{\int -3 \, dt} = e^{-3t} \]

\[ \frac{dy}{dt} - 3y = e^{7t} \]

\[ \Rightarrow y(t) = \frac{1}{e^{-3t}} \int e^{-3t} e^{7t} \, dt \]

\[ = e^{3t} \int e^{4t} \, dt = e^{3t} \left( \frac{e^{4t}}{4} + C \right) \]

\[ y(0) = 1 \]

\[ \Rightarrow 1 = 1 \cdot \left( \frac{1}{4} + C \right) \Rightarrow C = 1 - \frac{1}{4} = \frac{3}{4} \]

\[ y(t) = e^{3t} \left( \frac{e^{4t}}{4} + \frac{3}{4} \right) = \frac{e^{7t} + 3e^{3t}}{4} \]
6. (10 pts) A 200-gallon tank initially contains a mixture of 150 gallons of sugar water containing 1 pound of sugar per gallon. Sugar is added at a rate of 3 pounds per minute. Suppose that the mixture is kept well mixed and that sugar water is draining out at rate of 2 gallons per minute. Write an initial value problem that models the amount of sugar in the tank. Simplify your answer, but do not solve the problem.

\[ Q(t) = \text{amount of sugar in the tank at time } t \text{ measured in lbs} \]

Initially: \[ Q(0) = 150 \text{ lbs} \]

Rate in: \[ 3 \text{ lbs/min} \]

Rate out: \[ \frac{Q}{y \text{ gal/min}} \]

\[ y(t) = \text{amount of gallons of sugar water at time } t \]
\[ y(0) = 150 \]
\[ y(1) = 148 \]
\[ m = -2 \]
\[ y(t) = -2t + 150 \]

\[ \frac{dQ}{dt} = 3 - \frac{Q}{-2t+150} \cdot 2 = 3 - \frac{Q}{2(-t+75)} \]

\[ \Rightarrow \frac{dQ}{dt} = 3 - \frac{Q}{75-t} \]
\[ Q(0) = 150 \]
7. (15 pts) Consider the differential equation \( \frac{dy}{dt} = f(y) \), where the graph of \( f(y) \) is given below.

(a) (8 pts) Find the equilibrium points, sketch the phase line for this equation and classify the equilibrium points as sources, sinks or nodes.

**Equilibrium pts:** \( \frac{dy}{dt} = 0 \Rightarrow f(y) = 0 \\
\Rightarrow y = -2, 1, 3 \\
\begin{array}{c|c|c|c|c}
y & -2 & 1 & 3 \\
\hline
\frac{dy}{dt} & - & 0 & + & 0 & -
\end{array}

\begin{align*}
y = -2 & \quad \text{is a node} \\
y = 1 & \quad \text{is a source} \\
y = 3 & \quad \text{is a sink}
\end{align*}

Phase Line

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(b) (7 pts) Draw a sketch of the solution to this differential equation with initial value $y(1) = 0$ and discuss the long term behavior of this solution.

As $t \to \infty$, $y(t) \to -2$.

As $t$ increases, the solution approaches the equilibrium solution $y(t) = -2$. 
8. (12 pts) Consider the differential equation \( \frac{dy}{dt} = y^2 \).

(a) (6 pts) Show that \( y_1(t) = \frac{1}{1-t} \) and \( y_2(t) = \frac{1}{2-t} \) are both solutions to \( \frac{dy}{dt} = y^2 \).

\[
\frac{dy_1}{dt} = -(1-t)^{-2} \cdot (-1) = \frac{1}{(1-t)^2} = (y_1(t))^2
\]

\[
\frac{dy_2}{dt} = -(2-t)^{-2} \cdot (-1) = \frac{1}{(2-t)^2} = (y_2(t))^2
\]

\( \Rightarrow y_1(t), y_2(t) \) are both solutions to the diff. eq.

(b) (6 pts) What can you say about solutions of \( \frac{dy}{dt} = y^2 \) for which the initial condition \( y(0) \) satisfies \( \frac{1}{2} < y(0) < 1 \)?

\[
y_1(0) = \frac{1}{1-0} = 1 \quad y_2(0) = \frac{1}{2-0} = \frac{1}{2}
\]

By the Existence & Uniqueness thm, the solution \( y(t) \) of the IVP must satisfy

\[
y_2(t) < y(t) < y_1(t)
\]

i.e.

\[
\frac{1}{2-t} < y(t) < \frac{1}{1-t}
\]
9. (15 pts) Six differential equations and three phase lines are given below. Determine the equation that corresponds to each phase line and state briefly how you know your choice is correct.

\[(i) \frac{dy}{dt} = y^2|y - 1|, \quad (ii) \frac{dy}{dt} = y(1 - y), \quad (iii) \frac{dy}{dt} = y^2 - y, \quad (iv) \frac{dy}{dt} = y^2 - 2y, \quad (v) \frac{dy}{dt} = y^3 - y, \quad (vi) \frac{dy}{dt} = y - y^2.\]

\[\text{(a) (iii) (b) (v) (c) (i)}\]

\[(i) \frac{dy}{dt} = y^2|y - 1| \quad \text{Eq. pts: } y = 0, y = 1 \quad \text{None of them is (i)}\]
\[\frac{dy}{dt} = y(1 - y) \quad \text{Eq. pts: } y = 0, y = 1 \quad \text{None of them is (ii)}\]
\[\frac{dy}{dt} = y^2 - y = y(y - 1) \quad \text{Eq. pts: } y = 0, y = 1 \quad \Rightarrow (c)\]
\[\frac{dy}{dt} = y^2 - 2y = y(y - 2) \quad \text{Eq. pts: } y = 0, y = 2 \quad \Rightarrow (b)\]
\[\frac{dy}{dt} = y^3 - y = y(y^2 - 1) = y(y - 1)(y + 1) \quad \text{Eq. pts: } y = 0, y = -1 \quad \Rightarrow (c)\]
\[\frac{dy}{dt} = y - y^2 = y(1 - y) \quad \text{Eq. pts: } y = 0, y = 1 \quad \Rightarrow \text{None of them is (vi)}\]