Sampling theory and history
Sampling theory and history
Time- and bandlimiting: history, definitions and basic properties
Outline

Sampling theory and history
Time- and bandlimiting: history, definitions and basic properties
Connecting sampling and time- and bandlimiting
Outline

Sampling theory and history
Time- and bandlimiting: history, definitions and basic properties
Connecting sampling and time- and bandlimiting
Multiband-limiting
PART I: Sampling and Time- and Bandlimiting: Background
Bandlimiting

Fourier transform:
\[ \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi it\xi} \, dt \]

Bandlimiting:
\[ P \Sigma f(x) = (\hat{f}1\Sigma) \lor (x) \]

Paley-Wiener space:
\[ \text{PW} \Sigma = P \Sigma (L^2(\mathbb{R})) \]

\[ \text{PW} : \Sigma = [-1/2, 1/2] \]

Projection onto PW:
\[ f \mapsto \int \frac{\sin(\pi(t-s))}{\pi(t-s)} f(s) \, ds \]

Joe Lakey  Time- and bandlimiting
Bandlimiting

Fourier transform: $\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi it\xi} \, dt$
Bandlimiting

Fourier transform: \( \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \xi} \, dt \)

Bandlimiting: \( P_\Sigma f(x) = (\hat{f} \ 1_\Sigma)^\vee(x) \)
Fourier transform: $\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \xi} \, dt$

Bandlimiting: $P_\Sigma f(x) = (\hat{f} \cdot 1_\Sigma)^\vee(x)$

Paley-Wiener space: $PW_\Sigma = P_\Sigma(L^2(\mathbb{R}))$
Fourier transform: \( \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \xi} \, dt \)

Bandlimiting: \( P_{\Sigma} f(x) = (\hat{f} \mathbb{1}_{\Sigma})^\vee(x) \)

Paley-Wiener space: \( \text{PW}_{\Sigma} = P_{\Sigma}(L^2(\mathbb{R})) \)

\( \text{PW} : \Sigma = [-1/2, 1/2] \)
Bandlimiting

Fourier transform: \[ \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi it\xi} \, dt \]

Bandlimiting: \[ P_{\Sigma} f(x) = (\hat{f} \mathbb{1}_{\Sigma})^\vee(x) \]

Paley-Wiener space: \[ \text{PWL} = P_{\Sigma}(L^2(\mathbb{R})) \]

\( \text{PW} : \Sigma = [-1/2, 1/2] \)

\[ \mathbb{1}_{[-1/2,1/2]}^\vee = \frac{\sin \pi t}{\pi t} \]
Fourier transform: $\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi it\xi} \, dt$

Bandlimiting: $P_\Sigma f(x) = (\hat{f} 1_\Sigma) \check{\vee} (x)$

Paley-Wiener space: $\text{PW}_\Sigma = P_\Sigma(L^2(\mathbb{R}))$

$\text{PW} : \Sigma = [-1/2, 1/2]$

$1 \check{\vee}_{[-1/2,1/2]} = \frac{\sin \pi t}{\pi t}$

Projection onto PW: $f \mapsto \int \frac{\sin \pi(t-s)}{\pi(t-s)} f(s) \, ds$
The Shannon sampling theorem

If $f \in \mathcal{PW}$ then, with convergence in $L_2(\mathbb{R})$,

$$f(t) = \sum_{k=-\infty}^{\infty} f(k) \sin(\pi(t-k)) \pi(t-k)$$
The Shannon sampling theorem

Sampling theorem
If \( f \in \text{PW} \) then, with convergence in \( L^2(\mathbb{R}) \),

\[
f(t) = \sum_{k=\infty}^{\infty} f(k) \frac{\sin \pi(t - k)}{\pi(t - k)}
\]
Proof of the sampling theorem

\[ \hat{f}(\xi) = \sum_{k=-\infty}^{\infty} \langle \hat{f}, e^{2\pi i k \xi} \rangle_{L^2(T)} e^{2\pi i k \xi} \text{ on } [-\frac{1}{2}, \frac{1}{2}] \]

Restrict to \([-\frac{1}{2}, \frac{1}{2}]\); apply Fourier inversion:

\[ f(t) = \sum_{k=-\infty}^{\infty} \langle \hat{f}, e^{-2\pi i k \xi} \rangle_{L^2(T)} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i (t-k) \xi} d\xi = \sum_{k} f(k) \sin(\pi(t-k) \frac{\pi}{t-k}) \]

Joe Lakey

Time- and bandlimiting
Proof of the sampling theorem

\[ \hat{f}(\xi) = \sum_{k=-\infty}^{\infty} \langle \hat{f}, e^{2\pi ik \xi} \rangle_{L^2(\mathbb{T})} e^{2\pi ik \xi} \quad (\text{on } [-1/2, 1/2]) \]
Proof of the sampling theorem

\[ \hat{f}(\xi) = \sum_{k=-\infty}^{\infty} \langle \hat{f}, e^{2\pi ik\xi} \rangle_{L^2(\mathbb{T})} e^{2\pi ik\xi} \quad \text{(on } [-1/2, 1/2]) \]

Restrict to \([-1/2, 1/2]\); apply Fourier inversion:

\[ f(t) = \sum_{k=-\infty}^{\infty} \langle \hat{f}, e^{-2\pi ik\xi} \rangle_{L^2(\mathbb{T})} \int_{-1/2}^{1/2} e^{2\pi i(t-k)\xi} d\xi \]
Proof of the sampling theorem

\[ \hat{f}(\xi) = \sum_{k=-\infty}^{\infty} \langle \hat{f}, e^{2\pi ik\xi} \rangle_{L^2(\mathbb{T})} e^{2\pi ik\xi} \quad (\text{on } [-1/2, 1/2]) \]

Restrict to \([-1/2, 1/2]\); apply Fourier inversion:

\[ f(t) = \sum_{k=-\infty}^{\infty} \langle \hat{f}, e^{-2\pi ik\xi} \rangle_{L^2(\mathbb{T})} \int_{-1/2}^{1/2} e^{2\pi i(t-k)\xi} d\xi \]

\[ = \sum_{k=-\infty}^{\infty} \langle \hat{f}, e^{-2\pi ik\xi} \rangle_{L^2(\mathbb{T})} \frac{\sin \pi(t-k)}{\pi(t-k)} \]
Proof of the sampling theorem

\[ \hat{f}(\xi) = \sum_{k=-\infty}^{\infty} \langle \hat{f}, e^{2\pi ik\xi} \rangle_{L^2(\mathbb{T})} e^{2\pi ik\xi} \quad (\text{on } [-1/2, 1/2]) \]

Restrict to $[-1/2, 1/2]$; apply Fourier inversion:

\[
f(t) = \sum_{k=-\infty}^{\infty} \langle \hat{f}, e^{-2\pi ik\xi} \rangle_{L^2(\mathbb{T})} \int_{-1/2}^{1/2} e^{2\pi i(t-k)\xi} \, d\xi \\
= \sum_{k=-\infty}^{\infty} \langle \hat{f}, e^{-2\pi ik\xi} \rangle_{L^2(\mathbb{T})} \frac{\sin \pi(t-k)}{\pi(t-k)} \\
= \sum_{k} f(k) \frac{\sin \pi(t-k)}{\pi(t-k)} \]
Cauchy and Poisson: early contributors?
Cauchy and Poisson: early contributors?

Joe Lakey

Time- and bandlimiting
Nyquist (1924), Kotel’nikov (1933), Shannon (1949): information and communication
Nyquist (1924), Kotel’nikov (1933), Shannon (1949): information and communication
Nyquist (1924), Kotel’nikov (1933), Shannon (1949): information and communication
The problem: sinc not well localized
The problem: sinc not well localized
The problem: sinc not well localized
The engineer’s solution

Ignore the problem!
The engineer’s solution

Ignore the problem!

or perhaps oversample
A more mathematical approach . . .

Find a subspace of $\mathbf{PW}$ whose elements can be approximated locally by finite sinc series.
\[ Pf(x) = \left( \hat{f} \mathbb{1}_{[-1/2, 1/2]} \right)^\vee(x) \]
\[ Pf(x) = \left( \hat{f} \ 1_{[-1/2,1/2]} \right) (x) \]
\[ (Q_T f)(x) = 1_{[-T,T]}(x) \ f(x) \]
\[ Pf(x) = \left( \hat{f} \mathbb{1}_{[-1/2, 1/2]} \right)^\vee(x) \]

\[ (Q_T f)(x) = \mathbb{1}_{[-T, T]}(x) f(x) \]

\[ PQ_T P: \text{ self-adjoint,} \]
Problem: which 2nd order differential operators commute with $Q_T$?
Problem: which 2nd order differential operators commute with $Q_T$?

... with $PQ_T$
The “lucky accident”

$P_\Omega Q_T$ commutes with certain differential operator

\[
(4 T^2 - t^2) \frac{d^2}{dt^2} - 2t \frac{d}{dt} - t^2
\]
The “lucky accident”

\[ P_\Omega Q_T \] commutes with certain differential operator 
\[
(4T^2 - t^2) \frac{d^2}{dt^2} - 2t \frac{d}{dt} - t^2
\]

Eigenfunctions: Prolate Spheroidal Wave Functions
Prolate spheroidal wave functions
Prolate spheroidal wave functions

Study of heat flow (C. Niven, 1879)
Prolate spheroidal wave functions

Study of heat flow (C. Niven, 1879)

Relationships to Legendre, Bessel series: Stratton, 1935
Prolate spheroidal wave functions
Study of heat flow (C. Niven, 1879)
Relationships to Legendre, Bessel series: Stratton, 1935
Bouwkamp, 1946: expansions in Legendre series
Prolate spheroidal wave functions

Study of heat flow (C. Niven, 1879)

Relationships to Legendre, Bessel series: Stratton, 1935

Bouwkamp, 1946: expansions in Legendre series

$\varphi_n$ has $n$ zeros in $[-1, 1]$ (Kellogg, 1916 for general principle)
Figure: Even eigenvectors for one frequency interval. $N = 129$ point centered DFT. $S \sim 65 + [-16, 16]$; $\Sigma \sim 65 + [-16, 16]$. $c = \#T \times \#\Sigma / N = 8.44$. Plunge region $\sim 7 \leq n \leq 12$. 
THE PRESIDENT in the Chair.

The Present received were laid on the table, and thanks ordered for them.

The following Papers were read:


(Abstract.)

The object of the present paper is to investigate the expressions which present themselves in the mathematical treatment of the problem of the conduction of heat in an ellipsoid of revolution. The results obtained constitute a generalization of the corresponding solution for the sphere, and are found, in the first instance, for an ellipsoid whose major axis is the axis of revolution, but a slight alteration will render them applicable also to a planetary ellipsoid. We may effect the transformation to ellipsoidal co-ordinates of the general equation of conduction as follows: the semi-axes of the ellipsoid and hyperboloid through any point confocal to the surface being denoted by $c \cosh \alpha$, $c \sinh \alpha$, and $c \sin \beta$, and $c \sin \beta$, the co-ordinates of the point may be expressed by $c \cos \phi \sinh \alpha \sin \beta$, $c \sin \phi \sinh \alpha \sin \beta$, $c \cosh \alpha \cos \beta$, and the general equation of conduction becomes

$$\frac{d^2 \psi}{d\alpha^2} + \frac{d^2 \psi}{d\beta^2} + \cosh \alpha \frac{d\psi}{d\alpha} + \cot \beta \frac{d\psi}{d\alpha} + \left( \frac{1}{\sin \beta} + \frac{1}{\sinh \alpha \sin \beta} \right) \frac{d^2 \psi}{d\alpha^2} = \frac{c^2}{\kappa} (\cosh \alpha - \cos \beta) \frac{d\psi}{dt},$$

$\psi$ being the temperature at any point.

We may satisfy this equation by

$$\psi = (c \cos \phi) \omega \sin \phi \exp \left( \psi \right) \Omega^2 \left( 2 \right) \Omega^2 \left( 3 \right),$$

where

$$\frac{d^2 \phi}{d\phi^2} + \cot \beta \frac{d\phi}{d\alpha} - \cosh \alpha = \lambda^2 \sin \beta \sin \phi \exp \left( \psi \right),$$

$$\frac{d^2 \phi}{d\phi^2} + \cosh \alpha \frac{d\phi}{d\alpha} \omega \sin \phi \exp \left( \psi \right) = - \sqrt{2} \cosh \alpha \sin \phi \exp \left( \psi \right),$$

and

$$\psi = \omega \sin \phi \exp \left( \psi \right) \Omega^2 \left( 2 \right) \Omega^2 \left( 3 \right).$$
May 29, 1879.

THE PRESIDENT in the Chair.

The Present papers were read on the table, and thanks ordered for them.

The following Papers were read—

I. "On the Conduction of Heat in Ellipsoids of Revolution." By J. W. L. CLAIRMUR, F.R.S. Received May 7, 1879.

(Extract.)

The object of the present paper is to investigate the expressions which present themselves in the mathematical treatment of the problem of the conduction of heat in an ellipsoid of revolution. The results obtained constitute a generalization of the corresponding solution for the sphere, and are found, in the first instance, for an ellipsoid whose major axis is the axis of revolution, but a slight alteration will render them applicable also to a planar ellipsoid. We may effect the transformation to ellipsoidal co-ordinates of the general equation of conduction as follows: the semi-axes of the ellipsoid are hyperboloids through any point concentric to the surface being denoted by $c \cosh \alpha$, $c \sinh \alpha$, and $c \sin \beta$, the co-ordinates of the point may be expressed by $c \cosh \alpha \sinh \beta$, $c \sinh \alpha \sin \beta$, and the general equation of conduction becomes

$$\frac{d^2V}{d\alpha^2} + \frac{d^2V}{d\beta^2} + \cosh \alpha \frac{dV}{d\alpha} + \cot \beta \frac{dV}{d\beta} + \left( \frac{1}{\sinh^2 \alpha} + \frac{1}{\sin \beta} \right) \frac{d^2V}{d\beta^2} = \frac{c}{E} (c \cosh^2 \alpha - \cos^2 \beta) \frac{dV}{d\beta},$$

$V$ being the temperature at any point.

We may satisfy this equation by

$$V = (\cos m \phi + \sin m \phi) e^{-\gamma \sqrt{\lambda} (\beta) O^2 (\lambda)},$$

where

$$\frac{d^2 \phi}{d\beta^2} + \cot \beta \frac{d\phi}{d\beta} - \frac{\omega^2 \phi}{\sin \beta} = \lambda^2 \phi \cos \beta \phi - \omega^2 \phi,$$

$$\frac{d^2 \Omega}{d\alpha^2} + \coth \alpha \frac{d\Omega}{d\alpha} - \frac{\omega^2 \Omega}{\sinh^2 \alpha} = -\lambda^2 \cosh^2 \alpha \Omega - \omega^2 \Omega.$$
THE PRESIDENT in the Chair.

The Present was received were laid on the table, and thanks ordered for them.

The following Papers were read:


Abstract.

The object of the present paper is to investigate the expressions which present themselves in the mathematical treatment of the problem of the conduction of heat in an ellipsoid of revolution. The results obtained constitute a generalization of the corresponding solution for the sphere, and are found, in the first instance, for an ellipsoid whose major axis is the axis of revolution, but a slight alteration will render them applicable also to a planetary ellipsoid. We may effect the transformation to ellipsoidal co-ordinates of the general equation of conduction as follows: the semi-axes of the ellipsoid and hyperboloid through any point co-sestial to the surface being denoted by $c, b, c \sin \alpha$, and $c \cos \beta$, $c \sin \beta$, the co-ordinates of the point may be expressed by $c \cos \alpha \sin \beta \sin \gamma$, $c \sin \alpha \sin \beta \sin \gamma$, $c \cos \alpha \cos \beta$. The general equation of conduction becomes

$$\frac{d^2V}{ds^2} + \frac{d^2V}{dp^2} + \cosh \alpha \frac{d^2V}{d\phi^2} \cos \beta \frac{dV}{d\phi} + \frac{1}{\sin \beta} \frac{1}{\sin \gamma} \frac{d^2V}{dq^2} = 0.$$
THE PRESIDENT in the Chair.

The Papers received were laid on the table, and thanks ordered for them.

The following Papers were read:—


(Abstract.)

The object of the present paper is to investigate the expressions which present themselves in the mathematical treatment of the problem of the conduction of heat in an ellipsoid of revolution. The results obtained constitute a generalization of the corresponding solution for the sphere, and are found, in the first instance, for an ellipsoid whose major axis is the axis of revolution, but a slight alteration will render them applicable also to a planetary ellipsoid. We may effect the transformation to ellipsoidal co-ordinates of the general equation of conduction as follows: the semi-axes of the ellipsoid and hyperboloid through any point conformal to the surface being denoted by \( a, b, c \) and \( a, b, c \) and \( a, b, c \) and \( a, b, c \) and \( \alpha, \beta, \gamma \), the co-ordinates of the point may be expressed by \( \cos \alpha \sin \beta \), \( \sin \beta \), \( \cos \beta \), \( \cos \alpha \cos \beta \), \( \cos \alpha \sin \beta \), \( \cos \beta \), \( \cos \alpha \cos \beta \), and the general equation of conduction becomes

\[
\frac{\partial^2 \Phi}{\partial a^2} + \frac{\partial^2 \Phi}{\partial b^2} + \frac{\partial^2 \Phi}{\partial c^2} + \cot \alpha \frac{\partial \Phi}{\partial \alpha} + \cot \beta \frac{\partial \Phi}{\partial \beta} + \left( \frac{1}{\sin^2 \beta} + \frac{1}{\sinh^2 \gamma} \right) \frac{\partial^2 \Phi}{\partial \gamma^2}
\]
The Oscillation of Functions of an Orthogonal Set.*

By O. D. Kellogg.

1. Introductory.

The sets of orthogonal functions which occur in mathematical physics have, in general, the property that each changes sign in the interior of the interval on which they are orthogonal once more than its predecessor. So universal is this property that such sets are frequently referred to as sets of “oscillating functions.” The question arises, is this property of oscillation inherent in that of orthogonality? That it is not, is evidenced by a simple example. If the first function does not vanish, it is clear that the second must change signs, but the example shows that it does not follow that the third must change signs twice. Thus let \( \phi_0(x) = 1 \), while \( \phi_1(x) \) is defined on the interval \((0,1)\) as follows:

\[
\begin{align*}
\text{For } 0 \leq x < \frac{1}{3}, & \quad \phi_1(x) = 27x-8, \\
\frac{1}{3} \leq x < \frac{1}{2}, & \quad \phi_1(x) = 18x-5, \\
\frac{1}{2} \leq x < \frac{2}{3}, & \quad \phi_1(x) = -18x+13, \\
\frac{2}{3} \leq x \leq 1, & \quad \phi_1(x) = 1.
\end{align*}
\]

Let \( \phi_2(x) = \phi_1(1-x) \). Then the three functions are orthogonal on the interval \((0,1)\), while the second two change signs but once each.

If \( \phi_2(x) \) changes signs but once, say at \( x = a \), the function \( \phi_0(a)\phi_1(x) - \phi_1(a)\phi_0(x) \), which is orthogonal to \( \phi_2(x) \), can not have \( x = a \), where it vanishes, as the only point where it changes signs, since two orthogonal functions must be of the same sign in part of the interval and of opposite signs in part. Hence, if we make the supposition \( \phi_0(x_0)\phi_1(x_1) - \phi_0(x_1)\phi_1(x_0) > 0 \) for \( 0 < x_0 < x_1 < 1, \phi_0(x) \) must change signs twice in the interior of the interval.

These considerations suggest the determinant condition of the next paragraph. It is interesting to note that it is essentially the condition that a function \( c_0\phi_0(x) + c_1\phi_1(x) + \ldots + c_n\phi_n(x) \) can be found which will coincide with a given function \( f(x) \) at any \( n+1 \) interior points of the interval, the difference being merely the substitution of a definite sign for the being different from

* Read before the American Mathematical Society, November 29, 1913.
The Oscillation of Functions of an Orthogonal Set.*

By O. D. Kellogg.

1. Introductory.

The sets of orthogonal functions which occur in mathematical physics have, in general, the property that each changes sign in the interior of the interval on which they are orthogonal once more than its predecessor. So universal is this property that such sets are frequently referred to as sets of "oscillating functions." The question arises, is this property of oscillation inherent in that of orthogonality? That it is not, is evidenced by a simple example. If the first function does not vanish, it is clear that the second must change signs, but the example shows that it does not follow that the third must change signs twice. Thus let \( \varphi_0(x) = 1 \), while \( \varphi_1(x) \) is defined on the interval \((0, 1)\) as follows:

For \( 0 \leq x \leq \frac{1}{3} \), \( \varphi_1(x) = 27x - 8 \),

\( \frac{1}{3} \leq x \leq \frac{1}{2} \), \( \varphi_1(x) = 18x - 5 \),

\( \frac{1}{2} \leq x \leq \frac{2}{3} \), \( \varphi_1(x) = -18x + 13 \),

\( \frac{2}{3} \leq x \leq 1 \), \( \varphi_1(x) = 1 \).

Let \( \varphi_2(x) = \varphi_1(1 - x) \). Then the three functions are orthogonal on the interval \((0, 1)\), while the second two change signs but once each.

If \( \varphi_2(x) \) changes signs but once, say at \( x = a \), the function \( \varphi_0(a) \varphi_1(a) - \varphi_1(a) \varphi_0(a) \), which is orthogonal to \( \varphi_2(x) \), cannot have \( x = a \), where it vanishes, as the only point where it changes signs, since two orthogonal functions must be of the same sign in part of the interval and of opposite signs in part. Hence, if we make the supposition \( \varphi_0(x_0) \varphi_1(x_1) - \varphi_0(x_1) \varphi_1(x_0) > 0 \) for \( 0 < x_0 < x_1 < 1 \), \( \varphi_0(x) \) must change signs twice in the interior of the interval.

These considerations suggest the determinant condition of the next paragraph. It is interesting to note that it is essentially the condition that a function \( c_0 \varphi_0(x) + c_1 \varphi_1(x) + \ldots + c_n \varphi_n(x) \) can be found which will coincide with a given function \( f(x) \) at any \( n+1 \) interior points of the interval, the difference being merely the substitution of a definite sign for the being different from

* Read before the American Mathematical Society, November 29, 1913.
Eigenvalues of $PQ^T$: $\lambda_0 > \lambda_1 > \ldots$
Eigenvalue properties I

Eigenvalues of $PQ_T$: $\lambda_0 > \lambda_1 > \ldots$

$$\lambda_{\text{max}} = \lambda_0 = \|PQ_T\| = \sup_{f \in \mathcal{PW}, \|f\|=1} \|Q_T(f)\|^2$$
Eigenvalues of $PQ_T$: $\lambda_0 > \lambda_1 > \ldots$

$$\lambda_{\text{max}} = \lambda_0 = \|PQ_T\| = \sup_{f \in \mathcal{PW}, \|f\|=1} \|Q_T(f)\|^2$$

Uncertainty principle: $\lambda_{\text{max}} < 1$
For any $T > 0$ and $0 < \alpha < 1$ the number $N(\alpha)$ of eigenvalues of $PQ_T$ greater than $\alpha$ satisfies
Theorem

For any $T > 0$ and $0 < \alpha < 1$ the number $N(\alpha)$ of eigenvalues of $PQ_T$ greater than $\alpha$ satisfies

$$N(\alpha) = 2T$$
Theorem

For any $T > 0$ and $0 < \alpha < 1$ the number $N(\alpha)$ of eigenvalues of $PQ_T$ greater than $\alpha$ satisfies

$$N(\alpha) = 2T + \left( \frac{1}{\pi^2} \log \frac{1 - \alpha}{\alpha} \right) \log(T)$$
Theorem

For any $T > 0$ and $0 < \alpha < 1$ the number $N(\alpha)$ of eigenvalues of $PQ_T$ greater than $\alpha$ satisfies

$$N(\alpha) = 2T + \left( \frac{1}{\pi^2} \log \frac{1 - \alpha}{\alpha} \right) \log(T) + O(\log T).$$
Figure: *Eigenvalues for one frequency interval.* $N = 1025$ point centered DFT. $S \sim 513 + [-128, 128]$; $\Sigma \sim 513 + [-128, 128]$. $c = #T \times #\Sigma / N \approx 64$. Plunge region $\sim 61 \leq n \leq 69$. 
WHAT INDUSTRY WANTS A MATHEMATICIAN TO KNOW
AND HOW WE WANT THEM TO KNOW IT

Mathematics educators have become increasingly interested in recent years in the usefulness of the subject. This interest has taken many forms, such as, for example, the preparation of source books of applied problems, emphasis on applications at international congresses on mathematics education, Oxford seminars on industrial problems, internships in industry for students and faculty, and a desire to understand what mathematical scientists outside of education actually do and how they should be prepared for it. This paper contains the views of one industrial mathematician on this last topic.

The first question we need to examine is what mathematicians actually do in industry. The obvious answer, that is to solve specific mathematical problems bothering other people, is only part of the truth, in fact a relatively small part. One of the mathematician’s activities is indeed to solve mathematical problems precisely formulated in mathematical terms. For example, someone may want to know the length of the shortest tree connecting $n$ points in a unit square, either the expected length if these points are placed at random or the maximum possible length no matter how they are placed. Someone else may give you a 3-term recursion relation for a family of polynomials and ask you for the measure with respect to which these polynomials are orthogonal. Or they might just ask you to sum the following series. Is it wise to plunge in and immediately try to solve the problem? Maybe. The mathematician learns after a number of experiences to question why the visitor is interested in the particular problem. It turns out that it is by no means true that other people always ask the right mathematical question. This does not imply malevolence on their part. They have, after all, tried to solve the problem, but their mathematical knowledge may not have been sufficient to lead them to the best formulation.

Besides mathematical problems precisely formulated in mathematical terms, the mathematician will often be asked to investigate precisely formulated situations in some other field. What purely resistive circuits can you make by printed circuit techniques? Why does a strand of spaghetti slump sufficiently rapidly come up and hit your nose? Which parts of this composition were probably written by Mozart, and which by some graduate student? What is the best way to lay out a running track? What is the best strategy in playing blackjack against the house rules in Las Vegas, and can you win in the long run? These questions are all well formulated in the field of application, but the process of
WHAT INDUSTRY WANTS A MATHEMATICIAN TO KNOW
AND HOW WE WANT THEM TO KNOW IT

Mathematics educators have become increasingly interested in recent years in the usefulness of the subject. This interest has taken many forms, such as, for example, the preparation of source books of applied problems, emphasis on applications at international congresses on mathematics education, Oxford seminars on industrial problems, internships in industry for students and faculty, and a desire to understand what mathematical scientists outside of education actually do and how they should be prepared for it. This paper contains the views of one industrial mathematician on this last topic.

The first question we need to examine is what mathematicians actually do in industry. The obvious answer, that is to solve specific mathematical problems bothering other people, is only part of the truth, in fact a relatively small part. One of the mathematician's activities is indeed to solve mathematical problems precisely formulated in mathematical terms. For example, someone may want to know the length of the shortest tree connecting n points in a unit square, either the expected length if these points are placed at random or the maximum possible length no matter how they are placed. Someone else may give you a 3-term recursion relation for a family of polynomials and ask you for the measure with respect to which these polynomials are orthogonal. Or they might just ask you to sum the following series. Is it wise to plunge in and immediately try to solve the problem? Maybe. The mathematician learns after a number of experiences to question why the visitor is interested in the particular problem. It turns out that it is by no means true that other people always ask the right mathematical question. This does not imply malevolence on their part. They have, after all, tried to solve the problem, but their mathematical knowledge may not have been sufficient to lead them to the best formulation.

Besides mathematical problems precisely formulated in mathematical terms, the mathematician will often be asked to investigate precisely formulated situations in some other field. What purely resistive circuits can you make by printed circuit techniques? Why does a strand of spaghetti slump sufficiently rapidly come up and hit your nose? Which parts of this composition were probably written by Mozart, and which by some graduate student? What is the best way to lay out a running track? What is the best strategy in playing black-jack against the house rules in Las Vegas, and can you win in the long run? These questions are all well formulated in the field of application, but the process of
Mathematics educators have become increasingly interested in recent years in the usefulness of the subject. This interest has taken many forms, such as, for example, the preparation of source books of applied problems, emphasis on applications at international congresses on mathematics education, Oxford seminars on industrial problems, internships in industry for students and faculty, and a desire to understand what mathematical scientists outside of education actually do and how they should be prepared for it. This paper contains the views of one industrial mathematician on this last topic.

The first question we need to examine is what mathematicians actually do in industry. The obvious answer, that is to solve specific mathematical problems bothering other people, is only part of the truth, in fact a relatively small part. One of the mathematician's activities is indeed to solve mathematical problems precisely formulated in mathematical terms. For example, someone may want to know the length of the shortest tree connecting $n$ points in a unit square, either the expected length if these points are placed at random or the maximum possible length no matter how they are placed. Someone else may give you a 3-term recursion relation for a family of polynomials and ask you for the measure with respect to which these polynomials are orthogonal. Or they might just ask you to sum the following series. Is it wise to plunge in and immediately try to solve the problem? Maybe. The mathematician learns after a number of ex-
WHAT INDUSTRY WANTS A MATHEMATICIAN TO KNOW
AND HOW WE WANT THEM TO KNOW IT

Mathematics educators have become increasingly interested in recent years in the usefulness of the subject. This interest has taken many forms, such as, for example, the preparation of source books of applied problems, emphasis on applications at international congresses on mathematics education, Oxford seminars on industrial problems, internships in industry for students and faculty, and a desire to understand what mathematical scientists outside of education actually do and how they should be prepared for it. This paper contains the views of one industrial mathematician on this last topic.

The first question we need to examine is what mathematicians actually do in industry. The obvious answer, that is to solve specific mathematical problems bothering other people, is only part of the truth, in fact a relatively small part. One of the mathematician's activities is indeed to solve mathematical problems precisely formulated in mathematical terms. For example, someone may want to know the length of the shortest tree connecting $n$ points in a unit square, either the expected length if these points are placed at random or the maximum possible length no matter how they are placed. Someone else may give you a 3-term recursion relation for a family of polynomials and ask you for the measure with respect to which these polynomials are orthogonal. Or they might just ask you to sum the following series. Is it wise to plunge in and immediately try to solve the problem? Maybe. The mathematician learns after a number of ex-
Energy concentration:
\[ \phi_0 = \arg \max \left\{ \frac{\| Q^T f \|}{\| f \|} : f \in \mathcal{P}_W \right\} \]

Fourier covariance:
\[ \hat{\phi}_n (\xi/T) = \pm in \sqrt{T/\lambda_n} Q^T \phi_n (\xi) \]

Double orthogonality:
\[ \langle Q^T \phi_n, Q^T \phi_m \rangle = \langle \phi_n, P_{\Omega} Q^T \phi_m \rangle = \lambda_n \langle \phi_n, \phi_m \rangle = \lambda_n \delta_{nm} \]

\( \{ \phi_n \} \) forms a complete family in \( L^2 [-T, T] \).

Dilation:
\[ P_{\Omega} Q^T = D_{1/\Omega} (P_{\Omega} Q^T) D_{\Omega} \]
Landau, Slepian and Pollak, “Bell Labs” papers (1960s)

Energy concentration:

\[ \varphi_0 = \arg \max \left\{ \frac{\| Q_T f \|}{\| f \|} : f \in PW \right\} \]
Landau, Slepian and Pollak, “Bell Labs” papers (1960s)

Energy concentration:

$$\varphi_0 = \arg \max \{ \| Q_T f \| / \| f \| : f \in \text{PW} \}$$

Fourier covariance:

$$\hat{\varphi}_n(\xi / T) = \pm i^n \sqrt{T / \lambda_n} Q_T \varphi_n(\xi).$$
Eigenfunction properties II

Landau, Slepian and Pollak, “Bell Labs” papers (1960s)

Energy concentration:

\[ \varphi_0 = \text{arg max} \{ \| Q_T f \| / \| f \| : f \in PW \} \]

Fourier covariance:

\[ \hat{\varphi}_n(\xi / T) = \pm i^n \sqrt{T / \lambda_n} Q_T \varphi_n(\xi). \]

Double orthogonality

\[ \langle Q_T \varphi_n, Q_T \varphi_m \rangle = \langle \varphi_n, P_\Omega Q_T \varphi_m \rangle = \lambda_n \langle \varphi_n, \varphi_m \rangle = \lambda_n \delta_{nm}. \]
Eigenfunction properties II

Landau, Slepian and Pollak, “Bell Labs” papers (1960s)

Energy concentration:
\[ \varphi_0 = \arg \max \{ \| Q_T f \| / \| f \| : f \in PW \} \]

Fourier covariance:
\[ \hat{\varphi}_n(\xi/T) = \pm i^n \sqrt{T/\lambda_n} Q_T \varphi_n(\xi). \]

Double orthogonality
\[ \langle Q_T \varphi_n, Q_T \varphi_m \rangle = \langle \varphi_n, P_\Omega Q_T \varphi_m \rangle = \lambda_n \langle \varphi_n, \varphi_m \rangle = \lambda_n \delta_{nm}. \]

\{ \varphi_n \} forms a complete family in \( L^2[-T, T] \).
Landau, Slepian and Pollak, “Bell Labs” papers (1960s)

Energy concentration:
\[ \varphi_0 = \text{arg max}\{\|Q_T f\|/\|f\| : f \in \text{PW}\} \]

Fourier covariance:
\[ \hat{\varphi}_n(\xi/T) = \pm i^n \sqrt{T/\lambda_n} Q_T \varphi_n(\xi). \]

Double orthogonality
\[ \langle Q_T \varphi_n, Q_T \varphi_m \rangle = \langle \varphi_n, P_\Omega Q_T \varphi_m \rangle = \lambda_n \langle \varphi_n, \varphi_m \rangle = \lambda_n \delta_{nm}. \]

\{\varphi_n\} forms a complete family in \( L^2[-T, T] \).

Dilation \( P Q_\Omega T = D_{1/\Omega}(P_\Omega Q_T)D_\Omega \)
Slepian (1978): DPSS sequences
Slepian (1978): DPSS sequences
Thomson (1982): Multitaper method
Slepian (1978): DPSS sequences
Thomson (1982): Multitaper method
Applications and extensions I

Slepian (1978): DPSS sequences
Thomson (1982): Multitaper method
Wireless communications . . . since 2000
Why the 20 year latency?
Why the 20 year latency?
PART II: Sampling and Time-Frequency localization
Problem

Can one approximately recover the projection of $f \in \mathcal{PW}$ onto the space of “approximately time- and bandlimited” functions solely from samples near $[-T, T]$?
Theorem (Walter and Shen; Khare and George, 2003-2004)
Sampling and eigenfunctions: rectangle case

Theorem (Walter and Shen; Khare and George, 2003-2004)

If \[ A_{mk} = \int_{-T}^{T} \text{sinc} (t - m) \text{sinc} (t - k) \, dt. \]

\[ \lambda_n \phi_n (m) = \sum_k A_{mk} \phi_n (k) \]

i.e. samples of prolates are eigenvectors of \{A_{mk}\}.
Theorem (Walter and Shen; Khare and George, 2003-2004)

If
\[ A_{mk} = \int_{-T}^{T} \text{sinc}(t - m) \text{sinc}(t - k) \, dt. \]

then
\[ \lambda_n \varphi_n(m) = \sum_k A_{mk} \varphi_n(k) \]
Theorem (Walter and Shen; Khare and George, 2003-2004)

If \( A_{mk} = \int_{-T}^{T} \text{sinc}(t - m) \text{sinc}(t - k) \, dt \),

then \( \lambda_n \varphi_n(m) = \sum_k A_{mk} \varphi_n(k) \)

i.e. **samples of prolates are eigenvectors of** \( \{A_{mk}\} \).
Corollary (Second Shen-Walter formula)

If \( f \in PW \) then

\[
(PQ_T)(f)(t) = \sum_{n=0}^{\infty} \lambda_n
\]
Corollary (Second Shen-Walter formula)

If \( f \in PW \) then

\[
(PQ_T)(f)(t) = \sum_{n=0}^{\infty} \lambda_n \sum_{k=-\infty}^{\infty} f(k) \varphi_n(k) \varphi_n(t).
\]
Application: approximate time-frequency localization

Problem

For suitable $N(T) \approx T$ and $M(T) \approx T$, justify

$\text{span}\{\varphi_0, \ldots, \varphi_{N(T)}\} \approx \text{span}\{\text{sinc}(t-k) : |k| \leq M(T)\}$

when restricted to $[-T, T]$. 
Quadratic decay for PSWF samples

Shen–Walter: quadratic decay estimate,

\[ \sum_{|k| > T} (\varphi_n(k))^2 \leq C T \sqrt{1 - \lambda_n} \]
Shen–Walter: quadratic decay estimate,

$$\sum_{|k| > T} (\varphi_n(k))^2 \leq CT \sqrt{1 - \lambda_n}$$

The decay ought to be “linear” in $\lambda_n$ if

$$|k| > M(n, T) \approx T$$
\[ \varphi = Q_T \varphi + \tilde{Q}_T \varphi = \psi \ast (Q_T \varphi) + \psi \ast (\tilde{Q}_T \varphi) \]

\( \psi \): Fourier bump;
\[ \varphi = Q_T \varphi + \tilde{Q}_T \varphi = \psi \ast (Q_T \varphi) + \psi \ast (\tilde{Q}_T \varphi) \]

\[ \psi: \text{Fourier bump; Hardy-Littlewood implies . . .} \]

**Lemma**

\[ \sum (\psi \ast (\tilde{Q}_T \varphi_n)(k))^2 \leq C (1 - \lambda_n). \]
Lemma

ϕₙ: n-th eigenfunction of PQ
ψ: Fourier bump; Ψ: majorant

\[ \int_{|t| > M(n, T) - T - 1/2} \Psi(t) \, dt \leq \sqrt{1 - \lambda_n} \lambda_n (\ast) \]

Then

\[ \sum_{|k| > M(n, T)} (\langle Q T \phi_n, \psi^* \rangle(k))^2 \leq (1 - \lambda_n) \]

(\ast) if \n
\[ M(n, T) = (\pi/2 (T - n/2) + T)(1 + \log \gamma T) \]
Lemma
\[ \varphi_n: \text{n-th eigenfunction of } P Q_T \]
Lemma

\( \varphi_n: n\)-th eigenfunction of \( PQ_T \)

\( \psi: Fourier \) bump ; \( \Psi: \) majorant

Then

\[
\sum_{|k| > M(n,T)} \left| (\psi^* (Q_T \varphi_n))(k) \right|^2 \leq (1 - \lambda_n) \lambda_n.
\]

\((\ast)\) if \( M(n,T) = \left( \frac{\pi}{2} (T - n/2) + T \right)(1 + \log \gamma T) \).
Lemma

\( \varphi_n: n\text{-th eigenfunction of } PQ_T \)

\( \psi: Fourier \text{ bump} ; \Psi: \text{majorant} \)

\[ M(n, T) : \int_{|t| > M(n, T) - T - 1/2} \psi(t) \, dt \leq \sqrt{\frac{1 - \lambda_n}{\lambda_n}} \quad (\ast) \]
Lemma

\( \varphi_n : \) \emph{n-th eigenfunction of} \( PQ_T \)

\( \psi : \) \emph{Fourier bump} ; \( \Psi : \) \emph{majorant}

\[ M(n, T) : \int_{|t|>M(n, T) - T - 1/2} \Psi(t) \, dt \leq \sqrt{\frac{1 - \lambda_n}{\lambda_n}} \quad (\ast) \]

Then

\[ \sum_{|k|>M(n, T)} (((\psi \ast (Q_T \varphi_n))(k))^2 \leq (1 - \lambda_n). \]
Lemma

\( \varphi_n: n\text{-th eigenfunction of } PQ_T \)

\( \psi: Fourier bump \); \( \Psi: majorant \)

\[ M(n, T) : \int_{|t| > M(n, T) - T - 1/2} \Psi(t) \, dt \leq \sqrt{\frac{1 - \lambda_n}{\lambda_n}} \quad (\star) \]

Then

\[ \sum_{|k| > M(n, T)} \left( (\psi * (Q_T \varphi_n))(k) \right)^2 \leq (1 - \lambda_n). \]

\((\star)\) if \( M(n, T) = (\pi^2(T - n/2) + T)(1 + \log^\gamma T) \).
Corollary

Let $\varphi_n$ be the $n$-th eigenfunction of $PQ_T$. Then there is a $C > 0$ independent of $n$ such that

$$\sum_{|k| > M(n,T)} \varphi_n^2(k) \leq C(1 - \lambda_n).$$
Approximate time-localized projections: the solution

For $f \in \mathcal{PW}$ ...

$$f_N(t) = \sum_{n=0}^{N} \left( \sum_{k} f(k) \varphi_n(k) \right) \varphi_n(t)$$
Approximate time-localized projections: the solution

For $f \in \text{PW}$ . . .

$$f_N(t) = \sum_{n=0}^{N} \left( \sum_{k} f(k) \varphi_n(k) \right) \varphi_n(t)$$

$$f_{N,T} = \sum_{n=0}^{N} \left( \sum_{|k| \leq M(T)} f(k) \varphi_n(k) \right) \varphi_n(t)$$
Approximate time-localized projections: the solution

For $f \in PW$ ...

$$f_N(t) = \sum_{n=0}^{N} (\sum_{k} f(k) \varphi_n(k)) \varphi_n(t)$$

$$f_{N,T} = \sum_{n=0}^{N} (\sum_{|k| \leq M(T)} f(k) \varphi_n(k)) \varphi_n(t)$$

Proposition

$$\|Q_T(f_N - f_{N,T})\|^2 \leq C\|f\|^2 \sum_{n=0}^{N} \lambda_n(1 - \lambda_n).$$
Accurate estimates of integer samples of PSWFs

Want: integer samples of $\psi_n = \psi_n(k/T)$: dilate of $\psi_n$.

Use Legendre expansion of $\hat{\psi}_n$:

$$\sum_{\ell \geq 0} \beta_n^{\ell} \int_{-1}^{1} e^{2\pi i k \xi} P_{\ell}(\xi) \, d\xi = \mu_n \psi_n(k/T).$$

$P_{\ell}$: $\ell$-th Legendre polynomial, $L_{2\ell}[−1,1]$-norm one.
Accurate estimates of integer samples of PSWFs

Want: integer samples of \( \varphi_n = \psi_n(\cdot/T) \): dilate of \( \varphi_n \)
Accurate estimates of integer samples of PSWFs

Want: integer samples of $\varphi_n = \psi_n(\cdot/T)$: dilate of $\varphi_n$.

$\psi_n(k/T)$: Fourier coefficient of $\hat{\psi}_n$ on $[-1, 1]$.
Accurate estimates of integer samples of PSWFs

Want: integer samples of $\varphi_n = \psi_n(\cdot / T)$: dilate of $\varphi_n$

$\psi_n(k / T)$: Fourier coefficient of $\hat{\psi}_n$ on $[-1, 1]$

Use Legendre expansion of $\hat{\psi}_n$:

$$
\sum_{\ell \geq 0: \ell = n \mod 2} \beta^n_\ell \int_{-1}^{1} e^{2\pi i k \xi} P_\ell(\xi) d\xi = \mu_n \psi_n \left( \frac{k}{T} \right).
$$

$P_\ell$: $\ell$-th Legendre polynomial, $L^2[-1, 1]$-norm one.
Want: integer samples of $\varphi_n = \psi_n(\cdot/T)$: dilate of $\varphi_n$
$\psi_n(k/T)$: Fourier coefficient of $\hat{\psi}_n$ on $[-1, 1]$
Use Legendre expansion of $\hat{\psi}_n$:

$$\sum_{\ell \geq 0: \ell = n \mod 2} \beta^{n}_{\ell} \int_{-1}^{1} e^{2\pi ik\xi} P_{\ell}(\xi) d\xi = \mu_n \psi_n\left(\frac{k}{T}\right).$$

$P_{\ell}$: $\ell$-th Legendre polynomial, $L^2[-1, 1]$-norm one.
Legendre Fourier coefficients

$I_k, l = \cdots$

$I_k, \nu = \begin{cases} 
(\frac{-i}{2^{\nu}})^{\nu} \frac{J_{\nu+1}}{2} & \text{if } k \neq 0 \\
\delta_{0, \nu} & \text{if } k = 0
\end{cases}

\frac{\pi}{k}

Calculate using $besselj$. 
Legendre Fourier coefficients $I_{k,l} = \cdots$
Legendre Fourier coefficients $l_{k,l} = \cdots$

$$l_{k,\nu} = \begin{cases} 
(-i)^\nu \sqrt{\frac{2\nu+1}{2k}} \ J_{\nu+1/2}(\pi k), & k \neq 0 \\
\delta_{0,\nu}, & k = 0.
\end{cases}$$
Legendre Fourier coefficients $I_{k,l} = \cdots$

\[ I_{k,\nu} = \begin{cases} 
(-i)^\nu \sqrt{\frac{2\nu+1}{2k}} J_{\nu+1/2}(\pi k), & k \neq 0 \\
\delta_{0,\nu}, & k = 0.
\end{cases} \]

Calculate using \texttt{besselj}. 
Proposition (Karoui and Moumni, 2008)

For $k \neq 0$ and even $\ell_0 \geq 2([2Te] + 1)$,

$$\left| \psi_n \left( \frac{k}{T} \right) - \frac{1}{\mu_n} \sum_{\ell \leq \ell_0: \ell = n \mod 2} \beta^n_{\ell} l_{k,\ell} \right| \leq \frac{1}{2\ell_0} |\mu_n|^2.$$
Figure: Plots of $\varphi_{n}^{KM}$ (solid) and $\varphi_{n}^{loc}$ for $n = 0, 4, 8, 12, 16$ and $T = 5$. 
Figure: Plots of $\phi_{KM}^{15}$ (solid) and $\phi_{WS}^{15}$ (dashed) for $T = 5$. On the left, the time interval is $[-2T, 2T]$. On the right it is $[-T, T]$. 

Joe Lakey

Time- and bandlimiting
Figure: Plots of $\varphi_{15}^{\text{KM}}$ (solid) and $\varphi_{15}^{\text{WS}}$ (dashed) for $T = 5$. One the left, the time interval is $[-2T, 2T]$. On the right it is $[-T, T]$. 
Orthogonal eigenfunctions for symmetric bands
Orthogonal eigenfunctions for symmetric bands
Potential for "cognitive radio"
Orthogonal eigenfunctions for symmetric bands
Potential for “cognitive radio”
Potential eigenvectors of \( P_\Sigma Q_T \) as modulators
The “$\Sigma T$”-theorem, $N(\alpha)$, Multiple intervals

$S, \Sigma$: finite unions of intervals, $|S| = |\Sigma| = 1$. 
The “Σ T”-theorem, $N(\alpha)$, Multiple intervals

$S, \Sigma$: finite unions of intervals, $|S| = |\Sigma| = 1$. 
$A_a = P_{a\Sigma} Q_S P_{a\Sigma}$
The “$\Sigma T$”-theorem, $N(\alpha)$, Multiple intervals

$S$, $\Sigma$: finite unions of intervals, $|S| = |\Sigma| = 1$.

$A_a = P_{a\Sigma} Q_S P_{a\Sigma}$

$N(A_a, \alpha) = \# \{ \lambda(A_a) > \alpha \}$
The “$\Sigma T$”-theorem, $N(\alpha)$, Multiple intervals

$S, \Sigma$: finite unions of intervals, $|S| = |\Sigma| = 1$.

$A_a = P_{a\Sigma} Q_S P_{a\Sigma}$

$N(A_a, \alpha) = \#\{\lambda(A_a) > \alpha\}$

Landau and Widom (1980)

$N(A_a, \alpha) = a$
The “Σ T”-theorem, $N(\alpha)$, Multiple intervals

$S, \Sigma$: finite unions of intervals, $|S| = |\Sigma| = 1$.

$A_a = P_{a\Sigma}Q_SP_{a\Sigma}$

$N(A_a, \alpha) = \#\{\lambda(A_a) > \alpha\}$

Landau and Widom (1980)

$$N(A_a, \alpha) = a + \frac{N_SN_\Sigma}{\pi^2} \log\left(\frac{1 - \alpha}{\alpha}\right) \log a$$
The “Σ T”-theorem, $N(\alpha)$, Multiple intervals

$S, \Sigma$: finite unions of intervals, $|S| = |\Sigma| = 1$.

$A_a = P_a \Sigma Q_S P_a \Sigma$

$N(A_a, \alpha) = \#\{\lambda(A_a) > \alpha\}$

Landau and Widom (1980)

$$N(A_a, \alpha) = a + \frac{N_S N_\Sigma}{\pi^2} \log \left(\frac{1 - \alpha}{\alpha}\right) \log a + o(\log a)$$
The "Σ T"-theorem, \(N(\alpha)\), Multiple intervals

\(S, \Sigma\): finite unions of intervals, \(|S| = |\Sigma| = 1\).
\(A_a = P_{a\Sigma} Q_S P_{a\Sigma}\)
\(N(A_a, \alpha) = \#\{\lambda(A_a) > \alpha\}\)
Landau and Widom (1980)

\[N(A_a, \alpha) = a + \frac{N_S N_\Sigma}{\pi^2} \log \left(\frac{1 - \alpha}{\alpha}\right) \log a + o(\log a)\]

\(N_S N_\Sigma\): width of "plunge region"
Plunge width and $\sim N_S N_\Sigma$

“Separated at infinity”
“Separated at infinity”

$\phi_j$ frequency concentrated on $I_j$, $|I_j| = 1$
“Separated at infinity”

\( \phi_j \) frequency concentrated on \( l_j, |l_j| = 1 \)

\( \phi_j(t) = e^{2\pi i m_j t} \varphi_j(t) \) \( m_j = \overline{l_j}. \)
"Separated at infinity"

\( \phi_j \) frequency concentrated on \( l_j, \ |l_j| = 1 \)

\( \phi_j(t) = e^{2\pi im_j t} \varphi_j(t) \quad m_j = \bar{l}_j. \)

\( \varphi_j \) frequency concentrated on \([-1/2, 1/2]\).
“Separated at infinity”

$\phi_j$ frequency concentrated on $I_j$, $|I_j| = 1$

$\phi_j(t) = e^{2\pi i m_j t} \varphi_j(t)$ $m_j = \overline{I}_j$.

$\varphi_j$ frequency concentrated on $[-1/2, 1/2]$.

$$\langle Q\phi_j, Q\phi_k \rangle = \int_{-1/2}^{1/2} e^{2\pi i (m_j - m_k) t} \varphi_j(t) \varphi_k(t) \, dt$$

$$= \hat{\varphi}_j \ast \overline{\hat{\varphi}_k} \ast \text{sinc} (m_1 - m_2) = O(1/|m_j - m_k|)$$
Plunge width and $\sim N_S N_\Sigma$

"Separated at infinity"

$\phi_j$ frequency concentrated on $I_j$, $|I_j| = 1$

$\phi_j(t) = e^{2\pi i m_j t} \varphi_j(t) \quad m_j = \overline{I_j}$.

$\varphi_j$ frequency concentrated on $[-1/2, 1/2]$.

$$\langle Q\phi_j, Q\phi_k \rangle = \int_{-1/2}^{1/2} e^{2\pi i (m_j - m_k) t} \varphi_j(t) \overline{\varphi_k(t)} \, dt$$

$$= \hat{\varphi}_j \ast \hat{\varphi}_k \ast \text{sinc}(m_1 - m_2) = O(1/|m_j - m_k|)$$

Each $I_j$ gives one eigenvalue $\approx 1/2$
Plunge width and $\sim N_S N_\Sigma$

“Separated at infinity”

$\phi_j$ frequency concentrated on $I_j$, $|I_j| = 1$

$\phi_j(t) = e^{2\pi i m_j t} \varphi_j(t)$ $m_j = \bar{I}_j$

$\varphi_j$ frequency concentrated on $[-1/2, 1/2]$.

$$\langle Q\phi_j, Q\phi_k \rangle = \int_{-1/2}^{1/2} e^{2\pi i (m_j - m_k) t} \varphi_j(t) \overline{\varphi_k(t)} \, dt$$

$$= \hat{\varphi}_j * \hat{\varphi}_k * \text{sinc} (m_1 - m_2) = O(1/|m_j - m_k|)$$

Each $I_j$ gives one eigenvalue $\approx 1/2$

If each $I_j$ were very short: no large eigenvalues.
What about $N(P_{\Sigma}Q, 1/2)$?

Proposition (Landau (1993), Izu (2009))

Let $\Sigma = [-1/2, 1/2]$ and let $S$ be a union of $m$ pairwise disjoint intervals of total length $a$. Set

$$\nu = \max_{\alpha} \# \{ k \in \mathbb{Z} : (k, k + 1) \subset S + \alpha \},$$

$$\mu = \min_{\beta} \# \{ \ell \in \mathbb{Z} : (\ell, \ell + 1) \cap S + \beta \neq \emptyset \}.$$

Then the eigenvalues $\lambda_k$ of $QSP$ satisfy

$$\lambda_{\nu - 1} \geq 1/2 \geq \lambda_\mu.$$
Plunge width \( \sim N_{\Sigma} N_{\Sigma} \)
More discrete illustrations: random frequencies

Play/Pause Slow
Interval clumping
Corollary

When \( T = 1 \) and \( \Sigma \) is a union of integer intervals \([k, k + 1]\), \( \lambda_c = 1/2 \).

Conjecture

When \( \Sigma \) is a symmetric union of “grid intervals” of length \( 1/T \) (so \( c \in \mathbb{N} \)) one has \( \lambda_{c-k} + \lambda_{c+k} = 1 \), \( k = 1, \ldots, [c] \).
Figure: Σ: 10 symmetrized length 16 “grid intervals” $c = 40$ (real part)
Figure: Eigenvalues for DFT localization, \( N = 1024 \), \( T = 128 \), 10 symmetrized length 16 intervals \( c = 40 \) (real part), Note symmetry
Largest energy concentration for a given area?

- Donoho and Stark (1993): if $|\Sigma| = 1$ and $T \leq 0.8$ then . . .
Largest energy concentration for a given area?

- Donoho and Stark (1993): if $|\Sigma| = 1$ and $T \leq 0.8$ then ... 

- $\int_{-T/2}^{T/2} |f(t)|^2 dt \leq \int_{-T/2}^{T/2} |(\hat{f}^*)^\vee(t)|^2 dt$.

Optimal concentration: $\Sigma$ is an interval if $T$ is small enough.

Rearrangement inequality fails for large measure.
Donoho and Stark (1993): if $|\Sigma| = 1$ and $T \leq 0.8$ then ...

$$\int_{-T/2}^{T/2} |f(t)|^2 \, dt \leq \int_{-T/2}^{T/2} |(\hat{f}^*)(t)|^2 \, dt.$$ 

optimal concentration: $\Sigma$ is an interval if $T$ is small enough.
Largest energy concentration for a given area?

- Donoho and Stark (1993): if $|\Sigma| = 1$ and $T \leq 0.8$ then . . .

$$\int_{-T/2}^{T/2} |f(t)|^2 \, dt \leq \int_{-T/2}^{T/2} |(\hat{f}^\star)^\vee(t)|^2 \, dt.$$  

- Optimal concentration: $\Sigma$ is an interval if $T$ is small enough.

- Rearrangement inequality fails for large measure.
(S, Σ) supports information if
\[ \| P_{Σ} Q_{S} P_{Σ} \| \geq 1/2. \]
Information problem

- $(S, \Sigma)$ supports information if
  \[
  \|P_\Sigma Q_S P_\Sigma\| \geq \frac{1}{2}.
  \]
- ... at rate $N$: $N$ eigenvalues $\geq \frac{1}{2}$
Information problem

- \((S, \Sigma)\) supports information if 
  \[ \|P_\Sigma Q_S P_\Sigma\| \geq 1/2. \]
- ... at rate \(N\): \(N\) eigenvalues \(\geq 1/2\)
- Rationale: basis functions \(\sim\) codes
Information problem

- $(S, \Sigma)$ supports information if $\| P_\Sigma Q_S P_\Sigma \| \geq 1/2$.
- ... at rate $N$: $N$ eigenvalues $\geq 1/2$
- Rationale: basis functions $\sim$ codes
- Which pairs support information?
Theorem (Candès, Romberg, Tao)

Fix $N \geq 512$ and $\beta$ such that $1 \leq \beta \leq (3/8) \log N$.
Probabilistic bounds: finite case

Theorem (Candès, Romberg, Tao)

Fix $N \geq 512$ and $\beta$ such that $1 \leq \beta \leq (3/8) \log N$. Suppose that $S$ and $\Sigma$ are subsets of $\mathbb{Z}_N$, $S$ fixed and $\Sigma$ uniformly randomly generated subject to the constraint

$$|S| + |\Sigma| \leq \frac{N}{\sqrt{(\beta + 1) \log N}} \left( \frac{1}{\sqrt{6}} + o(1) \right).$$

Joe Lakey

Time- and bandlimiting
Theorem (Candès, Romberg, Tao)

Fix $N \geq 512$ and $\beta$ such that $1 \leq \beta \leq (3/8) \log N$. Suppose that $S$ and $\Sigma$ are subsets of $\mathbb{Z}_N$, $S$ fixed and $\Sigma$ uniformly randomly generated subject to the constraint

$$|S| + |\Sigma| \leq \frac{N}{\sqrt{\beta + 1} \log N} \left( \frac{1}{\sqrt{6}} + o(1) \right).$$

Then with probability at least $1 - O((\log N)^{1/2} / N^\beta)$, every signal $x$ frequency-supported in $\Sigma$ satisfies

$$\| x 1_S \|^2 \leq \frac{1}{2} \| x \|^2.$$
Theorem (Candès, Romberg, Tao)

Fix $N \geq 512$ and $\beta$ such that $1 \leq \beta \leq (3/8) \log N$. Suppose that $S$ and $\Sigma$ are subsets of $\mathbb{Z}_N$, $S$ fixed and $\Sigma$ uniformly randomly generated subject to the constraint

$$|S| + |\Sigma| \leq \frac{N}{\sqrt{\beta + 1} \log N} \left( \frac{1}{\sqrt{6}} + o(1) \right).$$

Then with probability at least $1 - O((\log N)^{1/2}/N^\beta)$, every signal $x$ frequency-supported in $\Sigma$ satisfies

$$\|x 1_S\|^2 \leq \frac{1}{2} \|x\|^2 \quad \text{or} \quad N(P_{\Sigma} Q_S, 1/2) = 0.$$
Figure: Norm of $P_\Sigma Q_T$ versus entropy, $N = 512$, $c = 8$
Problem (Open)

Describe projection onto localized eigenspaces of $P_\Sigma Q_T P_\Sigma$ in terms of samples of eigenvectors in the multiband case.
Among other things, described several formulas for interpolating approximate time-localized projections onto multiband Paley–Wiener spaces
THAT’S ALL FOLKS!