Recent progress on time and band limiting
SIAM Minisymposium, January 2013

Joe Lakey (w Jeff Hogan)\textsuperscript{1}

January 10, 2013
Overview of Issues

Approximation of time and band limited signals from their samples: which ones and how well?

Construction of time- and multi-band limited signals

Application to EEG

Joe Lakey (w Jeff Hogan)

Time- and bandlimiting
Approximation of time and band limited signals from their samples: which ones and how well?
Overview of Issues

**Approximation** of time and band limited signals from their samples: which ones and how well?  
**Construction** of time- and multi-band limited signals
Overview of Issues

Approximation of time and band limited signals from their samples: which ones and how well?
Construction of time- and multi-band limited signals
Application to EEG
More themes

Essentially, time- and band-limited signals can be approximated on \([-T, T]\) by sinc interpolants of samples on \([-MT, MT]\), where \(M \approx 10(1 + \log T)\).

Essentially, time-limited blue multiband signals can be constructed from each band.

Continuous eigenfunction problem can be reduced to discrete eigenvector problem.
Oversampling. *Essentially* time- and band-limited signals can be approximated on $[-T, T]$ by sinc interpolants of samples on $[-MT, MT]$, $M \approx 10(1 + \log T)$.
Oversampling *Essentially* time- and band-limited signals can be approximated on \([-T, T]\) by sinc interpolants of samples on \([-MT, MT]\), \(M \approx 10(1 + \log T)\). Essentially time-limited bluemultiband signals can be constructed from each band.
**Oversampling** *Essentially* time- and band-limited signals can be approximated on $[-T, T]$ by $sinc$ interpolants of samples on $[-MT, MT]$, $M \approx 10(1 + \log T)$.

Essentially time-limited bluemultiband signals can be constructed from each band.

Continuous eigenfunction problem can be reduced to discrete eigenvector problem.
Oversampling *Essentially* time- and band-limited signals can be approximated on \([-T, T]\) by \(sinc\) interpolants of samples on \([-MT, MT]\), \(M \approx 10(1 + \log T)\)

Essentially time-limited bluemultiband signals can be constructed from each band.

Continuous eigenfunction problem can be reduced to discrete eigenvector problem

Problems of *ill-conditioning* arise
Timelimiting and bandlimiting

Joe Lakey (w Jeff Hogan)
Bandlimiting

Fourier transform:
\[ \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \xi} \, dt \]

Bandlimiting:
\[ P_{\Sigma} f(x) = (\hat{f}_{\Sigma}) \lor (x) \]

Paley-Wiener space:
\[ \text{PW}_{\Sigma} = P_{\Sigma}(L^2(\mathbb{R})) \]
Fourier transform: \( \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \xi} \, dt \)
Bandlimiting

Fourier transform: $\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \xi} dt$

Bandlimiting: $P_{\Sigma} f(x) = (\hat{f} \mathbb{1}_\Sigma)^\vee(x)$
Bandlimiting

Fourier transform: \( \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \xi} \, dt \)

Bandlimiting: \( P_\Sigma f(x) = (\hat{f} \, 1_\Sigma) \vee (x) \)

Paley-Wiener space: \( PW_\Sigma = P_\Sigma(L^2(\mathbb{R})) \)
Time and band limiting: Bell Labs Theory

\[
(Pf)(x) = \left[\frac{-1}{2}, \frac{1}{2}\right] \lor (Qf)(x) = 1[\frac{-T}{2}, \frac{T}{2}]
\]

\[
PQ^T P: \text{self-adjoint, } \lambda_{\text{max}} = \lambda_0 = \|PQ^T\| = \sup_{f \in W, \|f\| = 1} \|Q^T(f)\|_2
\]

Uncertainty principle: \(\lambda_{\text{max}} < 1\)
(Pf)(x) = (\hat{f} \mathbb{1}_{[-1/2,1/2]})^\vee(x)
(Pf)(x) = (\hat{f} \mathbb{1}_{[-1/2, 1/2]})^\vee(x)
(Q_T f)(x) = \mathbb{1}_{[-T, T]}(x) f(x)
(Pf)(x) = (\hat{f} \mathbb{1}_{[-1/2, 1/2]})^{\vee}(x)
(Q_T f)(x) = \mathbb{1}_{[-T, T]}(x) f(x)

PQ_T P: self-adjoint,
\( (Pf)(x) = (\hat{f} \mathbb{1}_{[-1/2, 1/2]})^\vee(x) \)

\( (Q_T f)(x) = \mathbb{1}_{[-T, T]}(x) f(x) \)

\( PQ_T P \): self-adjoint,

\[ \lambda_{\text{max}} = \lambda_0 = \|PQ_T\| = \sup_{f \in PW, \|f\| = 1} \|Q_T(f)\|^2 \]
Time and band limiting: Bell Labs Theory

\[(Pf)(x) = (\hat{f} \mathbb{1}_{[-1/2, 1/2]})^\vee(x)\]

\[(Q_T f)(x) = \mathbb{1}_{[-T, T]}(x) f(x)\]

\[PQ_T P: \text{self-adjoint,}\]

\[\lambda_{\text{max}} = \lambda_0 = \|PQ_T\| = \sup_{f \in PW, \|f\|=1} \|Q_T(f)\|^2\]

Uncertainty principle: \(\lambda_{\text{max}} < 1\)
commutes with certain differential operator

Eigenfunctions: Prolate Spheroidal Wave Functions

Energy concentration: $\phi_0, \phi_1, \ldots$ are the bandlimited signals that are most time-limited to $[-T, T]$.

$\{\phi_n\}$ forms a complete, orthogonal family in $L^2[-T, T]$.

Joe Lakey (w Jeff Hogan)  Time- and bandlimiting
$P_\Omega Q_T$ commutes with certain differential operator . . .
$P_\Omega Q_T$ commutes with certain differential operator …

Eigenfunctions: Prolate Spheroidal Wave Functions
$P_\Omega Q_T$ commutes with certain differential operator …

Eigenfunctions: Prolate Spheroidal Wave Functions

Energy concentration: $\varphi_0, \varphi_1, \ldots$ are the bandlimited signals that are most time-limited to $[-T, T]$
$P_{\Omega} Q_T$ commutes with certain differential operator . . .

Eigenfunctions: Prolate Spheroidal Wave Functions

Energy concentration: $\varphi_0, \varphi_1, \ldots$ are the bandlimited signals that are most time-limited to $[-T, T]$

$\{\varphi_n\}$ forms a complete, orthogonal family in $L^2[-T, T]$. 
Plots generated on $[-1, 1]$ using “Bouwkamp’s method”

Figure: $\varphi_n$, $n = 0, 3, 10$, $c = \frac{\pi T \Omega}{2} = 5$
Theorem

For any $T > 0$ and $\Omega > 0$ and $0 < \alpha < 1$ the number $N(\alpha)$ of eigenvalues of $P_{\Omega} Q_T$ greater than $\alpha$ satisfies

$$N(\alpha) = 2\Omega T + \left(\frac{1}{\pi^2} \log \frac{1}{1-\alpha} \alpha\right) \log (\Omega T) + o(\log \Omega T).$$
The 2ΩT Theorem

**Theorem**

For any $T > 0$ and $\Omega > 0$ and $0 < \alpha < 1$ the number $N(\alpha)$ of eigenvalues of $P_\Omega Q_T$ greater than $\alpha$ satisfies

$$N(\alpha) = 2\Omega T$$
Theorem

For any $T > 0$ and $\Omega > 0$ and $0 < \alpha < 1$ the number $N(\alpha)$ of eigenvalues of $P_\Omega Q_T$ greater than $\alpha$ satisfies

$$N(\alpha) = 2\Omega T + \left( \frac{1}{\pi^2} \log \frac{1 - \alpha}{\alpha} \right) \log \left( \Omega T \right)$$
The $2\Omega T$ Theorem

**Theorem**

For any $T > 0$ and $\Omega > 0$ and $0 < \alpha < 1$ the number $N(\alpha)$ of eigenvalues of $P_{\Omega} Q_T$ greater than $\alpha$ satisfies

$$N(\alpha) = 2\Omega T + \left(\frac{1}{\pi^2} \log \frac{1 - \alpha}{\alpha}\right) \log (\Omega T) + o(\log \Omega T).$$
DFT illustration

Eigenvalues for 1025 points, normalized area of 64

Joe Lakey (w Jeff Hogan)
The classical sampling theorem

Bandlimiting:

\[ Pf(x) = (\hat{f}_{1/2}, 1/2) \lor (x) \]

Paley-Wiener space:

\[ PW = P(L^2(\mathbb{R})) \]

Sampling theorem: If \( f \in PW \) then, with convergence in your norm here,

\[ f(t) = \sum_{k=-\infty}^{\infty} f(k) \sin(\frac{\pi}{t-k}) \]

Joe Lakey (w Jeff Hogan)
The classical sampling theorem

Bandlimiting: $Pf(x) = (\hat{f} 1_{[-1/2, 1/2]})^\vee(x)$
The classical sampling theorem

Bandlimiting: \( Pf(x) = (\hat{f} \mathbb{1}_{[-1/2, 1/2]})^\vee(x) \)

Paley-Wiener space: \( \text{PW} = P(L^2(\mathbb{R})) \)

Sampling theorem: If \( f \in \text{PW} \) then, with convergence in \( \text{your norm here} \),

\[
f(t) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin \pi(t - k)}{\pi(t - k)}
\]
The classical sampling theorem

Bandlimiting: \( Pf(x) = (\hat{f} \mathbb{1}_{[-1/2, 1/2]})^\dagger(x) \)

Paley-Wiener space: \( \text{PW} = \mathcal{P}(L^2(\mathbb{R})) \)

Sampling theorem: If \( f \in \text{PW} \) then, with convergence in your norm here,

\[
f(t) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin \pi(t - k)}{\pi(t - k)}
\]
The classical sampling theorem

Bandlimiting: \( Pf(x) = (\hat{f} 1_{[-1/2, 1/2]})^\vee(x) \)

Paley-Wiener space: \( \text{PW} = P(L^2(\mathbb{R})) \)

Sampling theorem: If \( f \in \text{PW} \) then, with convergence in your norm here,

\[
f(t) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin \pi(t - k)}{\pi(t - k)}
\]
Can a time-limited and band-limited signal be approximated locally by a finite sinc series?
Problem (Statement 1)

Can one approximately recover the projection of \( f \in PW \) onto a subspace of “approximately time- and bandlimited” functions solely from samples near \([-T, T]\)?
Corollary (of Sampling theorem and defn of PSWFs)

If $f \in PW$ then

$$PQ_T f(t) = \sum_{n=0}^{\infty} \lambda_n$$
Corollary (of Sampling theorem and defn of PSWFs)

If $f \in PW$ then

$$PQ_T f(t) = \sum_{n=0}^{\infty} \lambda_n \sum_{k=-\infty}^{\infty} f(k) \varphi_n(k)$$
Corollary (of Sampling theorem and defn of PSWFs)

If \( f \in \text{PW} \) then

\[
PQ_T f(t) = \sum_{n=0}^{\infty} \lambda_n \sum_{k=-\infty}^{\infty} f(k) \varphi_n(k) \varphi_n(t).
\]
Problem (Statement 2)

For suitable $N(T) \approx T$ and $M(T) \approx T$, in what sense is

\[
\text{trunc span}\{\varphi_n : n \leq N(T)\} \\
\approx \text{trunc span}\{\text{sinc}(t - k) : |k| \leq M(T)\}?
\]
\{\varphi_n(k)\} as an eigenvector

Theorem (Walter and Shen; Khare and George, 2003-2004)
\{ \varphi_n(k) \} as an eigenvector

Theorem (Walter and Shen; Khare and George, 2003-2004)

\[ \lambda_n \varphi_n(m) = \sum_k A_{mk} \varphi_n(k); \quad A_{mk} = \int_{-T}^{T} \text{sinc} (t - m) \text{sinc} (t - k) \, dt \]
Quadratic decay of \( \{ \varphi_n(k) \} \)

Shen and Walter’s estimate:

\[
\sum |k| > T \varphi_n^2(k) \leq CT \sqrt{1 - \lambda_n}
\]

Desired estimate (Hogan, Lakey 2010)

\[
\sum |k| > M(T) \varphi_n^2(k) \leq C(1 - \lambda_n);
\]

\( M(T) = T (1 + \pi/2)(1 + \log \gamma T) \)

Estimate uses known decay on Fourier bump functions

Joe Lakey (w Jeff Hogan)
Quadratic decay of \( \{\varphi_n(k)\} \)

Shen and Walter’s estimate:

\[
\sum_{|k| > T} \varphi_2^n(k) \leq CT \sqrt{1 - \lambda_n}.
\]

Desired estimate (Hogan, Lakey 2010)

\[
\sum_{|k| > M(T)} \varphi_2^n(k) \leq C(1 - \lambda_n); \quad M(T) = T(1 + \pi^2)(1 + \log \gamma T).
\]

Estimate uses known decay on Fourier bump functions

Joe Lakey (w Jeff Hogan)

Time- and bandlimiting
Quadratic decay of \( \{ \varphi_n(k) \} \)

Shen and Walter’s estimate:

\[
\sum_{|k| > T} \varphi_n^2(k) \leq CT \sqrt{1 - \lambda_n}
\]
Quadratic decay of \( \{ \varphi_n(k) \} \)

Shen and Walter’s estimate:

\[
\sum_{|k| > T} \varphi_n^2(k) \leq CT \sqrt{1 - \lambda_n}
\]

Desired estimate (Hogan, Lakey 2010)
Quadratic decay of \( \{ \varphi_n(k) \} \)

Shen and Walter’s estimate:

\[
\sum_{|k| > T} \varphi_n^2(k) \leq CT \sqrt{1 - \lambda_n}
\]

Desired estimate (Hogan, Lakey 2010)

\[
\sum_{|k| > M(T)} \varphi_n^2(k) \leq C(1 - \lambda_n); \quad M(T) = T (1 + \pi^2)(1 + \log \gamma T)
\]
Quadratic decay of \( \{ \varphi_n(k) \} \)

Shen and Walter’s estimate:

\[
\sum_{|k| > T} \varphi_n^2(k) \leq CT \sqrt{1 - \lambda_n}
\]

Desired estimate (Hogan, Lakey 2010)

\[
\sum_{|k| > M(T)} \varphi_n^2(k) \leq C(1 - \lambda_n); \quad M(T) = T (1 + \pi^2)(1 + \log^\gamma T)
\]

Estimate uses known decay on Fourier bump functions
Local approximation by sinc series

Projection onto first $N$ prolates: For $f \in \text{PW}$ . . .

$$f_N(t) = \sum_{n=0}^{N} \left( \sum_{k} f(k) \varphi_n(k) \right) \varphi_n(t)$$
Local approximation by sinc series

Projection onto first $N$ prolates: For $f \in \text{PW}$...

$$f_N(t) = \sum_{n=0}^{N} \left( \sum_{k} f(k) \varphi_n(k) \right) \varphi_n(t)$$

Local sinc series
Local approximation by sinc series

Projection onto first $N$ prolates: For $f \in PW \ldots$

$$f_N(t) = \sum_{n=0}^{N} \left( \sum_{k} f(k) \varphi_n(k) \right) \varphi_n(t)$$

Local sinc series

$$f_{N,T} = \sum_{n=0}^{N} \left( \sum_{|k| \leq M(T)} f(k) \varphi_n^T(k) \right) \varphi_n^T(t); \varphi_n^T = \sum_{|k| \leq M(T)} \varphi_n(k) \text{sinc}(\cdot - k)$$
Local approximation by sinc series

Projection onto first $N$ prolates: For $f \in \text{PW}$ . . .

\[
f_N(t) = \sum_{n=0}^{N} \left( \sum_{k} f(k) \varphi_n(k) \right) \varphi_n(t)
\]

Local sinc series

\[
f_{N,T} = \sum_{n=0}^{N} \left( \sum_{|k| \leq M(T)} f(k) \varphi_n^T(k) \right) \varphi_n^T(t); \varphi_n^T = \sum_{|k| \leq M(T)} \varphi_n(k) \text{sinc}(\cdot-k)
\]

Proposition

\[
\| Q_T (f_N - f_{N,T}) \|^2 \leq \sum_{n=0}^{N} \lambda_n |\langle (f_N - f_{N,T}), \varphi_n \rangle|^2 \leq C \| f \|^2 \sum_{n=0}^{N} \lambda_n (1 - \lambda_n).
\]
Integer samples of PSWFs can be estimated accurately.
Integer samples of PSWFs can be estimated accurately

Karoui and Moumni, (ACHA, 2008):

Legendre Fourier coefficients: values of Bessel functions
Integer samples of PSWFs can be estimated accurately

Karoui and Moumni, (ACHA, 2008):
PSWF samples are integer values of Fourier transforms of $\hat{\phi}_n$
Integer samples of PSWFs can be estimated accurately

Karoui and Moumni, (ACHA, 2008):
PSWF samples are integer values of Fourier transforms of $\tilde{\phi}_n$
$\tilde{\phi}_n$ approximated on $[-1, 1]$ by Legendre series (small $c$)
Integer samples of PSWFs can be estimated accurately

Karoui and Moumnı, (ACHA, 2008):
PSWF samples are integer values of Fourier transforms of $\hat{\varphi}_n$
$\hat{\varphi}_n$ approximated on $[-1, 1]$ by Legendre series (small $c$)
Legendre Fourier coefficients: values of Bessel functions
Reconstruction from $M(T)$ versus from $T$ samples, $T = 5$
A more general case: multiband

The $2\Omega T$ theorem for multi bands is similar
A more general case: multiband

The $2\Omega T$ theorem for multi bands is similar. Plunge region width proportional to log area $\times$ number of intervals.
Unions of frequency supports

Proposition

\[ \Sigma = \Sigma_1 \cup \cdots \cup \Sigma_M \text{ compact}, \]
Proposition

$\Sigma = \Sigma_1 \uplus \cdots \uplus \Sigma_M$ compact, $\{\varphi_{n}^{\Sigma_{\nu}}\} \sim \lambda_{n}^{\Sigma_{\nu}} \sim P_{\Sigma_{\nu}} Q S P_{\Sigma_{\nu}}$
Proposition

\[ \Sigma = \Sigma_1 \uplus \cdots \uplus \Sigma_M \text{ compact, } \{ \phi^\Sigma_n \} \sim \lambda^\Sigma_n \sim P_{\Sigma_\nu} Q_S P_{\Sigma_\nu} \]

\[ \Lambda_{\Sigma_\nu} = \text{diag } \lambda^\Sigma_n \]
Proposition

\[ \Sigma = \Sigma_1 \uplus \cdots \uplus \Sigma_M \text{ compact}, \{ \varphi^{\Sigma_\nu}_n \} \sim \lambda^{\Sigma_\nu}_n \sim P_{\Sigma_\nu} Q_S P_{\Sigma_\nu} \]

\[ \Lambda_{\Sigma_\nu} = \text{diag} \, \lambda^{\Sigma_\nu}_n \]

\[ \Gamma^{\nu \mu}: \gamma^{\nu \mu}_{nm} = \langle Q_S \varphi^{\Sigma_\nu}_n, \varphi^{\Sigma_\mu}_m \rangle, \nu \neq \mu. \]
Proposition

\[ \Sigma = \Sigma_1 \cup \cdots \cup \Sigma_M \text{ compact}, \{ \varphi_{n}^{\Sigma_{\nu}} \} \sim \lambda_{n}^{\Sigma_{\nu}} \sim P_{\Sigma_{\nu}} Q_{S} P_{\Sigma_{\nu}} \]

\[ \Lambda_{\Sigma_{\nu}} = \text{diag} \lambda_{n}^{\Sigma_{\nu}} \]

\[ \Gamma^{\nu \mu} : \gamma_{nm}^{\nu \mu} = \langle Q_{S} \varphi_{n}^{\Sigma_{\nu}}, \varphi_{m}^{\Sigma_{\mu}} \rangle, \nu \neq \mu. \]

*Eigenvector–eigenvalue pairs* \( \psi \) and \( \lambda \) for \( P_{\Sigma} Q_{S} \):
Unions of frequency supports

Proposition

\[ \Sigma = \Sigma_1 \uplus \cdots \uplus \Sigma_M \text{ compact}, \{ \varphi_{\Sigma \nu}^n \} \sim \lambda_{\Sigma \nu}^n \sim P_{\Sigma \nu} Q_S P_{\Sigma \nu} \]

\[ \Lambda_{\Sigma \nu} = \text{diag} \lambda_{\Sigma \nu}^n \]

\[ \Gamma^{\nu \mu} : \gamma_{nm}^{\nu \mu} = \langle Q_S \varphi_{\Sigma \nu}^n, \varphi_{\Sigma \mu}^m \rangle, \nu \neq \mu. \]

Eigenvector–eigenvalue pairs \( \psi \) and \( \lambda \) for \( P_{\Sigma} Q_S \):

\[ \psi = \sum_{\nu=1}^M \sum_{n=0}^{\infty} \alpha_{\nu}^n \varphi_{\Sigma \nu}^n ; \]

\[ [\alpha_1, \ldots, \alpha_M]^T : \text{discrete eigenvector for the block matrix eigenvalue problem} \]

\[
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_M
\end{pmatrix}
\begin{pmatrix}
\Lambda_{\Sigma 1} & \bar{\Gamma}_{12} & \cdots & \bar{\Gamma}_{1M} \\
(\bar{\Gamma}_{12})^T & \Lambda_{\Sigma 2} & \bar{\Gamma}_{23} & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
(\bar{\Gamma}_{1M})^T & \cdots & (\bar{\Gamma}_{M-1,M})^T & \Lambda_{\Sigma M}
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_M
\end{pmatrix}.
\]
Figure: Matrix $\Gamma$ for $T = 2$ and THREE frequency intervals
Calculating cross terms

\[ \Gamma_{IJ} = \langle Q_T \phi_I, \phi_J \rangle = \sum_k \phi_n(k) \sum_\ell \phi_m(\ell) A(T; I, J)_{k\ell} \]

\[ A(T; I, J)_{k\ell} = \int_{T-T}^{T} e^{2\pi i (m_I - m_J) t} \text{sinc}(t-k) \text{sinc}(t-\ell) dt. \]
Calculating cross terms

\[ \Gamma_{n,m}^{I,J} = \langle Q_T \varphi_n^I, \varphi_m^J \rangle = \sum_k \varphi_n(k) \sum_\ell \varphi_m(\ell) A(T; I, J)_{k\ell}; \]

\[ A(T; I, J)_{k\ell} = \int_{-T}^{T} e^{2\pi i (m_I - m_J)t} \text{sinc}(t - k)\text{sinc}(t - \ell) \, dt. \]
Calculating cross terms

\[ \Gamma_{n,m}^{I,J} = \langle Q_T \varphi_n^{I}, \varphi_m^{J} \rangle = \sum_k \varphi_n(k) \sum_\ell \varphi_m(\ell) A(T; I, J)_{k\ell}; \]

\[ A(T; I, J)_{k\ell} = \int_{-T}^{T} e^{2\pi i (m_I - m_J) t} \text{sinc} (t - k) \text{sinc} (t - \ell) \, dt. \]
Proposition

As a bilinear form acting on \( \{ \varphi_n(k) \} \), \( \{ \varphi_m(\ell) \} \),

\[
A(T; I, J)_{k\ell} = i^{n+m} \sqrt{\lambda_m \lambda_n} \text{sinc} (2T(m_J - m_I) + k - \ell).
\]
Proposition

As a bilinear form acting on \( \{ \varphi_n(k) \}, \{ \varphi_m(\ell) \} \),

\[
A(T; I, J)_{k\ell} = i^{n+m} \sqrt{\lambda_m \lambda_n} \text{sinc} (2T(m_J - m_I) + k - \ell).
\]

Drawback: \( \Gamma \) ill-conditioned
Figure: PSWFs for $T = 2$ and $I = [-1/2, 1/2]$. Plot of the first six prolates generated by sinc-interpolating integer samples. The $n$th prolate has $n$ zeros on $[-T, T]$, $n = 0, 1, \ldots$
Figure: TMBLMs for $T = 2$, $I = [-1/2, 1/2]$ and $J = [2, 3]$ and $K = [5, 6]$. $n = \{0, 1, 2\}, \{3, 4, 5\}$ (three per each basic PSWF mode). Real parts solid.
Special case: Bandpass prolates

Time and bandlimiting
Special case: Bandpass prolates

Time limited to $[-1, 1]$, frequency limited to $c' \leq |\xi| \leq c$. 

Joe Lakey (w Jeff Hogan)  
Time- and bandlimiting
Special case: Bandpass prolates

Time limited to $[-1, 1]$, frequency limited to $c' \leq |\xi| \leq c$. Fourier covariance: $\hat{\varphi}_n$ is a truncated dilate of $\varphi_n$. 
Special case: Bandpass prolate

Time limited to $[-1, 1]$, frequency limited to $c' \leq |\xi| \leq c$. Fourier covariance: $\hat{\varphi}_n$ is a truncated dilate of $\varphi_n$.

$$R_{jk} = \frac{i^{k-j}}{\sqrt{\lambda_k \lambda_j}} \int_{-c'/c}^{c'/c} \varphi^c_k(\xi) \varphi^c_j(\xi) d\xi \quad (j, k \geq 0)$$
Special case: Bandpass prolates

Time limited to $[-1, 1]$, frequency limited to $c' \leq |\xi| \leq c$.

Fourier covariance: $\hat{\varphi}_n$ is a truncated dilate of $\varphi_n$.

$$R_{jk} = \frac{i^{k-j}}{\sqrt{\lambda_k \lambda_j}} \int_{-c'/c}^{c'/c} \varphi_k^c(\xi) \varphi_j^c(\xi) \, d\xi \quad (j, k \geq 0)$$

If $P_{c,c'} Q\psi = \lambda \psi$ and $\psi = \sum_n \alpha_n \varphi_n^c$, then

$$\lambda \alpha = (I - R) \Lambda \alpha; \quad \alpha = \{\alpha_n\}; \quad \Lambda = \text{diag}\{\lambda_n\}$$
Special case: Bandpass prolates

Time limited to $[-1, 1]$, frequency limited to $c' \leq |\xi| \leq c$.

Fourier covariance: $\hat{\varphi}_n$ is a truncated dilate of $\varphi_n$.

$$R_{jk} = \frac{i^{k-j}}{\sqrt{\lambda_k \lambda_j}} \int_{c'/c}^{c'/c} \varphi^c_k(\xi) \varphi^c_j(\xi) \, d\xi \quad (j, k \geq 0)$$

If $P_{c,c'} Q \psi = \lambda \psi$ and $\psi = \sum_n \alpha_n \varphi^c_n$, then

$$\lambda \alpha = (I - R) \Lambda \alpha; \quad \alpha = \{\alpha_n\}; \quad \Lambda = \text{diag}\{\lambda_n\}$$
Figure: $\psi_0^{c'}c$ for $c = 5\pi/2$. Symmetric bandpass prolates having largest energy concentration to $[-1,1]$. The thick curves: $c'/c = 0.02$ and the value $c'/c = 0.8$. 
EEG theory: brain uses phase in different bands for distributed cognitive processes
EEG theory: brain uses phase in different bands for distributed cognitive processes
Indicated by small deviations in phase difference among recruited regions.
Application to phase locking in EEG

EEG theory: brain uses phase in different bands for distributed cognitive processes
Indicated by small deviations in phase difference among recruited regions.
Different methods of measuring phase in band of interest: Filtered analytic signal; Gabor functions; Empirical modes
EEG theory: brain uses phase in different bands for distributed cognitive processes
Indicated by small deviations in phase difference among recruited regions.
Different methods of measuring phase in band of interest: Filtered analytic signal; Gabor functions; Empirical modes
Proposed here: Project onto span of time and bandpass limiteds

$PLV = \left| \int \text{analytic chan1 proj} \right| \left| \text{analytic chan1 proj} \right| \left| \text{analytic chan2 proj} \right| \left| \text{analytic chan2 proj} \right|$
EEG theory: brain uses phase in different bands for distributed cognitive processes
Indicated by small deviations in phase difference among recruited regions.
Different methods of measuring phase in band of interest: Filtered analytic signal; Gabor functions; Empirical modes
Proposed here: Project onto span of time and bandpass limiteds
Illustrated: gamma band (here 24 Hz – 40 Hz) 3 – 5 cycles
Application to phase locking in EEG

EEG theory: brain uses phase in different bands for distributed cognitive processes
Indicated by small deviations in phase difference among recruited regions.
Different methods of measuring phase in band of interest: Filtered analytic signal; Gabor functions; Empirical modes
Proposed here: Project onto span of time and bandpass limiteds
Illustrated: gamma band (here 24 Hz – 40 Hz) 3 – 5 cycles

\[ \text{PLV} = \left| \int \frac{\text{analytic chan1 proj}}{|\text{analytic chan1 proj}|} \frac{\text{analytic chan2 proj}}{|\text{analytic chan2 proj}|} \right| \]
EEG channel raw data

Joe Lakey (w Jeff Hogan)  Time- and bandlimiting
Channel projections, 6 eigenvectors, 24–40 Hz, 1/8 seconds.

Joe Lakey (w Jeff Hogan)

Time- and bandlimiting
Two-channel projection PLVs & averages vs time centers

Joe Lakey (w Jeff Hogan)  Time- and bandlimiting