Time-Frequency localization of Multiband signals
AMS Meeting # 1047, Urbana, March 27, 2009

Joe Lakey (w Scott Izu)\textsuperscript{1}

March 24, 2009
Fourier transform: \( \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \xi} \, dt \)
Time and frequency localization

- Fourier transform: \( \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi it\xi} \, dt \)

- \( P_{\Sigma} f(x) = (\hat{f} \mathbb{1}_\Sigma)^\vee(x) \); Paley-Wiener: \( \text{PW}_\Sigma = P_{\Sigma}(L^2(\mathbb{R})) \)
Fourier transform: \( \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi it\xi} dt \)

\( P_{\Sigma} f(x) = (\hat{f} 1_{\Sigma}) \vee (x) \); Paley-Wiener: \( PW_{\Sigma} = P_{\Sigma}(L^{2}(\mathbb{R})) \)

Fundamental Questions Include . . . :
Fourier transform:  \( \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \xi} \, dt \)

\[ P_\Sigma f(x) = (\hat{f} 1_\Sigma)^\vee(x); \text{ Paley-Wiener: } PW_\Sigma = P_\Sigma(L^2(\mathbb{R})) \]

**Fundamental Questions Include . . . :**

- Sampling theory of \( PW_\Sigma \)
- Time localization of \( PW_\Sigma \)
Time and frequency localization

- Fourier transform: \( \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi it\xi} dt \)
- \( P_{\Sigma} f(x) = (\hat{f} \mathbb{1}_{\Sigma})^\vee(x) \); Paley-Wiener: \( PW_{\Sigma} = P_{\Sigma}(L^2(\mathbb{R})) \)
- **Fundamental Questions Include . . . :**
  - Sampling theory of \( PW_{\Sigma} \)
  - Time localization of \( PW_{\Sigma} \)
  - \( Q_{\Sigma} f(x) = f(x) \mathbb{1}_S(x) \);
Fourier transform: \[ \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \xi} \, dt \]

\[ P_{\Sigma} f(x) = (\hat{f} \mathbbm{1}_{\Sigma})^\vee(x) \]; Paley-Wiener: \[ PW_{\Sigma} = P_{\Sigma}(L^2(\mathbb{R})) \]

Fundamental Questions Include . . . :

- Sampling theory of \( PW_{\Sigma} \)
- Time localization of \( PW_{\Sigma} \)
  - \( Q_{S} f(x) = f(x) \mathbbm{1}_{S}(x) \);
Fourier transform: \( \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi it\xi} dt \)

\( P_{\Sigma}f(x) = (\hat{f} \ 1_{\Sigma})^\vee(x) \); Paley-Wiener: \( \text{PW}_{\Sigma} = P_{\Sigma}(L^2(\mathbb{R})) \)

Fundamental Questions Include . . . :
- Sampling theory of \( \text{PW}_{\Sigma} \)
- Time localization of \( \text{PW}_{\Sigma} \)
  - \( Q_{S}f(x) = f(x) \ 1_{S}(x) \);
  - Eigenvalues of \( P_{\Sigma}Q_{S}P_{\Sigma} \): vs \( |S||\Sigma| \), linear distribution of \( \Sigma \)
Time and frequency localization

- Fourier transform: $\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \xi} \, dt$

- $P_{\Sigma} f(x) = (\hat{f} \cdot \mathbb{1}_{\Sigma})^\vee(x)$; Paley-Wiener: $\text{PW}_{\Sigma} = P_{\Sigma}(L^2(\mathbb{R}))$

- **Fundamental Questions Include . . . :**
  - Sampling theory of $\text{PW}_{\Sigma}$
  - Time localization of $\text{PW}_{\Sigma}$
    - $Q_S f(x) = f(x) \mathbb{1}_S(x)$
    - Eigenvalues of $P_{\Sigma} Q_S P_{\Sigma}$: vs $|S||\Sigma|$, linear distribution of $\Sigma$
    - Especially, $S$ an interval
Time and frequency localization

- Fourier transform: \( \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi it\xi} dt \)
- \( P_{\Sigma} f(x) = (\hat{f} \, 1_{\Sigma})^\vee(x) \); Paley-Wiener: \( PW_{\Sigma} = P_{\Sigma}(L^2(\mathbb{R})) \)

Fundamental Questions Include . . . :
- Sampling theory of \( PW_{\Sigma} \)
- Time localization of \( PW_{\Sigma} \)
  - \( Q_{S} f(x) = f(x) \, 1_{S}(x) \)
  - Eigenvalues of \( P_{\Sigma} Q_{S} P_{\Sigma} \): vs \( |S| \, |\Sigma| \), linear distribution of \( \Sigma \)
  - Especially, \( S \) an interval
- Sample based time-localized approximations
Time and frequency localization

- Fourier transform: \( \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \xi} \, dt \)
- \( P_{\Sigma} f(x) = (\hat{f} \mathbb{1}_{\Sigma})^\vee(x) \); Paley-Wiener: \( \text{PW}_{\Sigma} = P_{\Sigma}(L^2(\mathbb{R})) \)
- Fundamental Questions Include . . . :
  - Sampling theory of \( \text{PW}_{\Sigma} \)
  - Time localization of \( \text{PW}_{\Sigma} \)
    - \( Q_{\Sigma} f(x) = f(x) \mathbb{1}_S(x) \)
    - Eigenvalues of \( P_{\Sigma} Q_{\Sigma} P_{\Sigma} \): vs \( |S| |\Sigma| \), linear distribution of \( \Sigma \)
    - Especially, \( S \) an interval
  - Sample based time-localized approximations
    - \( \phi_n \) eigenvectors of \( P_{\Sigma} Q_{\Sigma} \)
Fourier transform: $\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi it\xi} \, dt$

$P_\Sigma f(x) = (\hat{f} \mathbb{1}_\Sigma)^\vee(x)$; Paley-Wiener: $PW_\Sigma = P_\Sigma(L^2(\mathbb{R}))$

Fundamental Questions Include . . . :

- Sampling theory of $PW_\Sigma$
- Time localization of $PW_\Sigma$
  - $Q_S f(x) = f(x) \mathbb{1}_S(x)$;
  - Eigenvalues of $P_\Sigma Q_S P_\Sigma$: vs $|S||\Sigma|$, linear distribution of $\Sigma$
  - Especially, $S$ an interval
- Sample based time-localized approximations
  - $\phi_n$ eigenvectors of $P_\Sigma Q_S$,
  - Quantify $\langle f, \phi_n \rangle$ in terms of $\{f(x_k)\}$
Fourier transform: \( \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi it\xi} \, dt \)

\( P_\Sigma f(x) = (\hat{f} \mathbb{1}_\Sigma)^\vee(x) \); Paley-Wiener: \( \text{PW}_\Sigma = P_\Sigma(L^2(\mathbb{R})) \)

Fundamental Questions Include . . . :

- Sampling theory of \( \text{PW}_\Sigma \)
- Time localization of \( \text{PW}_\Sigma \)
  - \( Q_S f(x) = f(x) \mathbb{1}_S(x) \);
  - Eigenvalues of \( P_\Sigma Q_S P_\Sigma \): vs \( |S||\Sigma| \), linear distribution of \( \Sigma \)
  - Especially, \( S \) an interval

- Sample based time-localized approximations
  - \( \phi_n \) eigenvectors of \( P_\Sigma Q_S \),
  - Quantify \( \langle f, \phi_n \rangle \) in terms of \( \{f(x_k)\} \)
  - . . . finite-dimensional approximations
Fourier transform: $\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi it\xi} dt$

$P_{\Sigma}f(x) = (\hat{f} \cdot 1_{\Sigma})^\vee(x)$; Paley-Wiener: $PW_{\Sigma} = P_{\Sigma}(L^2(\mathbb{R}))$

**Fundamental Questions Include . . . :**

- Sampling theory of $PW_{\Sigma}$
- Time localization of $PW_{\Sigma}$
  - $Q_{S}f(x) = f(x) \cdot 1_{S}(x)$;
  - Eigenvalues of $P_{\Sigma}Q_{S}P_{\Sigma}$: vs $|S||\Sigma|$, linear distribution of $\Sigma$
  - Especially, $S$ an interval
- Sample based time-localized approximations
  - $\phi_{n}$ eigenvectors of $P_{\Sigma}Q_{S}$,
  - Quantify $\langle f, \phi_{n} \rangle$ in terms of $\{f(x_k)\}$
  - . . . finite-dimensional approximations
- FFT version . . . communications applications
I. Time and frequency localization: Bell Labs Theory

\[ \lambda_{\text{max}} = \lambda_0 = \| P \Sigma Q S \| = \sup_{f \in \mathcal{P}, \| f \| = 1} \| Q S (f) \|_2 \]

Uncertainty principle: \( \lambda_{\text{max}} < 1 \) if \( |S| |\Sigma| < \infty \)
1. Time and frequency localization: Bell Labs Theory

- \( P_\Sigma Q_\Sigma P_\Sigma \): self-adjoint,
I. Time and frequency localization: Bell Labs Theory

- $P_\Sigma Q_\Sigma P_\Sigma$: self-adjoint,

  \[
  \lambda_{\text{max}} = \lambda_0 = \|P_\Sigma Q_\Sigma\| = \sup_{f \in PW_\Sigma, \|f\|=1} \|Q_\Sigma(f)\|^2
  \]
I. Time and frequency localization: Bell Labs Theory

- \( P_\Sigma Q_\Sigma P_\Sigma \): self-adjoint,

\[
\lambda_{\text{max}} = \lambda_0 = \| P_\Sigma Q_\Sigma \| = \sup_{f \in P W_\Sigma, \|f\|=1} \| Q_\Sigma(f) \|^2
\]

- Uncertainty principle: \( \lambda_{\text{max}} < 1 \) if \( |S||\Sigma| < \infty \)
Prolate spheroidal wave functions

\[ S = [-T/2, T/2]; \quad \Sigma = [-\Omega/2, \Omega/2], \quad \text{tr} P_\Omega Q_T = T\Omega \equiv c. \]
Prolate spheroidal wave functions

- $S = [-T/2, T/2]$; $\Sigma = [-\Omega/2, \Omega/2]$, $\text{tr } P_\Omega Q_T = T\Omega \equiv c$.
- Orthonormal eigenfunctions: $P_\Omega Q_T \phi_j = \lambda_j \phi_j$
Prolate spheroidal wave functions

- $S = [-T/2, T/2]$; $\Sigma = [-\Omega/2, \Omega/2]$, $\text{tr} P_\Omega Q_T = T\Omega \equiv c$.
- Orthonormal eigenfunctions: $P_\Omega Q_T \phi_j = \lambda_j \phi_j$
- $P_\Omega Q_T$ commutes with

\[
\left( T^2 - t^2 \right) \frac{d^2}{dt^2} - 2t \frac{d}{dt} - \Omega^2 t^2
\]
Prolate spheroidal wave functions

- $S = [-T/2, T/2]$; $\Sigma = [-\Omega/2, \Omega/2]$, $\text{tr} \ P_\Omega Q_T = T\Omega \equiv c$.
- Orthonormal eigenfunctions: $P_\Omega Q_T \phi_j = \lambda_j \phi_j$
- $P_\Omega Q_T$ commutes with

$$\left( T^2 - t^2 \right) \frac{d^2}{dt^2} - 2t \frac{d}{dt} - \Omega^2 t^2$$

- Eigenfunctions are Prolate Spheroidal Wave Functions

Joe Lakey (w Scott Izu)

Time-frequency multiband
Approximately $c = \Omega T$ eigenvalues close to one
Eigenvalue properties

- Approximately $c = \Omega T$ eigenvalues close to one
- Plunge region of width $\approx \log c$
Eigenvalue properties

- Approximately $c = \Omega T$ eigenvalues close to one
- Plunge region of width $\approx \log c$
- Transition about $j = [c]$: $\lambda_{[c]+1} \leq 1/2 \leq \lambda_{[c]-1}$
II. Discrete illustrations

Figure: *Eigenvalues for one frequency interval.* $N = 1025$ point centered DFT. $S \sim 513 + [-128, 128]; \Sigma \sim 513 + [-128, 128]$. 
$c = \# T \times \# \Sigma / N \approx 64$. Plunge region $\sim 61 \leq n \leq 69$. 
Figure: Even eigenvectors for one frequency interval. $N = 129$ point centered DFT. $S \sim 65 + [-16, 16]$; $\Sigma \sim 65 + [-16, 16]$. $c = \#T \times \#\Sigma/N = 8.44$. Plunge region $\sim 7 \leq n \leq 12$. 
Figure: Eigenvalues for two frequency intervals. $N = 1025$ point centered DFT. $S \sim 513 + [-128, 128]$; $\Sigma \sim 512 + [-64, 64] \cup \pm [128, 192]$. $c \approx \# T \times \# \Sigma / N \approx 64$. Plunge region $\sim 55 \leq n \leq 84$. 
Figure: Discrete eigenvectors, two “symmetric” frequency intervals. Normalized area 24.69
Figure: Random $\Sigma$, $N = 129$. $c = 24.69$. 
Figure: *Eigenvalues for random $\Sigma$. $N = 129$, $c = 24.69$. Flat around $c = |\Sigma|$*
Figure: Eigenvectors (real parts) for random $\Sigma$. $N = 129$, $c = 24.69$
Figure: Kernel of $P_\Sigma Q_T$, $N = 129$, $c = 24.69$ (real part)
The “Σ T”-theorem, $N(\alpha)$, Multiple intervals

The “Σ T”-theorem, $N(\alpha)$, Multiple intervals

- $S, \Sigma$: finite unions of intervals
The "$\Sigma T$"-theorem, $N(\alpha)$, Multiple intervals

- $S, \Sigma$: finite unions of intervals
- $\xi \in c\Sigma$: $\xi/c \in \Sigma$
The “Σ T”-theorem, $N(\alpha)$, Multiple intervals

- $S, \Sigma$: finite unions of intervals
- $\xi \in c\Sigma$: $\xi/c \in \Sigma$
- $A_c = P_{c\Sigma} Q_S P_{c\Sigma}$
The “Σ T”-theorem, \( N(\alpha) \), Multiple intervals

- \( S, \Sigma \): finite unions of intervals
- \( \xi \in c\Sigma: \xi/c \in \Sigma \)
- \( A_c = P_{c\Sigma} Q_S P_{c\Sigma} \)
- \( N(A_c, \alpha) = \#\{\lambda(A_c) > \alpha\} \)
The “Σ 𝑇”-theorem, $N(\alpha)$, Multiple intervals

- $S, \Sigma$: finite unions of intervals
- $\xi \in c\Sigma$: $\xi/c \in \Sigma$
- $A_c = P_c\Sigma Q_S P_c\Sigma$
- $N(A_c, \alpha) = \#\{\lambda(A_c) > \alpha\}$
- $N(A_c, \alpha) = c|S||\Sigma| + \frac{N_S N_{\Sigma}}{\pi^2} \log\left(\frac{1-\alpha}{\alpha}\right) \log c + o(\log c)$
The “Σ T”-theorem, $N(\alpha)$, Multiple intervals

- $S$, $\Sigma$: finite unions of intervals
- $\xi \in c\Sigma$: $\xi/c \in \Sigma$
- $A_c = P_{c\Sigma}QSP_{c\Sigma}$
- $N(A_c, \alpha) = \#\{\lambda(A_c) > \alpha\}$
- $N(A_c, \alpha) = c|S||\Sigma| + \frac{N_S N_\Sigma}{\pi^2} \log\left(\frac{1-\alpha}{\alpha}\right) \log c + o(\log c)$
- Area $c|S||\Sigma|$ for $\alpha = 1/2$ in limit.
The “Σ T”-theorem, $N(\alpha)$, Multiple intervals

- $S$, $\Sigma$: finite unions of intervals
- $\xi \in c\Sigma$: $\xi/c \in \Sigma$
- $A_c = P_{c\Sigma} Q_S P_{c\Sigma}$
- $N(A_c, \alpha) = \#\{\lambda(A_c) > \alpha\}$
- $N(A_c, \alpha) = c|S||\Sigma| + \frac{N_S N_{\Sigma}}{\pi^2} \log\left(\frac{1-\alpha}{\alpha}\right) \log c + o(\log c)$
- Area $c|S||\Sigma|$ for $\alpha = 1/2$ in limit.
- $N_S N_{\Sigma}$: width of “plunge region”
Plunge width $\sim N_S N_\Sigma$
Plunge width $\sim N_S N_\Sigma$

- Separated at infinity
Plunge width $\sim N_\Sigma N_\Sigma$

- Separated at infinity
- $\phi_j$ frequency concentrated on $l_j, |l_j| = 1$
Plunge width $\sim N_s N_\Sigma$

- Separated at infinity
- $\phi_j$ frequency concentrated on $l_j$, $|l_j| = 1$
- $\phi_j(t) = e^{2\pi i m_j t} \varphi_j(t)$ $m_j = \overline{l_j}$.
Plunge width $\sim N_S N_\Sigma$

- Separated at infinity
- $\phi_j$ frequency concentrated on $l_j$, $|l_j| = 1$
- $\phi_j(t) = e^{2\pi i m_j t} \varphi_j(t) \ m_j = \overline{l_j}$.
- $\varphi_j$ frequency concentrated on $[-1/2, 1/2]$. 
Separation: $\phi_j$ frequency concentrated on $I_j$, $|I_j| = 1$

\[ \phi_j(t) = e^{2\pi i m_j t} \psi_j(t) \quad m_j = \bar{I}_j. \]

$\varphi_j$ frequency concentrated on $[-1/2, 1/2]$.

\[
\langle Q\phi_j, Q\phi_k \rangle = \int_{-1/2}^{1/2} e^{2\pi i (m_j - m_k) t} \varphi_j(t) \bar{\varphi}_k(t) \, dt
= \hat{\varphi}_1 * \hat{\varphi}_2 * \text{sinc} (m_1 - m_2) = O(1/|m_1 - m_2|)
\]
Plunge width $\sim N_S N_\Sigma$

- Separated at infinity
- $\phi_j$ frequency concentrated on $l_j$, $|l_j| = 1$
  - $\phi_j(t) = e^{2\pi i m_j t} \varphi_j(t) \ m_j = l_j.$
- $\varphi_j$ frequency concentrated on $[-1/2, 1/2]$.

\[
\langle Q\phi_j, Q\phi_k \rangle = \int_{-1/2}^{1/2} e^{2\pi i (m_j - m_k) t} \varphi_j(t)\bar{\varphi}_k(t) \ dt
\]

\[
= \hat{\varphi}_1 \ast \hat{\varphi}_2 \ast \text{sinc} \ (m_1 - m_2) = O(1/|m_1 - m_2|)
\]

- Each $l_j$ gives one eigenvalue $\approx 1/2$
Plunge width $\sim N_{\mathcal{S}} N_{\Sigma}$

- Separated at infinity
- $\phi_j$ frequency concentrated on $l_j$, $|l_j| = 1$
- $\phi_j(t) = e^{2\pi i m_j t} \varphi_j(t)$ $m_j = \bar{l}_j$.  
- $\varphi_j$ frequency concentrated on $[-1/2, 1/2]$.

\[
\langle Q\phi_j, Q\phi_k \rangle = \int_{-1/2}^{1/2} e^{2\pi i (m_j - m_k) t} \varphi_j(t) \overline{\varphi_k(t)} \, dt
\]

\[
= \hat{\varphi}_1 \ast \hat{\varphi}_2 \ast \text{sinc} (m_1 - m_2) = O\left(1/|m_1 - m_2|\right)
\]

- Each $l_j$ gives one eigenvalue $\approx 1/2$
- If each $l_j$ were very short: no large eigenvalues.
More discrete illustrations: random frequencies

Play/Pause  Slow
Structured Fourier spectrum (DFT!)
When is area formula $\lambda_{[c]} \geq 1/2$ still valid?

**Proposition**

(IZU)

Let $\Sigma = [-1/2, 1/2]$ and let $S$ be a union of $m$ pairwise disjoint intervals of total length $c$. Set

$$
\nu = \max_\alpha \#\{k \in \mathbb{Z} : (k, k + 1) \subset S + \alpha\},
$$

$$
\mu = \min_\beta \#\{\ell \in \mathbb{Z} : (\ell, \ell + 1) \cap S + \beta \neq \emptyset\}.
$$

Then the eigenvalues $\lambda_k$ of $Q_SP$ satisfy

$$
\lambda_{\nu - 1} \geq 1/2 \geq \lambda_\mu.
$$
Corollary

When $T = 1$ and $\Sigma$ is a union of integer intervals $[k, k + 1]$, $\lambda_c = 1/2$.

Conjecture

When $\Sigma$ is a symmetric union of “grid intervals” of length $1/T$ (so $c \in \mathbb{N}$) one has $\lambda_{c-k} + \lambda_{c+k} = 1$, $k = 1, \ldots, [c]$. 
Figure: Eigenvalues for DFT localization, $N = 1024$, $T = 128$, 10 symmetrized length 16 intervals $c = 40$ (real part), Note symmetry
Figure: Σ: 10 symmetrized length 16 intervals $c = 40$ (real part)
Largest energy concentration for a given area?

- Donoho and Stark (1993): if $|\Sigma| = 1$ and $T \leq 0.8$ then ...
Largest energy concentration for a given area?

- Donoho and Stark (1993): if $|\Sigma| = 1$ and $T \leq 0.8$ then . . .

\[
\int_{-T/2}^{T/2} |f(t)|^2 \, dt \leq \int_{-T/2}^{T/2} |(\hat{f}^*)^\vee(t)|^2 \, dt.
\]
Largest energy concentration for a given area?

- Donoho and Stark (1993): if $|\Sigma| = 1$ and $T \leq 0.8$ then $\ldots$

$$\int_{-T/2}^{T/2} |f(t)|^2 dt \leq \int_{-T/2}^{T/2} |(\hat{f}^*)^\vee(t)|^2 dt.$$  

- optimal concentration: $\Sigma$ is an interval if $T$ is small enough.
Largest energy concentration for a given area?

- Donoho and Stark (1993): if $|\Sigma| = 1$ and $T \leq 0.8$ then . . .

$$\int_{-T/2}^{T/2} |f(t)|^2 dt \leq \int_{-T/2}^{T/2} |(\hat{f}^*)^\vee(t)|^2 dt.$$  

- optimal concentration: $\Sigma$ is an interval if $T$ is small enough.

- Rearrangement inequality fails for large measure.
(S, Σ) supports information if \( \|P_\Sigma Q_S P_\Sigma\| \geq 1/2 \).
Information problem

- $(S, \Sigma)$ supports information if $\|P_\Sigma Q_S P_\Sigma\| \geq 1/2$.
- ... at rate $N$: $N$ eigenvalues $\geq 1/2$
Information problem

- \((S, \Sigma)\) supports information if \(\|P_\Sigma Q_S P_\Sigma\| \geq 1/2\).
- ... at rate \(N\): \(N\) eigenvalues \(\geq 1/2\)
- Rationale: basis functions \(\sim\) codes
Information problem

- $(S, \Sigma)$ supports information if $\|P_{\Sigma} Q_{S} P_{\Sigma}\| \geq 1/2$.
- ... at rate $N$: $N$ eigenvalues $\geq 1/2$
- Rationale: basis functions $\sim$ codes
- Which pairs support information?
Theorem

(Candès, Romberg, Tao) Fix $N \geq 512$ and $\beta$ such that $1 \leq \beta \leq (3/8) \log N$. Suppose that $S$ and $\Sigma$ are subsets of $\mathbb{Z}_N$ whose sizes satisfy

$$|S| + |\Sigma| \leq M(N, \beta) = \frac{N}{\sqrt{\beta + 1} \log N} \left( \frac{1}{\sqrt{6}} + o(1) \right).$$

Then with probability at least $1 - O((\log N)^{1/2}/N^\beta)$, every signal $x$ frequency supported in $\Sigma$ satisfies

$$\|x 1_S\|^2 \leq \frac{1}{2} \|x\|^2.$$
The entropy of a partition $\mathcal{P}$ of a probability space $(X, \mathcal{B}, \mu)$ is
\[ E(\mathcal{P}) = -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P). \]

**Problem**

Let $S \sim [-T/2, T/2]$ and let $|\Sigma|$ and hence $|S||\Sigma|$ be fixed in the finite time-frequency plane. Establish a quantitative, probabilistic relationship between an appropriate entropy of $\Sigma$ and an appropriate norm of $A_{SS}$. 
Figure: Norm of $P_{\Sigma} Q_T$ versus entropy, $N = 512$, $c = 8$
IV. Sampling and Time-Frequency localization
Sampling and eigenfunctions: $\Omega T$ case

**Theorem**

*(Shen and Walter; Khare and George)*

- $\varphi_n \sim \lambda_n$ of $PQ_T P$. Then

$$\lambda_n \varphi_n(m) = \sum k A_{mk} \varphi_n(k).$$

where the doubly-infinite matrix $A$ has entries $A_{mk}$

$$A_{mk} = \int_{-T/2}^{+T/2} \text{sinc}(t - m) \text{sinc}(t - k) \, dt.$$
Sampling and eigenfunctions: $\Omega T$ case

**Theorem**

((Shen and Walter; Khare and George))

- $\varphi_n \sim \lambda_n$ of $PQ_T P$. Then

\[
\lambda_n \varphi_n(m) = \sum_k A_{mk} \varphi_n(k)
\]

- i.e. samples of $\varphi_n$ form $n$-th eigenvector of $\{A_{mk}\}$. 

Joe Lakey (w Scott Izu)  
Time-frequency multiband
Theorem
(Shen and Walter; Khare and George)

- $\varphi_n \sim \lambda_n$ of $PQ_T P$. Then

$$\lambda_n \varphi_n(m) = \sum_k A_{mk} \varphi_n(k)$$

- where the doubly-infinite matrix $A$ has entries $A_{mk}$ given by

$$A_{mk} = \int_{-T/2}^{T/2} \text{sinc} (t - m) \text{sinc} (t - k) \, dt.$$
Sampling and eigenfunctions: $\Omega T$ case

Theorem
(Shen and Walter; Khare and George)

$\varphi_n \sim \lambda_n$ of $PQ_T P$. Then

$$\lambda_n \varphi_n(m) = \sum_k A_{mk} \varphi_n(k)$$

where the doubly-infinite matrix $A$ has entries $A_{mk}$ given by

$$A_{mk} = \int_{-T/2}^{T/2} \text{sinc} (t - m) \text{sinc} (t - k) \, dt.$$ 

i.e. samples of $\varphi_n$ form $n$-th eigenvector of $\{A_{mk}\}$. 
... and

- Orthogonality of sample vectors:

$$\sum_{m=-\infty}^{\infty} \varphi_n(m) \varphi_\ell(m) = \delta_{n,\ell}$$
Problem
Quantify the sense in which the eigenvectors of the matrix $\tilde{A}$ obtained by truncating $A_{mk}$ to zero where $\max\{m, k\} > N$ approximate those of $A$. 
\[ A_{k\ell} = \int_{-T}^{T} \text{sinc} \,(x - k) \text{sinc} \,(x - \ell) \, dx. \]

**Proposition**

(Izu, L.)

(i) When \( \ell > k \geq T \):

\[ A_{k\ell} = \frac{(-1)^{k-\ell}}{\pi} \frac{2T}{(k + T)(\ell + T)} + O\left(\frac{1}{k^2(\ell - k)}\right), \quad \text{as } k, \ell \to \infty. \]

(ii) Let \( A_{k\ell}^{\text{trunc}} = A_{k\ell} \) if \( \max\{|k|, |\ell|\} \leq NT \) and \( A_{k\ell}^{\text{trunc}} = 0 \) otherwise. Set \( \tilde{A} = A - A_{k\ell}^{\text{trunc}}. \) Then \( \|\tilde{A}\|_{\ell^2 \to \ell^2} \approx C(NT)^{-1/2} \) where \( C \) is a fixed constant independent of \( N \) and \( T \).
**Problem**

*Quantify the sense in which the eigenvectors of the matrix \( \tilde{A} \) obtained by truncating \( A_{mk} \) to zero where \( \max\{m, k\} > N \) approximate those of \( A \).*
Observations

- **Shen and Walter**: Sample error for $\varphi_n$ decays like $1/(N\lambda_n)$ with factor depending on *area*
Observations

- **Shen and Walter**: Sample error for $\varphi_n$ decays like $1/(N\lambda_n)$ with factor depending on area.
- **Khare and George**: Truncation up to size $c$ works well for first $c$ eigenvectors . . . empirical.
Observations

- **Shen and Walter**: Sample error for $\varphi_n$ decays like $1/(N\lambda_n)$ with factor depending on area.
- **Khare and George**: Truncation up to size $c$ works well for first $c$ eigenvectors . . . empirical.
- **Levitina and Brandas**: extended to convolution with a prolate.
Observations

- **Shen and Walter**: Sample error for $\varphi_n$ decays like $1/(N\lambda_n)$ with factor depending on area.
- **Khare and George**: Truncation up to size $c$ works well for first $c$ eigenvectors . . . empirical.
- **Levitina and Brandas**: extended to *convolution with a prolate*.
- **Karoui and Moumni**: Replace truncation with Legendre approximations of PSWF samples (Bouwkamp’s method). Generalizable to multiband case?
Observations

- **Shen and Walter**: Sample error for $\varphi_n$ decays like $1/(N\lambda_n)$ with factor depending on area.
- **Khare and George**: Truncation up to size $c$ works well for first $c$ eigenvectors ... empirical.
- **Levitina and Brandas**: extended to *convolution with a prolate*.
- **Karoui and Moumni**: Replace truncation with Legendre approximations of PSWF samples (Bouwkamp’s method). Generalizable to multiband case?
- **Are numerical approximations better than theoretical bounds for $\lambda_n$, small $n$?**
Observations

- Shen and Walter: Sample error for $\varphi_n$ decays like $1/(N\lambda_n)$ with factor depending on area.
- Khare and George: Truncation up to size $c$ works well for first $c$ eigenvectors ... empirical.
- Levitina and Brandas: extended to convolution with a prolate.
- Karoui and Moumni: Replace truncation with Legendre approximations of PSWF samples (Bouwkamp’s method). Generalizable to multiband case?
- Are numerical approximations better than theoretical bounds for $\lambda_n$, small $n$?
- Generalizability to multiband?
Interpolation from truncation eigenvectors
Figure: Interpolation from truncation approximations. $T = 5$, $N = 10$: 21 terms (left). DPSS sequences $[E, V] = \text{dpss}(120, 10)$ (right). Approximation is excellent for PSWFs with $\lambda \approx 1$. 

Joe Lakey (w Scott Izu) 

Time-frequency multiband
Approximation and Sampling in the Multiband case (open problems)
Several methods for periodic nonuniform sampling in $\text{PW}_\Sigma$. 
Several methods for periodic nonuniform sampling in $\text{PW}_\Sigma$.

... Venkataramani and Bresler: interpolating functions in terms of spectral slicing
Several methods for periodic nonuniform sampling in $\text{PW}_\Sigma$.

... Venkataramani and Bresler: interpolating functions in terms of spectral slicing

**Problem 1.** Eigenfunction samples from truncation/interpolation matrix
Several methods for periodic nonuniform sampling in $PW_\Sigma$.

... Venkataramani and Bresler: interpolating functions in terms of spectral slicing

Problem 1. Eigenfunction samples from truncation/interpolation matrix

Problem 2. Quantify approximation rates in terms of an “alias per slice” criterion
Problem

Describe projection onto localized eigenspaces of $P_\Sigma Q^T P_\Sigma$ in terms of samples of eigenvectors in the multiband case.
First steps: Sampling and eigenfunctions

$S \subset \mathbb{R}$ and $\Sigma \subset \mathbb{R}$; $\psi_n$: $\forall f \in \text{PW}_\Sigma$, $f(t) = \sum_n f(x_n)\psi_n(t)$
First steps: Sampling and eigenfunctions

- $S \subset \mathbb{R}$ and $\Sigma \subset \mathbb{R}$; $\psi_n$: $\forall f \in PW_\Sigma$, $f(t) = \sum_n f(x_n)\psi_n(t)$
- Define $g_n(t) = (1_\Sigma)^\vee(x_n - t)$. 
First steps: Sampling and eigenfunctions

- $S \subset \mathbb{R}$ and $\Sigma \subset \mathbb{R}$; $\psi_n$: $\forall f \in PW_\Sigma$, $f(t) = \sum_n f(x_n)\psi_n(t)$
- Define $g_n(t) = (1_\Sigma)^\vee(x_n - t)$.
- $B: B_{nm} = \int_S g_n(t)\psi_m(t)\, dt$
First steps: Sampling and eigenfunctions

- $S \subset \mathbb{R}$ and $\Sigma \subset \mathbb{R}$; $\psi_n$: $\forall f \in PW_{\Sigma}, \ f(t) = \sum_n f(x_n)\psi_n(t)$
- Define $g_n(t) = (1\Sigma)^\vee(x_n - t)$.
- $B: B_{nm} = \int_S g_n(t)\psi_m(t) \, dt$
- Then

$$P_{\Sigma}Q_{\Sigma}f(x_n) = \int_S \left(\sum_m f(x_n)\psi_m(t)\right)g_n(t) \, dt = \sum_m B_{nm}f(x_m)$$
First steps: Sampling and eigenfunctions

- $S \subset \mathbb{R}$ and $\Sigma \subset \mathbb{R}$; $\psi_n: \forall f \in PW_\Sigma$, $f(t) = \sum_n f(x_n)\psi_n(t)$
- Define $g_n(t) = (\mathbb{1}_\Sigma)^\vee (x_n - t)$.
- $B: B_{nm} = \int_S g_n(t)\psi_m(t)\, dt$
- Then

$$P_\Sigma Q_S f(x_n) = \int_S \left( \sum_m f(x_n)\psi_m(t) \right) g_n(t)\, dt = \sum_m B_{nm} f(x_m)$$

- and

$$P_\Sigma Q_S f(t) = \sum_n (P_\Sigma Q_S f)(x_n)\psi_n(t) = \sum_n \left( \sum_m B_{nm} f(x_n) \right) \psi_n(t)$$
Theorem
(Izu) If $\varphi$ is a $\lambda$-eigenfunction of $P_{\Sigma}Q_{S}$ then $\{\varphi(x_{n})\}$ is a $\lambda$-eigenvector of $B$. Conversely, if $v$ is a $\lambda$-eigenvector of $B$ and if $\varphi(t) = \sum_{m} v_{m}g_{m}(t)$ converges then $\varphi$ is a $\lambda$-eigenfunction of $P_{\Sigma}Q_{S}$. 
Next steps . . .: Sampling of multiband signals

- Venkataramani and Bresler: periodic nonuniform sampling; Interpolating functions . . .
Next steps . . .: Sampling of multiband signals

- Venkataramani and Bresler: periodic nonuniform sampling; Interpolating functions . . .
- Many other approaches: Herley and Wong, Behmard Faridani and Walnut, Avdonin and Moran . . .
In multiband case still need
Summary

- In multiband case still need
  - *Analytical* eigenvalue estimates
In multiband case still need
- **Analytical** eigenvalue estimates
- **Numerical** evaluation of eigenfunctions
In multiband case still need
  - Analytical eigenvalue estimates
  - Numerical evaluation of eigenfunctions
  - Sampling projections/approximations
In multiband case still need
- Analytical eigenvalue estimates
- Numerical evaluation of eigenfunctions
- Sampling projections/approximations

Applications to communications
Summary

- In multiband case still need
  - Analytical eigenvalue estimates
  - Numerical evaluation of eigenfunctions
  - Sampling projections/approximations

- Applications to ... communications ...