Time-Frequency localization of Multiband signals
University of Arkansas, November 7, 2008

Joe Lakey (w Scott Izu)\textsuperscript{1}

November 3, 2008
Time and frequency localization

- Fourier transform: \( \hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \xi} \, dt \)
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- $P_{\Sigma} f(x) = (\hat{f} \mathbb{1}_{\Sigma})^\vee(x)$; Paley-Wiener: $\text{PW}_{\Sigma} = P_{\Sigma}(L^2(\mathbb{R}))$

Fundamental Questions:

- Sampling theory of $\text{PW}_{\Sigma}$
- Time localization of $\text{PW}_{\Sigma}$
- $Q_S f(x) = f(x) 1_S(x)$
- $P_{\Sigma} Q_S P_{\Sigma}$: self-adjoint, trace $|S|\Sigma$
- Eigenvalues of $P_{\Sigma} Q_S P_{\Sigma}$: vs $|S|\Sigma$
- Linear distribution of $\Sigma$
- Sample based time-localized approximations
- $\psi_n$ eigenvectors of $P_{\Sigma} Q_S$
- Quantify $\langle f, \psi_n \rangle$ in terms of $\{f(x_k)\}$
- Quantify finite-dimensional approximations
- DFT version of $P_{\Sigma} Q_S$ versus $R_{\ldots}$

Applications in spread spectrum communications...
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P \Sigma Q S \Sigma: self-adjoint, 

\lambda_{\text{max}} = \lambda_0 = \|P \Sigma Q S\| = \sup_{f \in PW, \|f\|_2 = 1} \|Q S (f)\|_2 

Uncertainty principle: \lambda_{\text{max}} < 1 if \|S\|_{\Sigma} < \infty 

Kernel: K(x, t) = 1 S(t)(1 \Sigma) \vee (x - t); 

Trace: \sum \lambda_j = \text{tr}(P \Sigma Q S) = \int K(t, t) dt = \|S\|_{\Sigma}.
$P_\Sigma Q_\Sigma P_\Sigma$: self-adjoint,
Time and frequency localization

\[ P_\Sigma Q_\Sigma P_\Sigma: \text{ self-adjoint,} \]

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- Kernel: $K(x, t) = 1_{S}(t)(1_{\Sigma})^{\vee}(x - t)$;
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$S = [-T/2, T/2]; \Sigma = [-\Omega/2, \Omega/2], \text{tr}P_\Omega Q_T = T\Omega \equiv c.$
Time and bandlimiting I: Prolate spheroidal wave functions

- $S = [-T/2, T/2]; \Sigma = [-\Omega/2, \Omega/2], \text{tr}P_\Omega Q_T = T\Omega \equiv c.$
- Orthonormal eigenfunctions: $P_\Omega Q_T\psi_j = \lambda_j\psi_j$
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- Orthonormal eigenfunctions: \( P_\Omega Q_T \psi_j = \lambda_j \psi_j \)
- \( P_\Omega Q_T \) commutes with

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(T^2 - t^2) \frac{d^2}{dt^2} - 2t \frac{d}{dt} - \Omega^2 t^2
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- Eigenfunctions are Prolate Spheroidal Wave Functions
- Fourier invariance ($\Omega = 1$): $\hat{\psi}_j(\xi / T) = i^{2j+1} \sqrt{T/\lambda_j} Q_T \psi_j(\xi)$,
- Double orthogonality:
  \[
  \int_{-T/2}^{T/2} \psi_j(t) \psi_k(t) \, dt = \lambda_j \delta_{jk}
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  \]
- Reproducing kernel
  \[
  \frac{\sin \pi \Omega (x - t)}{\pi (x - t)} = \sum_{j=0}^{\infty} \psi_j(x) \psi_j(t).
  \]
\[ \lambda_{\text{max}} = \lambda_0 = \| P_\Sigma Q_S \| = \sup_{f \in PW_\Sigma, \| f \| = 1} \| Q_S(f) \|^2 \]
\[ \lambda_{\text{max}} = \lambda_0 = \| P_{\Sigma} Q_S \| = \sup_{f \in \mathcal{PW}_{\Sigma}, \| f \| = 1} \| Q_S(f) \|^2 \]

- \( \psi_0 \): most energy localized in \([-T/2, T/2] \times [-\Omega/2, \Omega/2]\)
Approximately $c = \Omega T$ eigenvalues close to one
Eigenvalue properties

- Approximately \( c = \Omega T \) eigenvalues close to one
- Plunge region of width \( \approx \log c \)
Eigenvalue properties

- Approximately $c = \Omega T$ eigenvalues close to one
- Plunge region of width $\approx \log c$
- Transition about $j = [c]$: $\lambda_{[c]+1} \leq 1/2 \leq \lambda_{[c]-1}$
Figure: Eigenvalues for one frequency interval. \( N = 129 \) point centered DFT. \( S \sim 65 + [-16, 16]; \) \( \Sigma \sim 65 + [-8, 8]. \) \( c = \#T \times \#\Sigma/N = 8.63. \) Plunge region \( \sim 7 \leq n \leq 12. \)
Figure: Even eigenvectors for one frequency interval.

N = 129 point centered DFT. 

S \sim 65 + [−16, 16]; \Sigma \sim 65 + [−16, 16].

c = #T \times #\Sigma / N = 8.44.

Plunge region \sim 7 \leq n \leq 12.

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Time-frequency multiband
Discrete versions: two frequency intervals

Figure: Eigenvalues for one frequency interval. $N = 129$ point centered DFT. $S \sim 65 + [-16, 16]$; $\Sigma \sim 65 + [-8, 8] \cup \pm [24, 32]$. $c = \#T \times \#\Sigma / N = 8.95$. Plunge region $\sim 4 \leq n \leq 15$. 
Figure: Even eigenvectors for one frequency interval.

\[ N = 129 \text{ point centered DFT.} \]

\[ S \sim 65 + [-16, 16]; \Sigma \sim 65 + [-8, 8]. \]

\[ c = \frac{#T \times #\Sigma}{N} = 8.63. \]

Plunge region \( 7 \leq n \leq 12. \)

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Time-frequency multiband
Figure: Eigenvalues for two symmetric intervals. $N = 129$ centered DFT points; $T = 65 + [-32, 32]$ $\Sigma = 65 + [-8, 8] \cup 65 \pm [32, 48]$. Normalized area $\#T \times \#\Sigma / N = 24.69$. Plunge region: $17 \leq n \leq 33$
Figure: Discrete PSWFs, two frequency intervals. Normalized area 24.69
Figure: Random $\Sigma$, $N = 129$. $c = 24.69$. 
Figure: Eigenvalues for random $\Sigma$. $N = 129$, $\Sigma = 24.69$
Figure: Eigenvectors for random Σ. \( N = 129, \, c = 24.69 \)
Figure: Kernel of $P_\Sigma Q_T$, $N = 129$, $c = 24.69$ (real part)
Theorem

(i) For any $c = T > 0$ (and $\Omega = 1$) and $0 < \alpha < 1$ the number $N(\alpha)$ of eigenvalues of $PQ_c$ greater than $\alpha$ satisfies

$$N(\alpha) = c + \left( \frac{1}{\pi^2} \log \frac{1 - \alpha}{\alpha} \right) \log \left( \frac{c}{2} \right) + O(\log c).$$
Theorem

(i) For any $c = T > 0$ (and $\Omega = 1$) and $0 < \alpha < 1$ the number $N(\alpha)$ of eigenvalues of $PQ_c$ greater than $\alpha$ satisfies

$$N(\alpha) = c + \left( \frac{1}{\pi^2} \log \frac{1 - \alpha}{\alpha} \right) \log \left( \frac{c}{2} \right) + O(\log c).$$

(ii) For each $\eta > 0$ there is a $\rho$ such that, as $c \uparrow \infty$

$$\lambda_n(c) \to \begin{cases} 0, & \text{if } n = [(1 + \eta)c], \\ [1 + e^{\pi \rho}]^{-1}, & \text{if } n = [c + \frac{\rho}{\pi} \log \frac{\pi c}{2}], \\ 1, & \text{if } n = [(1 - \eta)c]. \end{cases}$$
Essentially time-and-bandlimited signals

\[ \left\| g \mathbb{1}_{\{|t| > T/2\}} \right\|_2^2 < \varepsilon : \epsilon - T - \text{timelimited} \]
Essentially time-and-bandlimited signals

\[ \| g \mathbb{1}_{\{|t| > T/2\}} \|_2^2 < \varepsilon: \varepsilon-T\text{-timelimited} \]

\[ \| \hat{g} \mathbb{1}_{\{|\omega| > \Omega/2\}} \|_2^2 < \varepsilon: \varepsilon-\Omega\text{-bandlimited} \]
Essentially time-and-bandlimited signals

- $\|g 1_{\{|t|>T/2\}}\|_2^2 < \varepsilon$: $\varepsilon$-$T$-timelimited
- $\|\hat{g} 1_{\{|\omega|>\Omega/2\}}\|_2^2 < \varepsilon$: $\varepsilon$-$\Omega$-bandlimited
- $\mathcal{F}_\varepsilon$: $\|g\| \leq 1$ and $g$ is $\varepsilon$-$T$-timelimited and $\varepsilon$-$\Omega$-bandlimited
Theorem

(Slepian) For $c = \Omega T$ large enough there are $N \approx c$ elements
$\{g_1, \ldots, g_N\}$ of $F_\varepsilon$ such that every element of $F_\varepsilon$ is within $2\varepsilon$ in
$L^2$-norm of some element of the span of $\{g_1, \ldots, g_N\}$.
The $\Omega T$-theorem, Part III: Multiple intervals

- Landau and Widom (1980)
The $\Omega T$-theorem, Part III: Multiple intervals

- Landau and Widom (1980)
- $S, \Sigma$: finite unions of intervals

$N(A_c, \alpha) = c|S| |\Sigma| + N_{\Sigma} N_{S} \pi^2 \log \left( \frac{1 - \alpha}{\alpha} \right) \log c + o(\log c)$

Area $c|S| |\Sigma|$ for $\alpha = 1/2$ in limit.

$\mu\nu$: width of "plunge region"
The $\Omega T$-theorem, Part III: Multiple intervals

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- $\xi \in c\Sigma$: $\xi/c \in \Sigma$
The $\Omega T$-theorem, Part III: Multiple intervals

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- $A_c = P_{c\Sigma}QSP_{c\Sigma}$

$N(A_c, \alpha) = |S|/|\Sigma| + N_{SN\Sigma} \pi^2 \log(1 - \alpha/\alpha) \log c + o(\log c)$

Area $c|S|/|\Sigma|$ for $\alpha = 1/2$ in limit.

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- $\xi \in c\Sigma$: $\xi/c \in \Sigma$
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- $N(A_c, \alpha) = \#\{\lambda(A_c) > \alpha\}$
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- Area $c|S||\Sigma|$ for $\alpha = 1/2$ in limit.
- $\mu \nu$: width of “plunge region”
Plunge width $\sim N_\Sigma N_\Sigma$

- Separated at infinity
Plunge width $\sim N^2 N$ Σ

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- $\psi_j$ frequency concentrated on $l_j$, $|l_j| = 1$
Plunge width $\sim N_S N_\Sigma$

- Separated at infinity
- $\psi_j$ frequency concentrated on $I_j$, $|I_j| = 1$
- $\psi_j(t) = e^{2\pi im_j t} \varphi_j(t) \ m_j = \bar{I}_j$. 

Each $I_j$ gives one eigenvalue $\approx 1/2$

If each $I_j$ were very short: no large eigenvalues.
Plunge width $\sim N_s N_\Sigma$

- Separated at infinity
- $\psi_j$ frequency concentrated on $l_j$, $|l_j| = 1$
- $\psi_j(t) = e^{2\pi im_j t} \varphi_j(t)$ $m_j = \overline{l_j}$.
- $\varphi_j$ frequency concentrated on $[-1/2, 1/2]$. 

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Time-frequency multiband
Plunge width $\sim N_\Sigma N_\Sigma$

- Separated at infinity
- $\psi_j$ frequency concentrated on $l_j$, $|l_j| = 1$
- $\psi_j(t) = e^{2\pi i m_j t} \varphi_j(t)$, $m_j = l_j$.
- $\varphi_j$ frequency concentrated on $[-1/2, 1/2]$.

\[ \langle \psi_j, \psi_k \rangle = \int_{-1/2}^{1/2} e^{2\pi i (m_j - m_k) t} \varphi_j(t) \overline{\varphi_k(t)} \, dt \]
\[ = \hat{\varphi}_1 * \hat{\varphi}_2 * \text{sinc} (m_1 - m_2) = O(1/|m_1 - m_2|) \]
Plunge width $\sim N_S N_\Sigma$

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- Each $l_j$ gives one eigenvalue $\approx 1/2$
Plunge width $\sim N_\Sigma N_\Sigma$

- Separated at infinity
- $\psi_j$ frequency concentrated on $I_j$, $|I_j| = 1$
- $\psi_j(t) = e^{2\pi im_j t} \varphi_j(t)$, $m_j = \bar{I}_j$.
- $\varphi_j$ frequency concentrated on $[-1/2, 1/2]$.

\[
\langle \psi_j, \psi_k \rangle = \int_{-1/2}^{1/2} e^{2\pi i(m_j-m_k)t} \varphi_j(t)\overline{\varphi_k(t)} \, dt \\
= \widehat{\varphi_1} * \widehat{\varphi_2} * \text{sinc}(m_1 - m_2) = O(1/|m_1 - m_2|)
\]

- Each $I_j$ gives one eigenvalue $\approx 1/2$
- If each $I_j$ were very short: no large eigenvalues.
When is area formula still valid?

**Proposition**

Let $\Sigma = [-1/2, 1/2]$ and let $S$ be a union of $m$ pairwise disjoint intervals of total length $c$. Set

$\nu = \max_\alpha \#\{k \in \mathbb{Z} : (k - 1/2, k + 1/2) \subset S + \alpha\}$ and

$\mu = \min_\beta \#\{\ell \in \mathbb{Z} : (\ell - 1/2, \ell + 1/2) \cap S + \beta \neq \emptyset\}$. Then the eigenvalues $\lambda_k$ of $QSP$ satisfy

$$\lambda_{\nu-1} \geq 1/2 \geq \lambda_\mu.$$
Largest concentration for a given area

- Donoho and Stark: if $|\Sigma| = 1$ and $T \leq 0.8$ then ...
Largest concentration for a given area

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\[
\int_{-T/2}^{T/2} |f(t)|^2 \, dt \leq \int_{-T/2}^{T/2} |(\hat{f}^\star(t))^\vee|^2 \, dt.
\]
Largest concentration for a given area

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- Optimal concentration: $\Sigma$ is an interval if $T$ is small enough.
Largest concentration for a given area

- Donoho and Stark: if $|\Sigma| = 1$ and $T \leq 0.8$ then ...

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- Optimal concentration: $\Sigma$ is an interval if $T$ is small enough.

- Rearrangement inequality fails for large measure.
$(S, \Sigma)$ supports information if $\|P_{\Sigma} Q S P_{\Sigma}\| \geq 1/2$. 
Information problem

- $(S, \Sigma)$ supports information if $\|P_\Sigma Q_S P_\Sigma\| \geq 1/2$.
- ... at rate $N$: $N$ eigenvalues $\geq 1/2$
Information problem

- \((S, \Sigma)\) supports information if \(\|Penate\ Q_\Sigma\ P_\Sigma\| \geq 1/2\).
- ... at rate \(N\): \(N\) eigenvalues \(\geq 1/2\)
- Rationale: basis functions \(\sim\) codes
- Which pairs support information?
Theorem

(Candès, Romberg, Tao) Fix $N \geq 512$ and $\beta$ such that $1 \leq \beta \leq (3/8) \log N$. Suppose that $S$ and $\Sigma$ are subsets of $\mathbb{Z}_N$ whose sizes satisfy

$$|S| + |\Sigma| \leq M(N, \beta) = \frac{N}{\sqrt{(\beta + 1) \log N}} \left( \frac{1}{\sqrt{6}} + o(1) \right).$$

Then with probability at least $1 - O((\log N)^{1/2} / N^\beta)$, every signal $x$ frequency supported in $\Sigma$ satisfies

$$\|x 1_S\|^2 \leq \frac{1}{2} \|x\|^2.$$
Problem

Fix $M_1 < M_2$ and $K_1 < K_2$ and suppose that $\Sigma \subset \mathbb{Z}_N$ is a union of $M$ intervals, $M_1 \leq M \leq M_2$ of total length $|\Sigma|$, $K_1 \leq |\Sigma| \leq K_2$ and that $S$ is a single interval of length $|S| \leq N_1 < N$, but that $S$, $\Sigma$ are otherwise “random.” Quantify the proportion of such rectangles such that $\| A_{S\Sigma} \|^2 \geq 1/2$. 
The entropy of a partition $\mathcal{P}$ of a probability space $(\mathcal{X}, \mathcal{B}, \mu)$ is

$$E(\mathcal{P}) = -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P).$$

**Problem**

Let $S \sim [-T/2, T/2]$ and let $|\Sigma|$ and hence $|S||\Sigma|$ be fixed in the finite time-frequency plane. Establish a quantitative, probabilistic relationship between the entropy of $\Sigma$ and $|A S \Sigma|$. 
Sampling and Time-Frequency localization
Theorem

(Shen and Walter; Khare and George)

\[ \varphi_n \sim \lambda_n \text{ of } PQ_T P. \text{ Then} \]

\[ \lambda_n \varphi_n(m) = \sum_k A_{mk} \varphi_n(k) \]

where the doubly-infinite matrix \( A \) has entries \( A_{mk} \) given by

\[ A_{mk} = \int_{-T/2}^{T/2} \text{sinc}(t - m) \text{sinc}(t - k) \, dt. \]

i.e. samples of \( \varphi_n \) form n-th eigenvector of \( \{A_{mk}\} \).
Sampling and eigenfunctions: $\Omega T$ case

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Sampling and eigenfunctions: $\Omega T$ case

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Further discrete observations:

- Orthogonality of sample vectors:

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- Reproduction in sample space
  \[
  \sum_{n=0}^{\infty} \varphi_n(m) \varphi_n(\ell) = \delta_{m,\ell}.
  \]
Problem

Quantify the sense in which the eigenvectors of the matrix $\tilde{A}$ obtained by truncating $A_{mk}$ to zero where $\max\{m, k\} > N$ approximate those of $A$. 
Interpolation from truncation eigenvectors
**Figure:** *Interpolation from approximations.* Interpolation of samples of truncation eigenvectors, $T = 5, N = 10$: 21 terms (left) and corresponding DPSS sequences $[E, V] = \text{dpss}(120, 10)$ (right)
Figure: *Errors vs interpolants.* Interpolant error of $2i$-th vectors between $T = 5$, $N = 10$ and $N = 21$ (left) and $2i + 2$-th vector magnitude (right)
Problem

*Describe projection onto localized eigenspaces of $P_{\Sigma} Q_T P_{\Sigma}$ in terms of samples of eigenvectors in the multiband case.*
Sampling and eigenfunctions: General setup

- $S \subset \mathbb{R}$ and $\Sigma \subset \mathbb{R}$; $\psi_n$: $\forall f \in \text{PW}_\Sigma$, $f(t) = \sum_n f(x_n)\psi_n(t)$
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- Then

\[
P_\Sigma Q_S f(x_n) = \int_S \left( \sum_m f(x_n)\psi_m(t) \right) g_n(t) \, dt = \sum_m B_{nm} f(x_m)
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- and

$$P_\Sigma Q_S f(t) = \sum_n\left( P_\Sigma Q_S f(x_n) \right) \psi_n(t) = \sum_n \left( \sum_m B_{nm} f(x_n) \right) \psi_n(t)$$
Theorem

If \( \varphi \) is a \( \lambda \)-eigenfunction of \( P_{\Sigma}Q_T \) then \( \{\varphi(x_n)\} \) is a \( \lambda \)-eigenvector of \( B \). Conversely, if \( \mathbf{v} \) is a \( \lambda \)-eigenvector of \( B \) and if
\[
\varphi(t) = \sum_m \nu_m g_m(t)
\]
converges then \( \varphi \) is a \( \lambda \)-eigenfunction of \( B \).
Sampling of multiband signals

- Many approaches: Herley and Wong, Behmard Faridani, Venkataramani Bresler ...
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- **Venkataramani and Bresler**: periodic nonuniform sampling; Interpolating functions
How to construct sampling/interpolating functions in the multiband setting?
Interpolation and Localization

▶ How to construct sampling/interpolating functions in the multiband setting?
▶ Consistent with localization in time?
Venkataramani and Bresler’s approach

\[ \Sigma = \bigcup_{k=1}^{m} [a_k, b_k] \text{ where } b_k < a_{k+1} \]
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- $\Sigma = \bigcup_{k=1}^{m} [a_k, b_k]$ where $b_k < a_{k+1}$
- $a_1 = 0$ and set $\tau = 1/b_m$: $\text{PW}\Sigma \subset \text{PW}[0,1/\tau]$
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- $n$-th term
  
  \[
  s_{kn} = \begin{cases} 
  s_n, & n = k + mL \\
  0, & \text{else}
  \end{cases}
  \]
Set $Q = 1/(L\tau)$: samples per slice per unit time.
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$Q$-periodic reversed Fourier series of $s_k$

$$S_k(\xi) = \sum_{n} s_{kn}e^{2\pi in\tau\xi} = Q \sum_{\ell=0}^{L-1} \hat{f} (\xi + \ell Q) e^{-2\pi ik\ell Q\tau}, \xi \in [0, Q).$$
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- Reconstruct from $P$ out of $L$ cosets $s_k$: $P/Q \gtrapprox |\Sigma|$: 
Set $Q = 1/(L\tau)$: samples per slice per unit time.

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- Reconstruct from $P$ out of $L$ cosets $s_k$: $P/Q \gtrapprox |\Sigma|$

- approximately $|\Sigma|$ samples per unit time
\[ 0 = \gamma_0 < \gamma_1 < \cdots < \gamma_M; \ M \leq 2m \]
Spectral slicing

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- endpoint moduli of $\Sigma$ modulo $Q$
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Spectral slicing

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- Endpoint moduli of $\Sigma$ modulo $Q$
- $\Gamma_\mu = [\gamma_{\mu-1}, \gamma_\mu]$.
- $\Gamma_\mu + \ell Q \subset \Sigma$ or $\Gamma_\mu + \ell Q \cap \Sigma = \emptyset$
Interpolating functions

\[ P = \max_{\mu=1,\ldots,M} \# I_\mu. \]
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\[
\hat{f}(\cdot + \ell Q) = \frac{1}{Q} \sum_{k=0}^{P-1} (F_\mu)_{\ell k}^{-1} S_k \quad \text{on } \Gamma_\mu
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- Inverse Fourier transforming:

$$f(t) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{P-1} f\left((k + nL)\tau\right) \varphi_k(t + nL\tau)$$
Interpolating functions

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Figure: Multiband spectrum $\Sigma \subset [0, 5]$
Figure: Pullback of multispectrum, $L = 5$, $T = 1/5$, $1/(LT) = 1$
Issues with spectral slicing: time-frequency localization.

- Choices of “$P$ out of $L$” and $F_{\mu}^{-1}$: noise/aliasing

\[ \Sigma = [0, \frac{1}{p}] \cup [1 - \frac{1}{q}, 1] \]

- Landau rate $|\Sigma| = \frac{p + q}{p q}$.

- Slicing rate $L = pq$ may be large. A spectral bin $\Gamma_\mu$ typically will have length on the order of $\frac{P}{m Q} \approx \frac{|\Sigma|}{m}$.

- $I \times J$ has to satisfy $||I|| ||J|| \geq 1$ to support information.

- Poor time localization of the $\phi_k$ is expected.
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Problem

Assuming $\Gamma_\mu T \geq 1$ for all $\mu = 1, \ldots, M$, find conditions under which the interpolating functions lie, or nearly lie, in the span of the first $|\Sigma| T$ eigenfunctions of $P_\Sigma Q_T$. 
In multiband case still need
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Applications to ... communications ...