Analytic Methods for Image Processing
January, 2008

Joe Lakey

January 23, 2008
Corruption and redemption

- Images as **Edges** plus **textures** plus **backgrounds**
Corruption and redemption

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- **Layers** complicate matters.
Corruption and redemption

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- Images are **corrupted** by
  - **geometric distortion**, 
  - **blurring**, 
  - **occlusion and noise**
- Images are restored by
  - **geometric transformations**, **registration** and other **editing**, 
  - **deblurring**, 
  - **inpainting** and **denoising**
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Math Methods in Image Processing

- **Segmentation**, including edge detection, histograms, partitioning/clustering
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- Mathematical **morphology** (erosion, dilation etc, lattice theory and topology)
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- PDE based methods: minimize some energy functional
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- **Compressed sensing** (finding right transformation)
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- **Hybrids**
Image statistics

- Distribution of pixel (gray) intensities
Image statistics

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- Histogram tells something
Image statistics

▶ Distribution of pixel (gray) intensities
▶ Histogram tells something
▶ Natural versus synthetic images
Image statistics

- Distribution of pixel (gray) intensities
- Histogram tells something
- **Natural** versus **synthetic** images
- Can one differentiate between **texture** and **noise**?
Granite histograms
Some natural images
Some synthetic images
Facets of Image Restoration
Mathematical Methods
Transform based compression and denoising
Compare and Compress

Image statistics

Symmlet; j=2, k=(1,1); FxF

Symmlet; j=2, k=(2,2); MxF

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Computational Methods
Transform based image compression

- Orthogonal transformation
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- Hard thresholding (compression): discard coefficient below threshold (percent energy)
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- Soft thresholding: decrease in proportion to magnitude
Transform based image compression

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- What comprises a good basis?
Fourier transform

Fast Fourier Transform
Fourier transform

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- Implementation of DFT: \( \frac{1}{\sqrt{N}} e^{2\pi i (j-1)(k-1)/N} \)
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- Elements not spatially localized
Facets of Image Restoration
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Fourier transform
Wavelet transform
Curvelet transform

real part of 128x128 Fourier matrix
Wavelet transforms

- Wavelet ONBs: \( \psi_{jk}(x) = 2^{j/2} \psi(2^j x - k) \)
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- Unconditional bases (best) for a variety of function spaces
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- Can have spatial (support) and frequency (smooth) localization
- 2 variables: tensor products or otherwise
- Unconditional bases (best) for a variety of function spaces
- Point singularities but not curves
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Fourier transform
Wavelet transform
Curvelet transform
Approximation in function spaces

- Rate of Approximation in $L^p$
Approximation in function spaces

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\[ \Sigma_N(B) = \left\{ \sum \alpha_n \varphi_n : \# \{ n : \alpha_n \neq 0 \} \leq N \right\} \]
Approximation in function spaces

- Rate of Approximation in $L^p$

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- $A^s_X = \{ f : N^s \sigma_N(f, B, \| \cdot \|_X) \to 0 \}.$
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- Stečkin numbers

\[ d_N(F, B) = \sup_{f \in F, \| f \|=1} \| f - S_N(f, B, \| \cdot \|_{L^2}) \|_{L^2}. \]
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\]

- Donoho’s heuristic: orthogonal, unconditional basis for $\mathcal{F}$ is minimax best
## Function space comparisons

<table>
<thead>
<tr>
<th>Space</th>
<th>Optimal basis</th>
<th>$d_n(\mathcal{F}, \mathcal{B})$</th>
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<td>$L^2$-Sobolev</td>
<td>$W_2^m$</td>
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<td>$\dot{B}_{1,1}^1$</td>
<td>wavelet</td>
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<td>$O(n^{-m})$</td>
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Computational Methods
Curvelet transforms

- Use spherical polar decomposition in frequency
Curvelet transforms

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- Meyer wavelet $\psi_{j,k}$ in scale,
Curvelet transforms

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- Meyer wavelet $\psi_{j,k}$ in scale,
- Periodic wavelet $w_{i,\ell}$ in angle
Curvelet transforms

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- Meyer wavelet $\psi_{j,k}$ in scale,
- Periodic wavelet $w_{i,\ell}$ in angle
- Ridgelets

$$\hat{\rho}_\lambda = \frac{1}{2} \frac{1}{\sqrt{|\xi|}} \left( \hat{\psi}_{j,k}(|\xi|)w_{i,\ell}(\theta) + \hat{\psi}_{j,k}(-|\xi|)w_{i,\ell}(\theta + \pi) \right)$$
Curvelet scheme

Subband code: $f \mapsto (P_0 f, Q_1 f, Q_2 f, \ldots)$ (Littlewood-Paley)
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- Smooth (angular) partitioning $h_R = w_R Q_s f$
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- Renormalization \( g_R = T_R^{-1} h_R \)
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- Ridgelet analysis $\alpha_{R,\lambda} = \langle g_R, \rho_{\lambda} \rangle$. 

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- Synthesis: reverse the steps.
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- Ridge: aspect ratio width $\approx$ length$^2$. 

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Fourier transform
Wavelet transform
Curvelet transform

Ridge in Square

It's Fourier Transform

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Fourier transform
Wavelet transform
Curvelet transform

Ridgelet Tiling

Fourier Transform within Tiling
Curvelet approximations

- Geometric curve $\Gamma \subset [0, 1]^2$
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Curvelet approximations

- Geometric curve $\Gamma \subset [0, 1]^2$
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- Adapted triangular wedges: $N$-term $O(N^{-2})$. Best possible.
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- Caveat: (adaptive) Mallat wavelets
Curvelet approximations

- Star-shaped objects with $C^2$ boundary
Curvelet approximations

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- Radius function $\rho(\theta)$, $0 \leq \theta \leq 2\pi$, $|\rho''| \leq C$
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- $\text{Star}^2(C)$: characteristic functions of star-shaped sets.
Curvelet approximations

- Star-shaped objects with $C^2$ boundary
- Radius function $\rho(\theta)$, $0 \leq \theta \leq 2\pi$, $|\rho''| \leq C$
- $\text{Star}^2(C)$: characteristic functions of star-shaped sets.
- Donoho, Johnstone (1995): any \textit{polynomial depth dictionary} (roughly finite parameter), the $N$-th term of any $\text{Star}^2(C) \geq \sim c/(N^2 \log N)$. 
Comparison for a texture

Zoom in of brodatz3

Zoom in of 2 pct Fourier brodatz3

Zoom in of 2 pct wavelet brodatz3

Zoom in of 2 pct curvelet brodatz3
2 % Fourier Compressions
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Transform based compression and denoising
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original image

reconstruction from large Fourier coefficients

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Side by side
Facets of Image Restoration
Mathematical Methods
Transform based compression and denoising
Compare and Compress

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Computational Methods
2 % Curvelet Compressions
Conclusions

- Several tools (orthogonal bases, dictionaries)
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- Compressibility depends on structure
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- Function spaces $\Leftrightarrow$ Optimality $\Leftrightarrow$ Asymptotic
Next time: Edge detection and Denoising

- Edge detection
Next time: Edge detection and Denoising

- Edge detection
- Soft thresholding
Next time: Edge detection and Denoising

- Edge detection
- Soft thresholding
- Modeling objects and textures (PDE)
Next time: Edge detection and Denoising

- Edge detection
- Soft thresholding
- Modeling objects and textures (PDE)
- Solvability