Real Analysis
Group Assignment 1
Fall 2004

Instructions: Turn in a solution of each problem. One solution per group. Three or four individuals per group. Each individual must write up at least two problems.

1. Let \( s \) denote the sum of the series \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \). Express the sum of the series \( 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \ldots \) in terms of \( s \). Do the same for the series \( 1 - \frac{1}{2} - \frac{1}{6} + \frac{1}{8} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \ldots \). Do the same for \( 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \frac{1}{13} + \ldots \).

2. Show that \( \sum_{n=1}^{\infty} \frac{3n - 2}{n(n+1)(n+2)} = 1 \).

3. Let \( \{r_n\} \) denote the rational numbers in \((0,1)\) enumerated in the form \( \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \ldots \). Show that \( \sum_{n=1}^{\infty} r_n \) diverges. Is there any enumeration of \( \mathbb{Q} \cap (0,1) \) whose corresponding series converges? Explain.

4. Determine whether each of the following sequences has a convergent series:
   \( a_k = \sqrt[k+1]{k+1} - \sqrt[k]{k} \); \( b_k = \sqrt{k} a_k \).

5. Show that if \( f(t) \) is monotonically decreasing and if \( c_n = \sum_{k=1}^{n} f(k) - \int_{1}^{n} f(t) \, dt \) then \( \lim_{n \to \infty} c_n \) exists.

6. Show that if \( \sum_{k=1}^{\infty} a_k^2/k \) converges then \( \frac{1}{N} \sum_{k=1}^{N} a_k \) tends to zero as \( N \to \infty \).

7. Determine the values of \( x \) for which the following series converge.
   (a) \( \frac{x}{2} + \frac{4x^2}{9} + \frac{9x^3}{28} + \frac{16x^4}{65} + \ldots \)
   (b) \( 1 + \frac{x}{3} + \frac{2x^2}{9} + \frac{(2\cdot3)x^3}{27} + \frac{(2\cdot3\cdot4)x^4}{81} + \ldots \)
   (c) \( \sum_{k=1}^{\infty} \frac{x^k(1-x)^k}{k} \)
   (d) \( \sum_{k=1}^{\infty} \sin \frac{x}{k} \)

8. Compute \( \sum_{n=3}^{\infty} \frac{1}{n^2-4} \) (Hint: \( 1 = \frac{1}{1} - \frac{1}{5} + \frac{1}{5} - \frac{1}{9} + \frac{1}{9} - \frac{1}{13} + \ldots \)).

9. Show that \( \sum_{n=1}^{\infty} \frac{1}{n^2+3n-4} = \frac{137}{300} \).

10. How many terms of the series \( S = \sum_{n=0}^{\infty} (-1)^n/(n+1)^2 \) are needed to approximate \( S \) accurate to 4 decimal places? Provide a value for this approximation.

11. Evaluate and describe convergence of the infinite product \( (1 - \frac{1}{4})(1 - \frac{1}{9})(1 - \frac{1}{16}) \cdots \)

12. Evaluate and describe convergence of the ‘continued radical’
\[ \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}} \]