1. (a) For which values of $c$ is $y(t) = \sin(t + c)$ a valid solution of the differential equation $y' = \sqrt{1 - y^2}$ having initial value $y(0) = 0$? (b) define the function

$$y(t) = \begin{cases} \sin x, & |x| \leq \pi/2 \\ 1, & x > \pi/2 \\ -1, & x < -\pi/2 \end{cases}$$

Is this function also a solution of $y' = \sqrt{1 - y^2}$? Does your conclusion contradict the uniqueness theorem?

2. Suppose that $y = f(t)$ is a solution of the differential equation $y' = y$ satisfying $y(0) = 0$. Without using prior knowledge that the solution is an exponential function, show that $f(s + t) = f(s)f(t)$ for all $s, t$.

3. Find solutions of the following differential equations satisfying $y(0) = 1$. In each case state, with justification, whether it is possible that other solutions exist. (i) $y' = y/x$ (ii) $y' = -xy$ (iii) $(x^2 + y^2)' = 0$ (iv) $y' = \sec y$ (v) $y' = -x/y$.

4. One end of a one-foot rubber band is attached to a wall while the other is stretched away from the wall at one foot per minute. Starting at the same time as the stretching begins, a bug starts to crawl from the wall along the rubber band at a constant rate of $b$ feet per minute relative to the band. Will the but ever reach the other end of the band and, if so, how will its arrival time depend on $b$? How will the solution change if the bug is already $x_0$ feet from the wall on the band?

5. Consider the following special case of Lotka-Volterra equations

$$\frac{dH}{dt} = -bHP; \quad \frac{dP}{dt} = bHP - cP$$

in which $b, c$ are positive constants. Give an interpretation of these equations in terms of spread of a disease between a healthy but susceptible subpopulation $H$ versus a contaminated population $P$ that can spread the disease through contact. Give interpretations of the constants $b, c$ in terms of rates of spread. You may assume the infected individuals are removed from the population either through death or through recovery with permanent immunity. Explain why $\dot{H} + \dot{P} = (c/b)\dot{H}/H$. Integrate this equation with respect to $t$. Show that the graph of $P = P(H)$ is concave down (assuming $P \geq 0$ and $H \geq 0$) with maximum value when $H = c/b$. Give a physical interpretation of this fact.
6. Two tanks, one of capacity 100 gallons and the other of capacity 200 gallons are initially full of fluid. The 100 gallon tank starts with nothing but pure water while the other starts with 10 lbs of salt dissolved in water. Solution flows through a pipe from the 100 gallon tank to the other at a rate of 1 gallon per minute while solution flows from the 200 gallon tank to the 100 gallon tank at a rate of 2 gallon per minute. Both tanks are allowed to overflow into a drain and the process terminates if one of the containers is empty. Which tank empties first, and at what time? Write down a system of differential equations and initial conditions whose solution describes the amount of salt in each tank at any given time. Show that the rate of decrease of the total amount of salt from the system at any time is equal to the concentration of salt in the 100 gallon tank.

7. Consider the motion of a particle fixed at a point on a rotating turntable having constant angular velocity \( \omega \) about \((0,0)\). The motion of the particle can be described in terms of centrifugal acceleration \( \omega^2(x,y) \) directed away from the center of rotation and a Coriolis acceleration perpendicular to the instantaneous direction of the particle’s motion and given by \( 2\omega(y, -x) \). Under these assumptions, give the following justifications for considering these two types of acceleration: (i) Show that \( x \) and \( y \) satisfy the system \( \ddot{x} = \omega^2x + 2\omega\dot{y} \) and \( \ddot{y} = \omega^2y - 2\omega\dot{x} \). (ii) Show that if we let \( x = x + iy \) the system can be written in complex coordinates as \( \ddot{z} + 2i\omega\dot{z} - \omega^2z = 0 \). (iii) Show that this equation has complex solutions \( z(t) = (c_1 + c_2t)e^{-i\omega t} \) and hence that the initial value problem \( x(0) = R; y(0) = 0 \) and \( \dot{x}(0) = u; \dot{y}(0) = v \) has solutions given in real form by

\[
\begin{align*}
  x(t) &= (R + ut) \cos \omega t + (v + \omega R)t \sin \omega t \\
  y(t) &= -(R + ut) \sin \omega t + (v + \omega R)t \cos \omega t
\end{align*}
\]

(iv) Conclude that motion relative to the turntable can be regarded as linear motion having constant velocity vector \( \vec{v} \) but with a correction for the rotation of the turntable. Express this velocity in terms of \( R, \omega, u, v \).

7. A projectile fired against a force of air resistance proportional to velocity (really speed) satisfies the uncoupled system

\[
\begin{align*}
  \ddot{x} &= -\frac{k}{m} \dot{x} \\
  \ddot{y} &= -\frac{k}{m} \dot{y} - g
\end{align*}
\]

Solve this system subject to each of the following initial conditions:

\[
x(0) = 0; \quad y(0) = 0; \quad \dot{x}(0) = z_0 > 0; \quad \dot{y}(0) = w_0 > 0
\]

Show that the trajectory rises to a unique maximum at \( t_{\text{max}} = (m/k) \ln(1 + kw_0/mg) \). Compute the coordinates \((x_{\text{max}}, y_{\text{max}}) = (x(t_{\text{max}}), y(t_{\text{max}}))\).

8. During World War I, Paris was bombarde by a gun located 75 miles away using shells that arrived in Paris 186 seconds after firing. Air resistance was negligible during a substantial
part of the 25 mile high trajectory. Using the gun’s maximum powder charge, a shell could
attain a muzzle velocity as high as 5500 feet per second. Since the shells had to be relatively
small to attain this high velocity they did not cause an enormous amount of physical damage.
Their purpose was mainly psychological. Neglecting air resistance altogether and using
\( g = 32 \text{ feet per second}^2 \), formulate a pair of differential equations for the \( x \) and \( y \) coordinates
of the shell. Use these equations to estimate the gun’s elevation angle \( \theta \), the muzzle velocity \( v_0 \)
and the maximum trajectory height. On the other, assuming a retarding force of magnitude
equal to a function \( a(y) > 0 \) of altitude \( y \) above ground times projectile speed \( v = (\dot{x}^2 + \dot{y}^2)^{1/2} \)
formulate a pair of equations satisfied by \( x \) and \( y \). Denote by \( m \) the mass of the shell. Solve
the equations under the hypothesis that \( a(y) \) is constant to obtain the muzzle velocity and
elevation angle. Compare these results to those in which air resistance was neglected.

9. According to Archimedes’ principle, the motion of a buoy in water is governed by the
equation

\[
m \frac{d^2 y}{dt^2} = -\rho S y
\]

where \( \rho \) is the mass density of water, \( S \) is the cross sectional area and \( y \) is the displacement
from its natural floating position. Assume that water weights 62.5 pounds per cubic foot. A
64 pound cylindrical buoy with a 2 foot radius floats in water with its axis vertical. Assume
the buoy is depressed 1/2 foot from equilibrium and then released. Find the amplitude and
period of the oscillations of the buoy.

10. Consider the differential equation \( \ddot{x} + 25x = 16 \cos 3t \). Find the general solution of this
equation. Verify that the solution satisfying \( x(0) = 0 = \dot{x}(0) \) is \( x_f = \cos 3t - \cos 5t \). Verify
that this can be rewritten \( 2 \sin 4t \sin t \). What is the beat frequency of this solution?

11. Find the eigenvectors and eigenvalues for the linear system

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = A \begin{bmatrix}
x \\
y
\end{bmatrix}
\]

for each of the following choices of \( A \):

\[
A_1 = \begin{bmatrix}
2 & 1 \\
-1 & 4
\end{bmatrix}; \quad A_2 = \begin{bmatrix}
1 & 4 \\
-3 & 2
\end{bmatrix}; \quad A_3 = \begin{bmatrix}
-2 & -3 \\
-3 & -2
\end{bmatrix}
\]

In each case, write the general solution of the system.

12. In each of the following systems, determine whether the system is Hamiltonian and if
so, determine the Hamiltonian function. (i) \( \dot{x} = x - 3y^2, \dot{y} = -y \), (ii) \( \dot{x} = \sin x \sin y, \dot{y} = -\cos x \sin y \), (iii) \( \dot{x} = x \cos y, \dot{y} = -y \cos x \), (iv) \( \dot{x} = 1, \dot{y} = y \).