Some applications of graph theory
More on the Four Color Problem
• The four color theorem was stated, but not proved, in 1853 by Francis Guthrie.
• The theorem asserts that four colors are enough to color any geographical map in such a way that no neighboring two countries are of the same color.
Other applications?

- Beyond the mathematical issues in connection with this theorem, coloring question have wider practical applications especially for mobile phones. They make it possible to effectively reduce the number of used broadcasting frequencies, equivalent to the colors.
A long road...

• Tait's conjecture:(1886) Every polyhedron has a Hamiltonian cycle

• ============== (along the edges) through ALL the vertices.
Polyhedra

• The Steinitz theorem (1894) for polyhedra states that every simple 3-connected planar graph is a polyhedral graph of a 3-dimensional polytope and vice versa.
If Tait’s conjecture were true...

- it would prove the 4-color theorem:
- any planar trivalent graph with a Hamiltonian cycle has a 3-coloring of the edges, which is equivalent to having a 4-coloring of faces.

Proof: color the Hamiltonian cycle alternately yellow and red. All other edges are diagonals of this polygon, & can be blue.

- code the edge colors as (0,1) (1,0) and (1,1); then color any face (0,0), and color the rest by crossing over edges using vector

- addition [mod 2]. This gives a face-coloring in (0,0) (0,1) (1,0) & (1,1).
Tutte’s counterexample

• A non-Hamiltonian 3-connected cubic graph given by Tutte (1946) as a counterexample to Tait's Hamiltonian graph conjecture using three copies of the Walther graph.

Four Colours Suffice

How the Map Problem Was Solved

Robin Wilson


Kenneth Appel and Wolfgang Haken in the 1970s

Wolfgang Haken, Robin Wilson & Kenneth Appel in October 2002
Appel and Haken’s proof (1972)

• reduced the infinitude of possible maps to 1,936 reducible configurations (later reduced to 1,476). These had to be checked one by one by computer ...

• This reducibility part of the work was independently double checked with different programs and computers. ...

• the unavoidability part of the proof was verified in over 400 pages of microfiche, which had to be checked by
Proof by ... computer

• Automated theorem proving has become a cottage industry.
• Human beings are still required to give meaning to what the computers do...
• Curiously, “proof assistants” are also used in studying software reliability.
Recent work on map coloring
Finding the shortest path
• Path from vertex V to vertex W
• f: edge weight function f(p)
• Minimize sum of f(p) over all possible paths
• Shortest path algorithms are applied in an obvious way to automatically find directions between physical locations, such as driving directions on web mapping websites like Mapquest.
• For example, if vertices represent the states of a puzzle like a Rubik's Cube and each directed edge corresponds to a single move or turn, shortest path algorithms can be used to find a solution that uses the minimum possible number of moves
• The **traveling salesman problem**
• find shortest path that goes through every vertex exactly once, and returns to the start.
• Unlike the shortest path problem, this problem is **NP-complete** and is believed not to be efficiently solvable
Dijkstra’s algorithm
• Dijkstra’s
• the running time is $O(|V|^2 + |E|) = O(|V|^2)$. 
Dijkstra’s algorithm

• Create distance list, previous vertex list, visited list, and current vertex.
• All values in distance list set to infinity except starting vertex to zero.
• All values in visited list are set to false.
• All values in the previous list are set to a special value signifying that they are undefined, such as null.
• Current vertex is set as the starting vertex.
• Mark the current vertex as visited.
• Update distance and previous lists based on those vertices which can be immediately reached from the current vertex.
• Update the current vertex to the unvisited vertex that can be reached by the shortest path from the starting vertex.
• Repeat (from step 6) until all nodes are visited.
function Dijkstra(Graph, source):
  2: for each vertex v in Graph: // Initialization
  3: dist[v] := infinity // initial distance from source to vertex v is set to infinite
  4: previous[v] := undefined // Previous node in optimal path from source
  5: dist[source] := 0 // Distance from source to source
  6: Q := the set of all nodes in Graph // all nodes in the graph are unoptimized - thus are in Q
  7: while Q is not empty: // main loop
    8: u := node in Q with smallest dist[ ]
    9: remove u from Q
  10: for each neighbor v of u: // where v has not yet been removed from Q.
   11: alt := dist[u] + dist_between(u, v)
    12: if alt < dist[v] // Relax (u,v)
       13: dist[v] := alt
       14: previous[v] := u
  15: return previous[ ]
Dijkstra demo on youtube
Dijkstra applets
SPA: the pineapple express
[Find the shortest path from Honolulu to London]

A. [Honolulu -> Chicago -> Boston -> London]
B. [Honolulu -> SF -> NY -> London]
C. [Honolulu -> LA -> NY -> London]
D. [Honolulu -> LA -> ATL -> London]
Traveling salesman problem

The case of Sweden
The Traveling Salesman Problem
World tour
Social Networks, Part II

• Every social network can be expressed as a graph.

• For example, consider the graph whose vertices are people and they are joined by an edge if the people have shared a handshake.

• What is the average path distance along this graph to the president of the United States? What is the maximum path distance?
• Small world graphs
Obama is 6'1 and was the 44th president of the US
• What is the probability of being two handshakes away from the president?
• President shakes hands about 65,000 times a year (100-250 per day).
• Assuming that most of the people the President meets are Americans, that means that about 1 in 4,615 Americans meets the President every year (300 million divided by 64,875).

• Warning: it’s not different people every time!
Thanks to George Hickenlooper, your instructor is two handshakes from the former president.
How many handshakes is your math teacher Barack Obama?

- A) 1 – he has shaken hands with Obama
- B) 2 – he has shaken hands with someone who has shaken hands with Obama
- C) at least three – he has never shaken hands with anyone who has shaken hands with Obama
Trump and Lakey
Lakey and Bill
The Venkers
Schlafly and Trump
Schlafly and Venker
So…

- Lakey -> Venker -> Schlafly -> Trump,
- Possibly…
- Lakey -> Venker -> Trump
How many handshakes is Lakey from Donald Trump

• A) He has shaken hands with Trump
• B) He may have shaken hands with someone who has shaken hands with Donald Trump
• C) He has definitely shaken hands with someone who has shaken hands with someone who has shaken hands with Donald Trump
• Small world graphs
What’s your Kevin Bacon number?

• Bela Lugosi, has a Bacon number of 3:
  • 1. Bela Lugosi was in Abbott and Costello Meet Frankenstein (1948) with Vincent Price
  • 2. Vincent Price was in The Raven (1963) with Jack Nicholson
  • 3. Jack Nicholson was in A Few Good Men (1992) with Kevin Bacon
A tiny portion of the movie-performer relationship graph
Cinema Finder

• Problem: Find an actor who has as large a Kevin Bacon number as possible
• http://oracleofbacon.org/
Which actor has the highest Kevin Bacon number

• A) Tom Hanks
• B) Will Smith
• C) George Hickenlooper
• D) Meryl Streep
• E) Leanne Lakey
What’s your Erdös number?

• Lakey has an Erdös number of 3:
  • 1. Lakey coauthored with Guido L. Weiss
  • 2. Weiss coauthored with Charles C.K. Chui
  • 3. Chui coauthored with Paul Erdös
Lakey’s Erdös number

• A) 1, Lakey wrote a paper with Erdös
• B) 2, Lakey wrote a paper with Charles C.K. Chui who wrote a paper with Erdös
• C) 3, Lakey wrote a paper with Guido Weiss, who wrote a paper with Charles C.K. Chui who wrote a paper with Erdös
• D) At least 4
Exercise:

• Draw a wheel graph that has 10 vertices in which each vertex is connected with its “nearest neighbor” and its “next nearest neighbor by an edge.

• What is the shortest path between vertex 1 and vertex 6? What is the maximum path distance between any two vertices on the graph?

• For each vertex \( n = 1 \cdots 5 \), flip a coin. If it comes up tails leave the connection with vertex \( n+1 \). If it comes up heads, change the edge between vertices \( n \) and \( n+1 \) to a an edge between vertex \( n \) and \( n+5 \).

• Find the maximum path distance between any two vertices on the new graph.