Algebras in type-2 fuzzy sets

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Type-1 fuzzy sets

$X = \{50, 60, 70, 80, 90\}$

A type-1 fuzzy subset of $X$ is a map $\text{COLD} : X \rightarrow [0, 1]$

The expert’s belief that 60 is cold is 0.8.
This is a map $\text{COLD} : X \rightarrow \{(a, b) \in [0, 1]^2 : a \leq b\}$.

The expert’s belief that 60 is cold is between $[0.6, 0.9]$. 

![Graph showing intervals for coldness]
Type-2 fuzzy sets

A type-2 fuzzy subset is $\text{COLD} : X \rightarrow \{f | f : [0, 1] \rightarrow [0, 1]\}$
Truth value algebras

The truth value algebras for fuzzy sets, interval valued fuzzy sets, and type-2 fuzzy sets are

\[ I = [0, 1] \]

\[ I^{[2]} = \{(a, b) : a \leq b \in I\} \]

\[ M = \{f \mid f : I \rightarrow I\} \]

\(I\) and \(I^{[2]}\) sit in \(M\) as characteristic functions of points and intervals.
Operations

I and \( I^2 \) are De Morgan algebras. One also considers t-norms and conorms on these.

**Definition (Zadeh)** Define the following operations on \( M \)

1. \((f \sqcap g)(x) = \bigvee \{(f(y) \land g(z)) : y \land z = x\}\)
2. \((f \sqcap g)(x) = \bigvee \{(f(y) \land g(z)) : y \lor z = x\}\)
3. \(f^*(x) = f(1 - x)\)
4. \(0(x) = 1\) if \( x = 0 \) and 0 otherwise
5. \(1(x) = 1\) if \( x = 1 \) and 0 otherwise

These are **convolutions** of the corresponding operations on \( I \). We can also convolute t-norms \( \triangle \) and conorms on \( I \).
Equations

Theorem  M satisfies the equations for De Morgan algebras except that absorption and distributivity are weakened to the following.

1. \( x \cap (x \cup y) = x \cup (x \cap y) \)

2. \( (x \cap y) \cup (x \cap z) \cup (y \cap z) = (x \cup y) \cap (x \cup z) \cap (y \cup z) \)

M is not a lattice.
The unbalanced distributive laws do not hold.
M is a type of thing known as a De Morgan Birkhoff system.
Theorem  The variety $V(M)$ is generated by a finite algebra. The variety generated by the reduct $(M, \sqcap, \sqcup)$ is generated by

\[
\begin{array}{c}
  d \\
  c \\
  b \\
  a \\
\end{array}
\quad \quad \quad
\begin{array}{c}
  d \\
  b \\
  c \\
  a \\
\end{array}
\]

\[
\sqcap \\
\sqcup
\]

Proof  $V(M)$ is generated by the complex algebra of any bounded chain with involution that has at least 5 elements.

So these varieties have solvable free word problems. We do not know if they are finitely based.
**Definition** A function $f : I \rightarrow I$ is convex normal if it goes up to 1, then down.

Convex normal functions are a not too restrictive setting for our desired use as belief functions.
A related algebra

**Theorem**  The convex normal functions are a subalgebra of $M$. For the quotient $L$ of this subalgebra modulo agreement c.a.e.

1. $L$ is a complete, completely distributive DeMorgan algebra
2. $L$ is a compact Hausdorff topological algebra
3. $\int_0^1 |f(x) - g(x)| \, dx$ is a metric on it

Further, the convolution $\triangle$ of any continuous t-norm on $I$ gives a commutative quantale structure $(L, \triangle, \vee)$. 
A purpose

Aim: extend the theory of fuzzy controllers to the type-2 setting.

An example

We have a room with a device in it to heat and cool the room and a sensor that measures approximate temperature. Our controller is to adjust the setting of the device.

\[ X = \{50, 60, 70, 80, 90\} \] possible temperatures
\[ Y = \{-2, -1, 0, +1, +2\} \] settings of the device

A setting of -2 puts lots of cold air in the room, +2 lots of hot air.
Type-1 fuzzy controllers

Make linguistic variables **Cold**, **Nice**, and **Hot** for temperature; **Air** and **Furnace** for settings. Experts give fuzzy sets for these.
Type-1 fuzzy controllers

We represent the fuzzy sets for temperature as a matrix

\[ P = \begin{pmatrix} 1 & .5 & 0 & 0 & 0 \\ 0 & .5 & 1 & .5 & 0 \\ 0 & 0 & 0 & .5 & 1 \end{pmatrix} \]

\[ \begin{array}{c|ccc|ccc} & 50 & 60 & 70 & 80 & 90 \\ \hline \text{Cold} & 1 & .5 & 0 & 0 & 0 \\ \text{Nice} & 0 & .5 & 1 & .5 & 0 \\ \text{Hot} & 0 & 0 & 0 & .5 & 1 \end{array} \]
Type-1 fuzzy controllers

And do the same for adjustments

\[ Q = \begin{pmatrix} 1 & .7 & .3 & 0 & 0 \\ 0 & 0 & .3 & .7 & 1 \end{pmatrix} \]

\[
\begin{array}{c|ccccc}
\hline
& -2 & -1 & 0 & 1 & 2 \\
\hline
\text{Air} & 1 & .7 & .3 & 0 & 0 \\
\text{Furnace} & 0 & 0 & .3 & .7 & 1 \\
\hline
\end{array}
\]
Type-1 fuzzy controllers

We are given a rule base that says what should be done in each case.

\[ R = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \]

<table>
<thead>
<tr>
<th></th>
<th>Cold</th>
<th>Nice</th>
<th>Hot</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>AIR</strong></td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td><strong>FURNACE</strong></td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Type-1 fuzzy controllers

Then if our sensor gives a reading of 80 for temperature we make a column vector $\hat{T}$ with a 1 in the spot for 80 and 0’s elsewhere and compute $Q^T RP(\hat{T})$

\[
\begin{pmatrix}
1 & 0 \\
.7 & 0 \\
.3 & .3 \\
0 & .7 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & .5 & 0 & 0 & 0 \\
0 & .5 & 1 & .5 & 0 \\
0 & 0 & 0 & .5 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
.5 \\
.3 \\
.2 \\
0
\end{pmatrix}
\]

The result is a fuzzy subset of $Y = \{-2, -1, 0, 1, 2\}$ that we then “defuzzify” to get an adjustment to the device.
Matrix multiplication computes entries as sums of products.

This multiplication was done using $\cdot$ as product and $\lor$ as sum. It can be done using any continuous t-norm $\Delta$ as product and $\lor$ as sum. This requires

\[ x \Delta \lor y_i = \lor (x \Delta y_i) \]

to obtain associativity of matrix multiplication.
Ordinary fuzzy controllers live in the symmetric monoidal category of matrices over \((I, \triangle, \lor)\).

Objects: sets
Morphisms: matrices composed by multiplication

Tensor product is ordinary product of sets and Kronecker products of matrices. It allows to have more dependent or independent variables in the controller.
Type-2 fuzzy controllers

Do exactly the same with the category of matrices over \((L, \Delta, V)\).
Practicality

Implementations would require some restriction on the functions $f : I \to I$ (taking $n$ values, or with $n$ linear pieces)

Algorithms for $\cap$, $\cup$ of convex normal functions are linear in $n$. 
Thanks for listening.

Papers at www.math.nmsu.edu/~jharding