BOUNDARY BEHAVIOR OF SPECIAL COHOMOLOGY CLASSES ARISING FROM THE WEIL REPRESENTATION

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Abstract. In our previous paper [10], we established a theta correspondence between vector-valued holomorphic Siegel modular forms and cohomology with local coefficients for local symmetric spaces attached to real orthogonal groups. This correspondence is realized via theta functions associated to explicitly constructed “special” Schwartz forms. These forms give rise to relative Lie algebra cocycles for the orthogonal group with values in the Weil representation tensored with the coefficients.

In this paper, we systematically study certain complexes associated to the Weil representation and introduce a "local" restriction map for the Weil representation to a face of the Borel-Serre enlargement of the underlying orthogonal symmetric space. We then compute the restriction of the special cocycles. As a consequence we obtain a formula for the (global) restriction of the associated theta function to the Borel-Serre boundary of orthogonal local symmetric spaces. The global restriction is again a theta series as in [10] for a smaller orthogonal group and a larger coefficient system.

1. Introduction

Let $V$ be a rational non-degenerate quadratic space of dimension $m$ and signature $(p, q)$ and let $G = \text{SO}(V)$. We let $G = G(\mathbb{R})_0 = \text{SO}_0(V(\mathbb{R}))$. Let $X_V = X = G/K$ be the symmetric space of $G$ with $K = K_V$ a maximal compact subgroup. We let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be the associated Cartan decomposition of the Lie algebra of $G$ associated to a Cartan involution $\theta$. We let $\Gamma$ be an appropriate congruence subgroup of $G(\mathbb{Z}) \cap G$ and write $Y = \Gamma \backslash X$ for the associated locally symmetric space. We let $G' = \text{Mp}(n, \mathbb{R})$ denote the metaplectic covering group of the symplectic group $\text{Sp}(n, \mathbb{R})$, and we let $K'$ be the 2-fold covering of $U(n)$ in $G'$. Note that $K'$ admits a character $\text{det}^{-1/2}$ of $K'$. We let $\mathcal{W} = \bigwedge^n(\mathbb{C}^n)^*$ and define an action of $K'$ on the tensor power $T^j(U)$ by requiring that $K'$ acts on $T^j(U)$ by $\text{det}^{j-(p-q)/2}$. Thus we have twisted the natural action by a character depending on the signature of the quadratic space $V$. Furthermore, we let $\mathcal{W}_{n, V}$ be the $K'$-finite vectors of the Weil representation of $G' \times G$.

Every partition $\lambda$ of a non-negative integer $\ell'$ into at most $n$ parts gives rise to a dominant weight $\lambda$ of $\text{GL}(n)$. We explicitly realize the corresponding irreducible representation of highest weight $\lambda$ as the image $\mathcal{S}_\lambda(\mathbb{C}^n)$ of the Schur functor $\mathcal{S}_\lambda(\cdot)$ for (a standard filling of) the Young diagram $D(\lambda)$ associated to $\lambda$ applied to the tensor space $T^\ell'(\mathbb{C}^n)$. We can apply the same Schur functor to $T^\ell'(V_\mathbb{C})$ to obtain the space

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\[ S_{\lambda}(V_C). \] We let \( \mathcal{H} : T^{\ell'}(V_C) \to V_C^{[\ell]} \) be the orthogonal projection to the harmonic \( \ell' \)-tensors \( V_C^{[\ell']} \). Then applying \( \mathcal{H} \) to \( S_{\lambda}(V_C) \) one obtains the irreducible representation \( S_{\lambda|\lambda'}(V_C) \) with highest weight \( \lambda' \) of \( O(V_C) \), where \( \lambda' \) has the same nonzero entries as \( \lambda \). We therefore do not distinguish between \( \lambda \) and \( \lambda' \) and write \( S_{\lambda|\lambda'}(V_C) \). Finally, we also write \( i(\lambda) \) for the number of nonzero entries of \( \lambda \) (or \( \lambda' \)).

The main point of our previous work [10] was the construction of certain \((g, K)\)-cocycles

\[
\varphi^V_{nq,\lambda} \in C^{q, nq, \lambda}_{\lambda}(\varphi) \quad \text{and} \quad \varphi^V_{nq,[\lambda]} \in C^{q, nq,[\lambda]}(\varphi),
\]

where \( C^{\bullet,\lambda}_{\lambda}(\varphi) \) is the complex given by

\[
C^{j,r,\lambda}_V = \left[ T^j(U) \otimes S_{\lambda}(\mathbb{C}^n)^* \otimes W_{n,V} \otimes \mathcal{A}^r(A) \otimes S_{\lambda}(V_C) \right] K' \times G
\]

\[
z \simeq \left[ T^j(U) \otimes S_{\lambda}(\mathbb{C}^n)^* \otimes W_{n,V} \otimes \bigwedge^r (p_C^\perp) \otimes S_{\lambda}(V_C) \right] K' \times K_V.
\]

Here \( \mathcal{A}^{nq}(X) \) denotes the differential \( nq \)-forms on \( X \). The isomorphism is given by evaluation at the base point of \( X \). Furthermore, \( K' \) acts on the first three tensor factors, while \( G \) (resp. \( K \)) acts on the last three. The differential is the usual relative Lie algebra differential for the action of \( O(V) \). Finally, \( C^{j,r,\lambda}_V \) is obtained from \( C^{j,r,\lambda}_V \) by replacing \( S_{\lambda}(V_C) \) with the irreducible representation \( S_{\lambda}(V_C) \), and we let \( \pi_{\lambda} \) be the natural map from \( C^{j,r,\lambda}_V \) to \( C^{j,r,\lambda}_V \) induced by the harmonic projection \( \mathcal{H} \). Then \( \varphi^V_{nq,[\lambda]} = \pi_{\lambda}\varphi^V_{nq,\lambda} \). These forms vanish if \( i(\lambda) > \min(p, [\frac{m}{2}]) \) or \( n > p \).

These classes generalize the work of Kudla and Millson (see e.g. [14]) to the case of nontrivial coefficients systems \( S_{\lambda}(V_C) \). In [10], we showed how theta series \( \varphi^V_{nq,[\lambda]} \) associated to \( \varphi^V_{nq,\lambda} \) give rise to vector-valued Siegel modular forms whose Fourier expansions involve periods over certain “special” cycles with coefficients.

The purpose of this paper is to study the boundary behavior of these classes.

We let \( P = P(\mathbb{R})_0 \) be the connected component of the identity of the real points of a rational parabolic subgroup \( P \) in \( G \) stabilizing a flag \( F \) of totally isotropic rational subspaces in \( V \). We can choose \( F \) in such a way such that the \( \theta \)-stable subgroup of \( P \) forms a Levi subgroup. We let \( P = NAM \) be the associated (rational) Langlands decomposition, and we let \( m \) and \( n \) be the Lie algebras of \( M \) and \( N \) respectively. We set \( p_M = p \cap m \). Let \( E \) be the largest element in the isotropic flag with dimension \( \ell \). We let \( W = E^\perp / E \), which is naturally a quadratic space of signature \((p-\ell,q-\ell)\), and we realize \( W \) as a subspace of \( V \) such that the Cartan involution \( \theta \) for \( O(V) \) restricts to one for \( O(W) \). Then \( M \) splits naturally into the product of \( SO_0(W(\mathbb{R})) \) with a product of special linear groups of subquotients of \( E(\mathbb{R}) \). We let \( c(P) = NM/K_P \) be the associated face of the Borel-Serre enlargement of \( X \) with \( K_P = M \cap K \), see [3].

We consider an analogous complex \( A^{\bullet,\lambda}_P \) (similarly \( A^{\bullet,\lambda}_P \)) at the boundary given by

\[
A^{j,r,\lambda}_P = \left[ T^j(U) \otimes S_{\lambda}(\mathbb{C}^n)^* \otimes W_{n,W} \otimes \mathcal{A}^r(e(P)) \otimes S_{\lambda}(V_C) \right] K' \times N M
\]

\[
z \simeq \left[ T^j(U) \otimes S_{\lambda}(\mathbb{C}^n)^* \otimes W_{n,W} \otimes \bigwedge^r (n \oplus p_M)^* \otimes S_{\lambda}(V_C) \right] K' \times K_P.
\]
with coefficients in the Weil representation for \( G' \times O(W_\mathbb{R}) \). We introduce a local restriction map of complexes

\[
(1.4) \quad r_P : C^\bullet_V \to A^\bullet_P,
\]

induced by a natural (restriction) map \( \bigwedge^n(p^*) \to \bigwedge^n(n \oplus p_M)^* \) and by a \( g' \times NM \)-intertwiner from \( \mathcal{W}_{n,V} \) to \( \mathcal{W}_{n,W} \) using a so-called mixed model of \( \mathcal{W}_{n,V} \). We explicitly describe the local restriction \( r_P \phi_{nq,\lambda}^V \) and show that it actually lies in a particular subcomplex (with a refined grading) \( B_p^\bullet \) of \( A_p^\bullet \). The complex \( B_p^\bullet \) consists of the those elements in \( A_p^\bullet \) which are pullbacks for a certain bundle \( e(P) \to e_W(P) \) (see section 5.4). Hence there is a natural inclusion

\[
(1.5) \quad B_{P}^{j \ell_{1}, r_{2} \lambda} \subset A_{P}^{j \ell_{1} + r_{2} \lambda}.
\]

We then construct an inclusion map \( \iota_P \) of complexes from the relative Lie algebra complex \( C_{W}^{\bullet, \lambda} \) for \( W \) into \( A_{P}^{\bullet, \lambda} \) with the property

\[
(1.6) \quad \iota_P : C_{W}^{j - \ell, r \omega_n + \lambda} \hookrightarrow B_{P}^{j \ell, r n \ell, \lambda}.
\]

Here \( \omega_n = (1, \ldots, 1) \) is the \( n \)-th fundamental weight for \( GL(n) \), so that the Young diagram associated to \( \ell \omega_n \) is an \( n \) by \( \ell \) rectangle. The map \( \iota_P \) is a \( G' \times O(W_\mathbb{R}) \)-intertwiner (but note the shift by \( \ell \) in the first degree).

**Theorem 1.1.** We have

\[
\begin{align*}
r_P(\varphi_{nq,\lambda}^V) &= \iota_P(\varphi_{n(q-\ell), \ell \omega_n + \lambda}^V), \\
r_P(\varphi_{nq,[\lambda]}^V) &= \pi_{[\lambda]} \circ \iota_P(\varphi_{n(q-\ell), \ell \omega_n + \lambda}^V).
\end{align*}
\]

In particular, if \( n > p - \ell \) or \( i(\lambda) > \min(p, \lceil \frac{n}{2} \rceil) - \ell \), then \( r_P(\varphi_{nq,[\lambda]}^V) = 0 \).

Very roughly, the theorem says that the local restriction ”enlarges” the coefficient system by placing an \( n \) by \( \ell \) rectangle on the left of the Young diagram corresponding to \( \lambda \) to obtain a bigger Young diagram corresponding to \( \ell \omega_n + \lambda \). Note however that \( \varphi_{nq,\lambda}^V \) does not restrict physically to the Borel-Serre enlargement \( \overline{X} \) of the symmetric space \( X \) by the faces \( e(P) \).

The connection between the local restriction map and the global restriction to a face of the Borel-Serre compactification \( \overline{Y} \) is the following. We let \( e'(P) \) be the corner corresponding to \( e(P) \) in the Borel-Serre compactification \( \overline{Y} \) of \( Y \), and let \( r_{e'(P)} \) be the restriction map from \( Y \) to the corner \( e'(P) \). We let \( \theta_{\varphi_{nq,[\lambda]}^V}(g') \) with \( g' \in G' \) be the theta series associated to the Schwartz form \( \varphi_{nq,[\lambda]}^V \) and a suitable integral lattice in \( V \) viewed as a differential \( nq \)-form on \( Y \). Similarly, we define \( \theta_{\varphi_{nq,\lambda}^V}(g') \).

**Theorem 1.2.** The forms \( \theta_{\varphi_{nq,[\lambda]}^V} \) and \( \theta_{\varphi_{nq,\lambda}^V} \) extend to differential forms on the Borel-Serre compactification \( \overline{Y} \). Moreover,

\[
r_{e'(P)}(\theta_{\varphi_{nq,[\lambda]}^V}) = \theta_{r_P(\varphi_{nq,[\lambda]}^V)}
\]

and analogously for \( \theta_{\varphi_{nq,\lambda}^V} \). Furthermore,

\[
r_{e'(P)}(\theta_{\varphi_{nq,[\lambda]}^V}) = 0
\]
if \( n > p - \ell \) or \( i(\lambda) > \min(p, \lfloor \frac{m}{2} \rfloor) - \ell \). In particular, if \( n = p \) or \( i(\lambda) = \min(p, \lfloor \frac{m}{2} \rfloor) \), then the differential form \( \theta_{\nu_{\eta} \nu, [\lambda]} \) is rapidly decreasing.

**Remark 1.3.** The map \( \iota_P \) induces a global map on theta series. Then Theorem 1.1 and Theorem 1.2 can be summarized that the restriction of our theta series for \( O(V) \) to a face of the Borel-Serre compactification “is” the theta series for \( O(W) \) corresponding to an enlarged coefficient system (in the sense that the restriction is the image under the global \( \iota_P \) of such a series).

One motivation for our work arises from the problem to extend the Kudla-Millson lift (and our extension to non-trivial coefficient systems) from the cohomology of compact supports to the full cohomology of the space \( Y \).

We can now obtain such an extension in the following way. The theta series \( \theta_{\nu, \nu_{\eta} \nu, [\lambda]} \) gives rise to a non-holomorphic vector valued Siegel modular form with values in the closed \( S_{[\lambda]}(V) \)-valued differential \( nq \)-forms of \( Y \). Here \( S_{[\lambda]}(V) \) denotes the local system associated to \( S_{[\lambda]}(V_C) \). The main result of [10] is that the cohomology class \( [\theta_{\nu, \nu_{\eta} \nu, [\lambda]}] \) is a holomorphic vector-valued Siegel modular form with values in \( H^{nq}(Y, S_{[\lambda]}(V)) \), and its Fourier coefficients are represented by certain, totally geodesic, ‘special’ cycles. More precisely, for \( \eta \in Z^{(p-n)q}_{c}(Y, S_{[\lambda]}(V)^*), \) the space of compactly supported closed differential \((p-n)q\)-forms with values in \( S_{[\lambda]}(V)^* \), we can define its lift

\[
\Lambda_{nq, [\lambda]}(\eta) = \int_Y \eta \wedge \theta_{\nu_{\eta} \nu, [\lambda]}.
\]

Then by [10], this integral transform gives rise to a map

\[
\Lambda_{nq, [\lambda]} : H^{(p-n)q}_{c}(Y, S_{[\lambda]}(V)^*) \longrightarrow \text{Mod}(\Gamma', S'_{\lambda} \otimes \det^{-\frac{m}{2}})
\]

from the cohomology of compact supports to \( \text{Mod}(\Gamma', S'_{\lambda} \otimes \det^{-\frac{m}{2}}) \), the space of vector-valued holomorphic Siegel modular forms of type \( \det \frac{m}{2} \otimes S_{\lambda} \) for a certain congruence subgroup \( \Gamma' \) of \( \text{Sp}(n, \mathbb{Z}) \).

Now note that Theorem 1.2 allows us to extend the lift \( \Lambda_{nq, [\lambda]} \) to \( Z^{(p-n)\ell}(\overline{Y}, S_{\lambda}(V)^*) \), the space of closed differential \((p-n)\ell\)-forms with values in \( S_{\lambda}(V)^* \) which extend to the Borel-Serre compactification \( \overline{Y} \). The image of this extended lift lies in \( \mathcal{A} = \mathcal{A}(\Gamma', S'_{\lambda} \otimes \det^{-\frac{m}{2}}) \), the space of not necessarily holomorphic automorphic forms for \( \Gamma' \) and type \( \det \frac{m}{2} \otimes S_{\lambda} \) (The holomorphicity proofs for the lift of [14] and [10] heavily depend on \( \eta \) being rapidly decreasing, see also [8]). We denote by \( \mathcal{A}_0 = \mathcal{A}_0(\Gamma', S'_{\lambda} \otimes \det^{-\frac{m}{2}}) \) the image of the exact forms under the lift \( \Lambda_{nq, [\lambda]} \). Using the homotopy equivalence of \( Y \) and \( \overline{Y} \), we then obtain in the natural fashion a cohomological lift

\[
\Lambda_{nq, [\lambda]} : H^{(p-n)q}((Y, S_{[\lambda]}(V)^*) \longrightarrow \mathcal{A}/\mathcal{A}_0.
\]

It is an important problem to find a different, more explicit description of this extended lift and its Fourier coefficients. For a discussion of these points for hyperbolic manifolds, see [9]. We hope to come back to these kind of issues in the near future.
The paper is organized as follows. In section 2, we establish the basic notation of the paper. In particular, we introduce the locally symmetric space \( Y \) and its Borel-Serre compactification and give an explicit description of the parabolic subgroups of \( G \). In section 3, we briefly review some basics of the representation theory of \( GL(n, \mathbb{C}) \) and \( O(V) \) via the Schur functor \( S(\cdot) \). In section 4, we study various models of the Weil representation and introduce the Weil representation restriction map \( \tau^W_P \) from \( \mathcal{S}(V(\mathbb{R}))^n \) to \( \mathcal{S}(W(\mathbb{R}))^n \) and study \( \tau^W_P \) for a certain class of Schwartz functions. In section 5, we introduce the complexes \( C_V^*, A_P^*, \) and \( B_P^* \) and define the map \( \iota_P \) from \( C_V^* \) to \( B_P^* \). In sections 6 and 7, we study the special Schwartz form \( \varphi_{nq,\lambda} \) and establish the local restriction formula, Theorem 1.1. Finally, in section 8, we introduce the theta series \( \theta_{\varphi_{nq,\lambda}} \) and prove Theorem 1.2.

2. Basic Notations

2.1. Orthogonal Symmetric Spaces. Let \( V \) be a rational vector space of dimension \( m = p + q \) and let \((\cdot,\cdot)\) be a non-degenerate symmetric bilinear form on \( V \) with signature \((p,q)\). We choose a rational decomposition of \( V = V_+ \oplus V_- \) into a positive and a negative definite subspace. We fix a standard orthogonal basis \( e_1,\ldots,e_p, e_{p+1},\ldots,e_m \) of \( V(\mathbb{R}) \) such that \((e_\alpha, e_\alpha) = 1 \) for \( 1 \leq \alpha \leq p \) and \((e_\mu, e_\mu) = -1 \) for \( p + 1 \leq \mu \leq m \).

We let \( G = SO(V) \) viewed as an algebraic group over \( \mathbb{Q} \). We let \( G := G(\mathbb{R})_0 \) be the connected component of the identity of \( G(\mathbb{R}) \) so that \( G \simeq SO_0(p,q) \). We let \( K \) be the maximal compact subgroup of \( G \) stabilizing \( V_+ \) (and \( V_- \)). Thus \( K \simeq SO(p) \times SO(q) \). Let \( X = G/K \) be the symmetric space of dimension \( pq \) associated to \( G \). We realize \( X \) as the space of negative \( q \)-planes in \( V(\mathbb{R}) \):

\[
X \simeq \{ z \in V(\mathbb{R}) : \dim z = q; (\cdot,\cdot)_z < 0 \}.
\]

Thus \( z_0 = V_- \) is the base point of \( X \). Furthermore, we can also interpret \( X \) as the space of minimal majorants for \((\cdot,\cdot)\). That is, \( z \in X \) defines a majorant \((\cdot,\cdot)_z\) by \((x,x)_z = -(x,x)\) if \( x \in z \) and \((x,x)_z = (x,x)\) if \( x \in z^\perp \). We write \((\cdot,\cdot)_0\) for the majorant associated to the base point \( z_0 \).

The Cartan involution \( \theta_0 \) of \( G \) corresponding to the basepoint \( z_0 \) is obtained by conjugation by the matrix \( I_{p,q} \). We will systematically abuse notation below and write \( \theta_0(v) \) for the action of the linear transformation of \( V \) with matrix \( I_{p,q} \) relative to the above basis acting on \( v \in V \). Let \( g \) be the Lie algebra of \( G \) and \( \mathfrak{k} \) be the one of \( K \). We obtain the Cartan decomposition

\[
g = \mathfrak{k} \oplus \mathfrak{p},
\]

where

\[
\mathfrak{p} = \text{span}\{X_{\alpha\mu} := e_\alpha \wedge e_\mu ; 1 \leq \alpha \leq p, p + 1 \leq \mu \leq m \}.
\]

Here \( w \wedge w' \in \bigwedge^2 V(\mathbb{R}) \) is identified with an element of \( g \) via

\[
(w \wedge w')(v) = (w,v)w' - (w',v)w.
\]
Thus \( p \simeq \text{Hom}(V_-(\mathbb{R}), V_+(\mathbb{R})) \), and via the standard basis of \( V(\mathbb{R}) \) we have
\[
\begin{aligned}
(2.5) \quad p &\simeq \left\{ \begin{pmatrix} 0 & X \\ tX & 0 \end{pmatrix} : X \in M_{p,q}(\mathbb{R}) \right\}.
\end{aligned}
\]

We let \( \{\omega_{\alpha}\} \) be the dual basis of \( p^* \) corresponding to \( \{X_{\alpha}\} \). Finally note that we can identify \( p \) with the tangent space \( T_{z_0}(X) \) at the base point \( z_0 \) of \( X \).

Let \( L \subset V \) be an even \( \mathbb{Z} \)-lattice of full rank, i.e., \( (x,x) \in 2\mathbb{Z} \) for \( x \in L \). In particular, \( L \subset L^\# \), the dual lattice. We denote by \( \Gamma(L) \) the stabilizer of the lattice \( L \) and fix a (neat) subgroup \( \Gamma \) of finite index in \( \Gamma(L) \cap G \) which acts trivially on \( L^\# / L \). We let \( Y = \Gamma \backslash X \) be the locally symmetric space. We assume that \( Y \) is non-compact. It is well known that this is the case if and only if \( V \) has an isotropic vector over \( \mathbb{Q} \). We let \( r \) be the Witt rank of \( V \), i.e., the dimension of a maximal totally isotropic subspace of \( V \) over \( \mathbb{Q} \).

Let \( F \) be an isotropic subspace of \( V \) of dimension \( \ell \). Then we can describe the \( \ell \)-dimensional isotropic subspace \( \theta_0(F) \) as follows. For \( U \) a subspace of \( V \), let \( U^\perp \), resp. \( U^\perp_0 \) be the orthogonal complement of \( U \) for the form \((\ , \ )\), resp. \((\ , \ )_0\). Then
\[
\theta_0(F) = (F^\perp)^\perp_0.
\]

We fix a maximal totally isotropic subspace \( E_r \) and choose a basis \( u_1, u_2, \ldots, u_r \) of \( E_r \). Let \( E'_r = \theta_0(E_r) \). We pick a basis \( u'_1, \ldots, u'_r \) of \( E'_r \) such that \((u_i, u'_j) = -\frac{1}{2}\delta_{ij}\).

More generally, we let
\[
(2.6) \quad E_\ell := \text{span}\{u_1, \ldots, u_\ell\},
\]
and we call \( E_\ell \) a standard totally isotropic subspace. Furthermore, we set \( E'_\ell = \theta_0(E_\ell) = \text{span}(u'_1, \ldots, u'_\ell) \). Note that \( E'_\ell \) can be naturally identified with the dual space of \( E_\ell \). We can assume that with respect to the standard basis of \( V(\mathbb{R}) \) we have
\[
(2.7) \quad e_\alpha = u_\alpha - u'_\alpha \quad \text{and} \quad e_{m+1-\alpha} = u_\alpha + u'_\alpha
\]
for \( \alpha = 1, \ldots, \ell \). We let
\[
(2.8) \quad W_\ell = E^\perp_\ell / E_\ell,
\]
and note that \( W_\ell \) is a non-degenerate space of signature \((p-\ell, q-\ell)\). We can realize \( W_\ell \) as a subspace of \( V \) by
\[
(2.9) \quad W_\ell = (E_\ell \oplus E'_\ell)^\perp,
\]
where the orthogonal complement is either with respect to \((\ , \ )\) or \((\ , \ )_0\). This gives a \( \theta_0 \)-invariant Witt splitting
\[
(2.10) \quad V = E_\ell \oplus W_\ell \oplus E'_\ell.
\]
Note that with these choices \( \theta_0 \) restricts to a Cartan involution for \( O(W_\ell(\mathbb{R})) \). We obtain a Witt basis \( u_1, \ldots, u_\ell, e_{\ell+1}, \ldots, e_{m-\ell}, u'_\ell, \ldots, u'_\ell \) for \( V(\mathbb{R}) \). We will denote coordinates with respect to the Witt basis with \( y_i \) and coordinates with respect to the standard basis with \( x_i \). Note that with respect to the Witt basis, the bilinear form \((\ , \ )\) has Gram matrix
\[
(2.11) \quad (\ , \ ) \sim \begin{pmatrix} 1_{W_\ell} & -\frac{1}{2}J \\ -\frac{1}{2}J & -1_{E_\ell} \end{pmatrix}
\]
with $J = \begin{pmatrix} 1 & \cdots & 1 \\ & & \\
\end{pmatrix}$.

We often drop the subscript $\ell$ and just write $E$, $E'$, and $W$. We also write $W_+ = W \cap V_+$ and $W_- = W \cap V_-.$

2.2. Parabolic Subgroups. We now describe the parabolic subgroups of $G$. We follow in part [3]. We let $F$ be a flag of totally isotropic subspaces $0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_k \subset F_k^1 \subset \cdots \subset F_l^1 \subset V$ of $V$ over $\mathbb{Q}$. Then we let $P = P_F$ be the parabolic subgroup of $G$ stabilizing the flag $F$:

\begin{equation}
(2.12) \quad P_F = \{ g \in G ; g F_i = F_i \}.
\end{equation}

We let $N_P$ be the unipotent radical of $P$. It acts trivially on all the quotients of the flag. We let $L_P = N_P \backslash P$ and let $S_P$ be the split center of $L_P$ over $\mathbb{Q}$. Note that $S_P$ acts by scalars on each quotient. Let $M_P = \cap_{x \in \chi(L_P)} \text{Ker}(\chi^2)$. We let $N = N_P$ and $L = L_P$ be their respective real points in $G$, and let $P = P_F = (L_F(\mathbb{R}))_0$, $M = M_P = (M_P(\mathbb{R}))_0$, and $A = A_P = (S_P(\mathbb{R}))_0$ be the connected component of the identity in $M_P(\mathbb{R})$ and $S_P(\mathbb{R})$ respectively.

By conjugation, we can assume that the flag $F$ consists of standard totally isotropic subspaces $E_i$ (2.6) and call $P_F$ a standard $\mathbb{Q}$-parabolic. In that case, using the Cartan involution $\theta_0$, we realize $L_P$ (and also $S_P$, $M_P$) as $\theta_0$-stable subgroups of $P$:

\begin{equation}
(2.13) \quad L_P = P \cap \theta_0(P).
\end{equation}

Then $M_P$ is the semi-simple part of the centralizer of $S_P$ in $P$. We will regularly drop the subscripts $F$, $P$, and $P$. We obtain the (rational) Langlands decomposition of $P$:

\begin{equation}
(2.14) \quad P = NAM \simeq N \times A \times M,
\end{equation}

and we write $n$, $a$, and $m$ for their respective Lie algebras. The map $P \to N \times A \times M$ is equivariant with the $P$-action defined by

\begin{equation}
(2.15) \quad n'a'm'(n, a, m) = (n' \text{Ad}(a'm'))(n), a'a, m'm).
\end{equation}

We let $F$ be a standard rational totally isotropic flag $E_{i_1} \subset \cdots \subset E_{i_k} = E_\ell = E$ and assume that the last (biggest) totally isotropic space in the flag $F$ is equal to $E_\ell$ for some $\ell$.

Let $U_{ij}$ be the orthogonal complement of $E_{ij}$ in $E_{ij+1}$ and $U'_{ij}$ be the orthogonal complement of $E'_{ij}$ in $E'_{ij+1}$ and let $W = W_\ell = (E_\ell \oplus E'_\ell)^\perp$. We obtain a refinement of the Witt decomposition of $V$ such that all the subspaces $U_{ij}, U'_{ij}$, and $W$ are mutually orthogonal for $(\ , \ )_0$ and defined over $\mathbb{Q}$:

\begin{equation}
(2.16) \quad V = \left( \bigoplus_{i_j=1}^k U_{ij} \right) \oplus W \oplus \left( \bigoplus_{i_j=1}^k U'_{ij} \right).
\end{equation}

Then $L_P$ is the subgroup of $P$ that stabilizes each of the subspaces in the above decomposition of $V$. In what follows we will describe matrices in block form relative to the above direct sum decomposition of $V$. 

We first note that we naturally have $O(W) \times \text{GL}(E) \subset O(V)$ via
\begin{equation}
\{ \begin{pmatrix} g & h \\ \tilde{g} & \end{pmatrix} ; h \in O(W), g \in \text{GL}(E) \},
\end{equation}
where $\tilde{g} = Jg^*J$ and $g^* = \frac{1}{2} g^{-1}$. In particular, we can view the corresponding Lie algebras $\mathfrak{o}(W(\mathbb{R}))$ and $\mathfrak{gl}(E(\mathbb{R}))$ as subalgebras of $\mathfrak{g}$. Namely,
\begin{equation}
\mathfrak{o}(W(\mathbb{R})) \simeq \text{span}\{e_i \wedge e_j ; \ell < i < j \leq m - \ell \},
\end{equation}
\begin{equation}
\mathfrak{gl}(E(\mathbb{R})) \simeq \text{span}\{u_i \wedge u'_j ; i, j \leq \ell \}.
\end{equation}
via the identification $\mathfrak{g} \simeq \bigwedge^2 V(\mathbb{R})$.

We let $S$ be the maximal $\mathbb{Q}$-split torus of $G$ given by
\begin{equation}
\mathcal{S} = \left\{ a(t_1, \ldots, t_r) := \left( \begin{smallmatrix} \text{diag}(t_1, \ldots, t_r) \\ \text{diag}(t_1^{-1}, \ldots, t_r^{-1}) \end{smallmatrix} \right) \right\}.
\end{equation}
We write $t = (t_1, \ldots, t_r)$ and $\tilde{t}J = (t_r, \ldots, t_1)$. Note
\begin{equation}
\exp(2u_i \wedge u'_i) = a(0, \ldots, 0, 1, 0, \ldots, 0).
\end{equation}

The set of simple rational roots for $G$ with respect to $P$ and $\mathcal{S}$ is given by $\Delta = \Delta(S, G) = \{\alpha_1, \ldots, \alpha_r\}$, where
\begin{equation}
\alpha_i(a) = t_it_{i+1}^{-1}, \quad (1 \leq i \leq r - 1)
\end{equation}
\begin{equation}
\alpha_i(a) = \begin{cases} t_r & \text{if } W_r \neq 0 \\ t_{r-1}t_r & \text{if } W_r = 0. \end{cases}
\end{equation}

We write $\Phi(P, A_{P'})$ for the roots of $P$ with respect to $A_P$ and $\Delta(P, A_{P'})$ for the simple roots of $P$ with respect to $A_{P'}$, which are those $\alpha \in \Delta$ which act nontrivially on $S_{P'}$. We have
\begin{equation}
\Delta(P, A_{P'}) = \{\alpha_{i_1}, \ldots, \alpha_{i_k}\}.
\end{equation}

We let $Q$ be the standard maximal parabolic stabilizing the single totally isotropic rational subspace $E_\ell$ of dimension $\ell \leq r$. In that case, we have $S_P = S = \{a(t)\}$ with $a(t) = a(t, \ldots, t)$.

We now consider the isotropic flag $F$ in $V$ as a flag $F(E) : (0) \subset E_{i_1} \subset \cdots \subset E_{i_k} = E$ inside $E$. We let $P'$ be the parabolic subgroup of $\text{GL}(E)$ stabilizing $F(E)$. Then for the real points $P' = (P'(\mathbb{R}))_0$, we have
\begin{equation}
P' = N_{P'}AM_{P'},
\end{equation}
with unipotent radical $N_{P'}$ and Levi factor
\begin{equation}
M_{P'} = \prod_{j=1}^{k} \text{SL}(U_{i_j}(\mathbb{R})).
\end{equation}
Here $A$ is as above, viewed as a subgroup of $\text{GL}_+(E(\mathbb{R}))$. Note $M_{Q'} \simeq \text{SL}(E(\mathbb{R}))$.

Returning to $P$, we have
\begin{equation}
L \simeq \left\{ \begin{pmatrix} g & h \\ \tilde{g} & \end{pmatrix} ; h \in O(W), g = \text{diag}(g_1, \ldots, g_k) \in \prod_{j=1}^{k} \text{GL}(U_{i_j}), \det g \det h = 1 \right\}.
\end{equation}
Thus
\[ M \simeq SO_0(W(\mathbb{R})) \times M_{P^e}. \]
We also define
\[ p_M = p \cap m, \]
and we write
\[ p_M = p_W \oplus p_E, \]
where \( p_E = \mathfrak{s}(E) \cap p \) and
\[ p_W = \mathfrak{o}_W \cap p = \text{span}\{X_{\alpha \mu} = e_\alpha \wedge e_\mu; \ell + 1 \leq \alpha \leq p, p + 1 \leq \mu \leq m - \ell\}. \]

We naturally view \( N_{P^e} \subset \text{SL}(E) \) as a subgroup of the unipotent radical \( N_P \) via
\[ n' \mapsto N(n') := \begin{pmatrix} n' & 1 \\ \tilde{n}' & 1 \end{pmatrix} \in N_{P^e}. \]

We then have a semidirect product decomposition
\[ N_{P^e} = N_{P^e} \rtimes N_{Q}, \]
where \( Q \) is as above the maximal parabolic that stabilizes \( E \). Furthermore, we let \( Z_Q \) be the center of \( N_Q \subseteq N_{P^e} \). It is given by
\[ Z_Q = \left\{ z(b) := \begin{pmatrix} 1 & b \\ 1 & 1 \end{pmatrix}; J'bj = -b \right\}. \]

Then for the coset space \( N_{P^e}/(N_{P^e} \rtimes Z_Q) \), we have
\[ N_{P^e}/(N_{P^e} \rtimes Z_Q) \simeq N_Q/Z_Q \simeq W \otimes E \]
as vector spaces. Explicitly, the basis of \( E \) gives rise to an isomorphism \( W \otimes E \simeq W^\ell \).

Then for \( (w_1, \ldots, w_\ell) \in W^\ell \), the corresponding coset is represented by
\[ n(w_1, \ldots, w_\ell) := \begin{pmatrix} I_\ell & 2(\cdot, w_1) & w_1^2 \\ \vdots & \ddots & \vdots \\ I_W & w_\ell & w_1 \end{pmatrix}. \]

Here we write \( w_i^2 = (w_i, w_i) \) for short.

On the Lie algebra level, we let \( \mathfrak{z}_Q \) be the center of \( \mathfrak{n}_Q \subseteq \mathfrak{n}_{P^e} \), whence corresponding to (2.34)
\[ \mathfrak{z}_Q \simeq \bigwedge^2 E(\mathbb{R}). \]

We let \( \mathfrak{n}_{P^e} \) be the Lie algebra of \( N_{P^e} \); thus \( \mathfrak{n}_{P^e} \subseteq E \wedge E' = \mathfrak{g}(E(\mathbb{R})). \) Corresponding to (2.36), we can realize \( W(\mathbb{R}) \otimes E(\mathbb{R}) \) as a subspace of \( \mathfrak{n} \). Namely, we obtain an embedding
\[ W(\mathbb{R}) \otimes E(\mathbb{R}) \hookrightarrow \mathfrak{n}, \]
\[ w \otimes u \rightarrow 2w \wedge u =: n_u(w), \]
and we denote this subspace by \( n_W \), which we frequently identify with \( W(\mathbb{R}) \otimes E(\mathbb{R}) \). Furthermore, this embedding is \( \mathfrak{o}(W(\mathbb{R})) \oplus \mathfrak{gl}(E(\mathbb{R})) \)-equivariant, i.e.,
\[ [X, n_u(w)] = n_u(Xw) \quad [Y, n_u(w)] = n_Y u(w) \]
for \( X \in \mathfrak{o}(W(\mathbb{R})) \) and \( Y \in \mathfrak{gl}(E(\mathbb{R})) \). We easily see
\[ \exp(n_u(w)) = n(0, \ldots, w, \ldots, 0). \]

A standard basis of \( n_W \) is given by
\[ Y_{\alpha_i} := n_u(e_{\alpha}) = 2e_{\wedge}u_i \quad Y_{\mu_i} := n_u(e_{\mu}) = 2e_{\wedge}u_i \]
with \( 1 \leq i \leq \ell, \ell + 1 \leq \alpha \leq p, \) and \( p + 1 \leq \mu \leq m - \ell \). We denote the elements of the corresponding dual basis by \( \nu_{\alpha_i} \) and \( \nu_{\mu_i} \).

Summarizing, we obtain

**Lemma 2.1.** We have a direct sum decomposition (of vector spaces)
\[ n_P = n_{P'} \oplus n_W \oplus 3Q. \]
Furthermore, the adjoint action of \( \mathfrak{o}(W(\mathbb{R})) \oplus \mathfrak{gl}(E(\mathbb{R})) \) on \( n_P \) induces an action on the space \( n_P/(n_{P'} \oplus 3Q) \cong n_W \) such that
\[ n_W \cong W(\mathbb{R}) \otimes E(\mathbb{R}) \]
as \( \mathfrak{o}(W(\mathbb{R})) \oplus \mathfrak{gl}(E(\mathbb{R})) \)-representations.

### 2.3. The Maurer Cartan forms and horospherical coordinates.

The Langlands decomposition of \( P \) gives rise to the (rational) horospherical coordinates on \( X \) associated to \( P \) by
\[ \sigma = \sigma_P : N \times A \times X_P \longrightarrow X \]
given by
\[ \sigma(n, a, m) = na \cdot m z_0. \]
Here
\[ X_P = M_P/K_P \]
is the boundary symmetric space associated to \( P \). Here \( K_P = M \cap K \).

We note that the boundary symmetric space \( X_P \) always factors into a product of symmetric spaces for special linear groups and one orthogonal factor, namely, the symmetric space associated to \( \text{SO}(W) \). We call the associated symmetric space \( X_W \) the **orthogonal factor** in the boundary symmetric space \( X_P \). We have
\[ X_P = X_W \times \prod_{j=1}^k X_{U_{i_j}}, \]
where \( X_{U_{i_j}} \) denotes the symmetric space associated to \( \text{SL}(U_{i_j}) \).

We now describe how the basic cotangent vectors \( \omega_{\alpha\mu} = (e_{\alpha} \wedge e_{\mu})^* \in p^* \cong T_{z_0}(X) \) look like in \( NAM \) coordinates. We extend \( \sigma \) to \( N \times A \times M \times K \rightarrow G \) by \( \sigma(n, a, m, k) = namk \), and this induces an isomorphism between the left-invariant
forms on $NAM$ (which we identify with $n^* \oplus a^* \oplus p^*_M$) and the horizontal left-invariant forms on $G$ (which we identify with $p^*$). Thus we have an isomorphism

$$\sigma^*: p^* \longrightarrow n^* \oplus a^* \oplus p^*_M.$$  

Explicitly, we have

**Lemma 2.2.** Let $1 \leq i \leq \ell$. For the preimage under $\sigma^*$ of the elements in $n^*_W$ coming from $W_+ \otimes E$, we have

$$\sigma^* \omega_{\alpha m+1-i} = \nu_{ai},$$

where $\ell + 1 \leq \alpha \leq p$. Furthermore, for the ones coming from $W_- \otimes E$, we have

$$\sigma^* \omega_{\alpha m} = -\nu_{ai},$$

where $p + 1 \leq \mu \leq m+1-\ell$. On $p^*_M$, the map $\sigma^*$ is the identity. In particular, for $\omega_{\alpha \mu} \in p^*_W$,

$$\sigma^* \omega_{\alpha \mu} = \omega_{\alpha \mu},$$

where $\ell + 1 \leq \alpha \leq p$ and $p + 1 \leq \mu \leq m - \ell$. In particular, for $\ell + 1 \leq \alpha \leq p$ and any $\mu \geq p + 1$, we have

$$\sigma^* \omega_{\alpha \mu} \in p^*_W \oplus n^*_W.$$

The remaining elements of $p^*$ are of the form $\omega_{\alpha \mu}$ with $p + 1 \leq \mu \leq m+1-\ell$. These elements are mapped under $\sigma^*$ to $n^*_P \oplus a^* \oplus p^*_E \subset \mathfrak{gl}(E(R))^*$. 

**2.4. Borel-Serre Compactification.** We now describe the Borel-Serre compactification of $X$ and of $Y = \Gamma \backslash X$. We follow [3], III.9. We first partially compactify the symmetric space $X$. For any rational parabolic $P$, we define the boundary component

$$e(P) = N_P \times X_P \simeq P/A_P K_P.$$

Then as a set the (rational) Borel-Serre enlargement $X^{BS} = X$ is given by

$$X = X \cup \bigsqcup_P e(P),$$

where $P$ runs over all rational parabolic subgroups of $G$. As for the topology of $X$, we first note that $X$ and $e(P)$ have the natural topology. Furthermore, a sequence of $y_j = \sigma_P(n_j, a_j, z_j) \in X$ in horospherical coordinates of $X$ converges to a point $(n, z) \in e(P)$ if and only if $n_j \to n$, $z_j \to z$ and $\alpha(a_j) \to \infty$ for all roots $\alpha \in \Phi(P, A_P)$. For convergence within boundary components, see [3], III.9.

With this, $X$ has a canonical structure of a real analytic manifold with corners. Moreover, the action of $G(\mathbb{Q})$ extends smoothly to $X$. The action of $g = kp = k \cdot n, a, z \cdot m \in KMAN = G$ on $e(P)$ is given by

$$g \cdot (n', z') = k \cdot (Ad(\alpha)(nn'), mz') \in e(Ad(k)P) = e(Ad(g)P)$$

with $k \cdot (n', z') = (Ad(k)n, Ad(k)mK_{Ad(k)P}) \in e(Ad(k)P)$. 

Finally,

$$Y := \Gamma \backslash X.$$


is the Borel-Serre compactification of \( Y = \Gamma \backslash X \) to a manifold with corners. If \( P_1, \ldots, P_k \) is a set of representatives of \( \Gamma \)-conjugacy classes of rational parabolic subgroups of \( G \), then

\[
\Gamma \backslash X = \Gamma \backslash X \cup \bigcup_{i=1}^{k} \Gamma_{P_i} \backslash e(P_i),
\]

with \( \Gamma_{P_i} = \Gamma \cap M \). We will write \( e'(P) = \Gamma_{P} \backslash e(P) \).

We now describe Siegel sets. For \( t \in \mathbb{R}_+ \), let

\[
A_{P,t} = \{ a \in A_P; \alpha(a) > t \text{ for all } \alpha \in \Delta(P, A_P) \},
\]

and for bounded sets \( U \subset N_P \) and \( V \subset X_P \), we define the Siegel set

\[
\mathfrak{S}_{P,U,V} = U \times A_{P,t} \times V \subset N_P \times A_P \times X_P.
\]

Note that for \( t \) sufficiently large, two Siegel sets for different parabolic subgroups are disjoint. Furthermore, if \( P_1, \ldots, P_k \) are representatives of the \( G(\mathbb{Q}) \)-conjugacy classes of rational parabolic subgroups of \( G \), then there are Siegel sets \( \mathcal{S}_i \) associated to \( P_i \) such that the union \( \bigcup \pi(S_i) \) is a fundamental set for \( \Gamma \). Here \( \pi \) denotes the projection \( \pi : X \to \Gamma \backslash X \).

3. Review of representation theory for general linear and orthogonal groups

In this section, we will briefly review the construction of the irreducible finite dimensional (polynomial) representations of \( \text{GL}(\mathbb{C}^n) \) and \( \text{O}(V) \). Here, in this section, we assume that \( V \) is complex space of dimension \( m \). Basic references are [7], §4.2 and §6.1, [11], §9.3.1-9.3.4 and [1], Ch. V, §5 to which we refer for details.

3.1. Representations of \( \text{GL}_n(\mathbb{C}) \). Let \( \lambda = (b_1, b_2, \ldots, b_n) \) be a partition of \( \ell' \). We assume that the \( b_i \)'s are arranged in decreasing order. We will use \( D(\lambda) \) to denote the Young diagram associated to \( \lambda \). We will identify the partition \( \lambda \) with the dominant weight \( \lambda \) for \( GL(n) \) in the usual way. A standard filling \( \lambda \) of the Young diagram \( D(\lambda) \) by the elements of the set \( [\ell] = \{1, 2, \ldots, \ell'\} \) is an assignment of each of the numbers in \( [\ell'] \) to a box of \( D(\lambda) \) so that the entries in each row strictly increase when read from left to right and the entries in each column strictly increase when read from top to bottom. A Young diagram equipped with a standard filling will be also called a standard tableau.

We let \( s(t(\lambda)) \) be the idempotent in the group algebra of the symmetric group \( S_{\ell'} \) associated to a standard tableau \( T \) with \( \ell' \) boxes corresponding to a standard filling \( t(\lambda) \) of a Young diagram \( D(\lambda) \). Note that \( S_{\ell'} \) acts on the space of \( \ell' \)-tensors \( T^{\ell'}(\mathbb{C}^n) \) in the natural fashion on the factors of \( T^{\ell'}(\mathbb{C}^n) \). Therefore \( s(t(\lambda)) \) gives rise to a projection operator in \( \text{End}(T^{\ell'}(\mathbb{C}^n)) \), which by slight abuse of notation we also denote by \( s(t(\lambda)) \). We write

\[
\mathcal{S}_{t(\lambda)}(\mathbb{C}^n) = s(t(\lambda))(T^{\ell'}(\mathbb{C}^n)).
\]
We have a direct sum decomposition

\[(3.2) \quad T^{\ell'}(\mathbb{C}^n) = \bigoplus_{\lambda} \bigoplus_{t(\lambda)} S_{t(\lambda)}(\mathbb{C}^n),\]

where \(\lambda\) runs over all partitions of \(\ell'\) and \(t(\lambda)\) over all standard fillings of \(D(\lambda)\). This gives the decomposition of \(T^{\ell'}(\mathbb{C}^n)\) into irreducible constituents, i.e., for every standard filling \(t(\lambda)\), the \(GL(\mathbb{C}^n)\)-module \(S_{t(\lambda)}(\mathbb{C}^n)\) is irreducible with highest weight \(\lambda\). In particular, \(S_{t(\lambda)}(\mathbb{C}^n)\) and \(S_{t'(\lambda)}(\mathbb{C}^n)\) are isomorphic for two different standard fillings \(t(\lambda)\) and \(t'(\lambda)\). We denote this isomorphism class by \(S_{\lambda}(\mathbb{C}^n)\) (or if we do not want to specify the standard filling).

Explicitly, we let \(A\) be the standard filling of a Young diagram \(D(A)\) corresponding to the partition \(\lambda\) with less than or equal to \(n\) rows and \(\ell'\) boxes by \(1, 2, \ldots, \ell'\) obtained by filling the rows in order beginning at the top with \(1, 2, \ldots, \ell'\). We let \(R(A)\) be the subgroup of \(S_{\ell'}\) which preserves the rows of \(A\) and \(C(A)\) be the subgroup that preserves the columns of \(A\). We define elements \(r(A)\) and \(c(A)\) by

\[(3.3) \quad r(A) = \sum_{s \in R(A)} s \quad \text{and} \quad c(A) = \sum_{s \in C(A)} \text{sgn}(s)s.\]

Let \(h(A)\) be the product of the hook lengths of the boxes in \(D(A)\), see \([7]\), page 50. Then the idempotent \(s(A)\) is given

\[(3.4) \quad s(A) = \frac{1}{h(A)} c(A)r(A).\]

We will also need the ”dual” idempotent \(s(A)^*\) given by

\[(3.5) \quad s(A)^* = \frac{1}{h(A)} r(A)c(A).\]

We let \(\varepsilon_1, \ldots, \varepsilon_n\) denote the standard basis of \(\mathbb{C}^n\) and \(\theta_1, \ldots, \theta_n \in (\mathbb{C}^n)^*\) be its dual basis. We set

\[(3.6) \quad \varepsilon_A = \varepsilon_1^{b_1} \otimes \cdots \otimes \varepsilon_n^{b_n}\]

and let \(\theta_A\) be the corresponding element in \(T^{\ell'}(\mathbb{C}^n)^*\). Then \(s(A)(\varepsilon_A)\) is a highest weight vector in \(S_{A}(\mathbb{C}^n)\), see \([11]\), §9.3.1. We have

**Lemma 3.1.**

\[s(A)^*\theta_A(s(A)\varepsilon_A) = \frac{|R(A)|}{h(A)}.\]

Here \(|R(A)|\) is the order of \(R(A)\).

**Proof.** We compute

\[s(A)^*\theta_A(s(A)\varepsilon_A) = \theta_A(s(\varepsilon_A)A^2\varepsilon_A) = \theta_A(s(A)\varepsilon_A) = \frac{|R(A)|}{h(A)} \theta_A(c(A)\varepsilon_A) = \frac{|R(A)|}{h(A)} \theta_A(\varepsilon_A).\]

The last equation holds because \(\theta_A(q\varepsilon_A) = 0\) for any nontrivial \(q\) in the column group of \(A\) as the reader will easily verify. We have used \(r(A)\varepsilon_A = |R(A)|\varepsilon_A\) (since all row permutations fix \(\varepsilon_A\)) and \(s(A) = \frac{1}{h(A)} c(A)r(A)\). \(\square\)
3.2. **Enlarging the Young diagram.** The following will be important later.

We let $B = B_{n,\ell}$ be the standard tableau with underlying shape $D(B)$ an $n$ by $\ell$ rectangle with the standard filling obtained by putting 1 through $\ell$ in the first row, $\ell+1$ through $2\ell$ in the second row etc. Then $D(B)$ is the Young diagram corresponding to the dominant weight $\ell \varpi_n$. Here $\varpi_n = (1,1,\ldots,1)$ is the $n$-th fundamental weight for $GL(n)$. We note that we have $\varepsilon_B = \varepsilon_1^\ell \otimes \cdots \otimes \varepsilon_n^\ell$ and $\theta_B = \theta_1^\ell \otimes \cdots \otimes \theta_n^\ell$. Also note

\[ h(B) = \prod_{j=1}^{n} (\ell + j - 1)! \, . \]

**Lemma 3.2.**

1. $s(B)T^{n\ell}(\mathbb{C}^n)$ is one-dimensional and $s(B)T^{n\ell}(\mathbb{C}^n) = \mathbb{C}s(B)\varepsilon_B$ as $GL(n,\mathbb{C})$-modules.

2. $s(B)^*T^{n\ell}(\mathbb{C}^n)^*$ is one-dimensional and $s(B)^*T^{n\ell}(\mathbb{C}^n)^* = \mathbb{C}s(B)^*\theta_B$ as $GL(n,\mathbb{C})$-modules. In particular, we have $s(B)^*T^{n\ell}(\mathbb{C}^n)^* \cong \left( \bigwedge^n (\mathbb{C}^n)^* \right) \otimes \ell$.

We let $A$ be the standard filling of the Young diagram $D(\lambda)$ as above. Then $B|A$ denotes the standard tableau with underlying shape $D(B|A)$ given by making the shape of $A$ about $B$ (on the right), using the above filling for $B$ and filling $A$ in the standard way (as above) with $n\ell + 1$ through $n\ell + \ell'$. For example, if $B = \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}$ and $A = \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 \\ 7 & 8 & 9 \end{array}$, then $B|A = \begin{array}{cccc} 1 & 2 & 3 & 10 & 11 & 12 \\ 4 & 5 & 6 & 13 & 14 \\ 7 & 8 & 9 \end{array}$.

We have an idempotent $s(B|A)$ in the group ring of $S_{n\ell+\ell'}$ and the $n\ell + \ell'$ tensor $\varepsilon_{B|A} \in T^{n\ell+\ell'}(\mathbb{C}^n)$, which gives rise to a highest weight vector $s(B|A)\varepsilon_{B|A}$ in $s(B|A)(T^{n\ell+\ell'}(\mathbb{C}^n))$. Note

\[ \varepsilon_B \otimes \varepsilon_A = \varepsilon_{B|A}. \]

**Lemma 3.3.** There is a positive number $c(A, B)$ such that $s(B)\varepsilon_B \otimes s(A)\varepsilon_A = c(A, B)s(B|A)\varepsilon_{B|A}$.

**Proof.** Since the Young diagrams $D(B)$ and $D(A)$ are abutted along their vertical borders, we see

\[ c(B|A) = (c(B) \otimes 1_{\ell'}) \circ (1_{n\ell} \otimes c(A)) = (1_{n\ell} \otimes c(A)) \circ (c(B) \otimes 1_{\ell'}). \]

Also (for any standard tableau $C$)

\[ r(C)\varepsilon_C = |R(C)|\varepsilon_C. \]

Then we compute (using the three equations (3.8), (3.9),(3.10))
for different fillings $t$ and obtain the irreducible $O(\ell)$-module

\begin{align*}
\text{Corollary 3.4.} & \quad \text{Under the identification of } T^{n\ell}(\mathbb{C}^n) \otimes T^{\ell'}(\mathbb{C}^n) \to T^{n\ell+\ell'}(\mathbb{C}^n) \text{ given by tensor multiplication, we have the equality of maps}
\end{align*}

\begin{align*}
s(B) \otimes s(A) & = s(B|A).
\end{align*}

That is,

\begin{align*}
S_{B}(\mathbb{C}^n) \otimes S_{A}(\mathbb{C}^n) & = S_{B|A}(\mathbb{C}^n)
\end{align*}

as (physical) subspaces of $T^{n\ell+\ell'}(\mathbb{C}^n)$. The same statements hold for the dual space $S_{B|A}(\mathbb{C}^{n+\ell'})^*$ etc.

\begin{proof}
Since $S_{B}(\mathbb{C}^n)$ is one-dimensional, the tensor product $S_{B}(\mathbb{C}^n) \otimes S_{A}(\mathbb{C}^n)$ defines an irreducible representation for $GL_n(\mathbb{C})$ (under the tensor multiplication map $T^{n\ell}(\mathbb{C}^n) \otimes T^{\ell'}(\mathbb{C}^n)$ inside $T^{n\ell+\ell'}(\mathbb{C}^n)$). But by Lemma 3.3 it has nonzero intersection with the irreducible $GL_n(\mathbb{C})$-representation $S_{B|A}(\mathbb{C}^n)$ inside $T^{n\ell+\ell'}(\mathbb{C}^n)$. Hence the two subspaces coincide.
\end{proof}

\begin{enumerate}
\item[3.3.] \textbf{Representations of $O(V)$}. We extend the bilinear form $(\cdot,\cdot)$ on $V$ to $T^{\ell'}(V)$ as the $\ell'$-fold tensor product and note that the action of $S_{\ell'}$ on $T^{\ell'}(V)$ is by isometries. We let $\lambda = (b_1,\ldots,b_k)$ be a dominant weight of $SO(V)$ where $k = \left\lceil \frac{n}{2} \right\rceil$. Here our coordinates are relative to the standard basis $\{e_i\}$ of $[6]$, Planche II and IV. We will also use $\lambda$ to denote the corresponding partition of $\ell' = \sum b_i$. We again have the Schur functors $s(t(\lambda))(\cdot)$ as above and obtain representations $S_{t(\lambda)}(V)$, which are irreducible as $GL(V)$ representations. Here, by abuse of notation, we also considered $\lambda$ as a dominant weight for $GL(V)$ (by considering $\lambda = (b_1,\ldots,b_k,0,\ldots,0)$). This is justified since the Schur projectors are identical. However, of course, $S_{\lambda}(V)$ is not irreducible as an $O(V)$-representation.

We let $V^{[\ell']}$ be the space of harmonic $\ell'$-tensors (which are those $\ell'$-tensors which are annihilated by all contractions with the form $(\cdot,\cdot)$). We let $\mathcal{H}$ be the orthogonal projection $\mathcal{H}: T^{\ell'}(V) \to V^{[\ell']}$ onto the harmonic $\ell'$-tensors of $V$. Note that the space of harmonic $\ell'$-tensors is invariant under the action of $S_{\ell'}$. We then define the harmonic Schur functor $S_{t(\lambda)}(V)$ by

\begin{align}
S_{t(\lambda)}(V) & = \mathcal{H}S_{t(\lambda)}(V)
\end{align}

and obtain the irreducible $O(V)$-module $S_{t(\lambda)}(V)$ with highest weight $\lambda$. Of course, for different fillings $t(\lambda)$ of $D(\lambda)$, these representations are all isomorphic and we write $S_{\lambda}(V)$ for the isomorphism class.
4. The Weil Representation

We review different models of the Weil representation. In this section, $V$ denotes a real quadratic space of signature $(p, q)$ and dimension $m$.

We let $V'$ be a real symplectic space of dimension $2n$. We denote by $G' = \text{Mp}(n, \mathbb{R})$ the metaplectic cover of the symplectic group $\text{Sp}(V') = \text{Sp}(n, \mathbb{R})$ and let $\mathfrak{g}'$ be its Lie algebra. We let $K'$ be the inverse image of the standard maximal compact $U(n) \subset \text{Sp}(n, \mathbb{R})$ under the covering map $\text{Mp}(n, \mathbb{R}) \to \text{Sp}(n, \mathbb{R})$. Note that $K'$ admits a character $\det^{1/2}$, i.e., its square descends to the determinant character of $U(n)$. The embedding of $U(n)$ into $\text{Sp}(n, \mathbb{R})$ is given by $A + iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$.

We write $W_n,V$ for (an abstract model of) the $K'$-finite vectors of the restriction of the Weil representation of $\text{Mp}(V') \otimes V$ to $\text{Mp}(n, \mathbb{R}) \times O(V)$ associated to the additive character $t \mapsto e^{2\pi it}$.

4.1. The Schrödinger model. We let $V'_1$ be a Langrangian subspace of $V'$. Then $V \otimes V'_1$ is a Langrangian subspace of $V' \otimes V$ (which is naturally a symplectic space of dimension $2nm$). The Schrödinger model of the Weil representation consists of the space of (complex-valued) Schwartz functions on the Langrangian subspace $V'_1 \otimes V \cong V^n$. We write $S(V^n)$ for the space of Schwartz functions on $V^n$ and write $\omega = \omega_{n,V}$ for the action.

The Siegel parabolic $P' = M'N'$ has Levi factor
\begin{equation}
M' = \left\{ m'(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} ; a \in \text{GL}(n, \mathbb{R}) \right\}
\end{equation}
and unipotent radical
\begin{equation}
N' = \left\{ n'(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} ; b \in \text{Sym}_n(\mathbb{R}) \right\}.
\end{equation}
It is well known that we can embed $P'$ into $\text{Mp}(n, \mathbb{R})$ ([16]), and the action of $P'$ on $S(V^n)$ is given by
\begin{align}
\omega(m'(a)) \varphi(x) &= (\det a)^{m/2} \varphi(xa) \quad (\det a > 0), \\
\omega(n'(b)) \varphi(x) &= e^{\pi i \text{tr}(b(x,x))} \varphi(x)
\end{align}
with $x = (x_1, \ldots, x_n) \in V^n$. The central $\mathbb{C}^1$ acts by
\begin{equation}
\omega((1, t)) \varphi = \begin{cases} t \varphi & \text{if } m \text{ is odd} \\
\varphi & \text{if } m \text{ is even} \end{cases}
\end{equation}
for all $t \in \mathbb{C}^1$. The orthogonal group $G$ acts on $S(V^n)$ via
\begin{equation}
\omega(g) \varphi(x) = \varphi(g^{-1}x),
\end{equation}
which commutes with the action $G'$. The standard Gaussian is given by
\begin{equation}
\varphi_0(x) = e^{-\pi \text{tr}(x,x)/\alpha} \in S(V^n)\langle K \rangle.
\end{equation}
Here $(x, x)$ is the inner product matrix $(x_i, x_j)_{ij}$. Note that $\varphi_0$ has weight $\frac{p-q}{2}$, i.e.,
\begin{equation}
\omega(k') \varphi_0 = \det \frac{p-q}{2}(k') \varphi_0
\end{equation}
for $k' \in K'$.
We let $S(V^n)$ be the space of smooth, i.e., $K'$-finite, vectors inside the space of Schwartz functions on $V^n$. It consists of those Schwartz functions of the form $p(x)\varphi_0(x)$, where $p$ is a polynomial function on $V^n$.

4.2. The mixed model and the definition of local restriction for the Weil representation. We now describe a different model for the Weil representation, the so-called mixed model. Furthermore, we will define a "local" restriction $\iota_P^W$ from $S(V^n)$ to the space of Schwartz functions $S(W^n)$ for $W$, a (real) subspace of signature $(p - \ell, q - \ell)$.

4.2.1. The mixed model. We let $E = E_\ell$ be one of the standard totally isotropic subspaces of $V$, see (2.6). As before, we identify the dual space of $E_\ell$ with $E'_\ell$. Accordingly, for the decomposition $V = E \oplus W \oplus E'$, we write for $x \in V^n$,

$$x = \begin{pmatrix} u \\ x_W \\ u' \end{pmatrix}$$

with $u \in E^n$, $u' \in (E')^n$, and $x_W \in W^n$. We then have an isomorphism of two models of the Weil representation given by

$$S(V^n) \longrightarrow S((E')^n) \otimes S(W^n) \otimes S((E')^n)$$

$$\varphi \longmapsto \hat{\varphi}$$

given by the partial Fourier transform operator

$$\hat{\varphi}\left(\begin{pmatrix} \xi \\ x_W \\ u' \end{pmatrix}\right) = \int_{E^n} \varphi\left(\begin{pmatrix} u \\ x_W \\ u' \end{pmatrix}\right) e^{4\pi i (u,\xi)} du$$

with $\xi, u' \in (E')^n$ and $x_W \in W^n$.

We will need some formulae relating the action of $\omega$ in the two models. For this it is very convenient to use the bases $u_1, \ldots, u_\ell$ of $E$ and $u'_1, \ldots, u'_\ell$ of $E'$ to identify $E^n$ and $(E')^n$ with $M_{\ell,n}(\mathbb{R})$. Note that the pairing $(\ ,\ )$ between $E$ and $E'$ is then given by $-\frac{\ell}{2} u Ju'$. (This also explains the "4" in the definition of the Fourier transform).

Remark 4.1. The mixed model is nothing but the Schrödinger model for a different Langrangian subspace of the big symplectic space $V' \otimes V$ underlying the dual pair $\text{Sp}(n) \times O(V)$, namely, $S((V'_1 \otimes W) \oplus (V' \otimes E')) \simeq S(W^n) \otimes S((E'')^{2n})$. In this model, $O(W)$ acts via its natural linear action on the first factor, and $\text{Mp}(n, \mathbb{R})$ via its natural action on the second factor, as we will also see below.

For the actions in the mixed model, we have

Lemma 4.2. Let $\left(\begin{pmatrix} \xi \\ x_W \\ u' \end{pmatrix}\right) \in (E' \oplus W \oplus E')^n$.

(i) Let $n \in N_Q$ and write $n^{-1} = n_W(w)z(b)$ with $n_W(w) = n_W(w_1, \ldots, w_\ell)$ as in (2.36) and $z(b) \in Z_Q$ as in (2.34). Then

$$\tilde{m}\phi((\xi, x_W, u')) = e (-2(n_W(w)x_W + n_W(w)z(b)u'), \xi) \hat{\varphi}^{(\ell)}((\xi, w + n_W(w)u', u')).$$
Proof. (i) For \( g \in \text{SL}(E) \subset G \) (in particular, \( g \in N_{P'} \) or \( g \in M_{P'} \)) we have
\[
\hat{\varphi}(g^t(\xi, x_w, u')) = \hat{\varphi}(g(\xi, x_w, \tilde{gu}')
\]
with \( \tilde{g} = J g^* J \) and \( g^* = t^* g^{-1} \).

(ii) For \( t = (t_1, \ldots, t_\ell) \), set \( tJ = (t_{\ell}, \ldots, t_1) \) and \( |t| = t_1 \cdot t_2 \cdots t_\ell \). Then for \( a(t) \), we have
\[
\tilde{a}(t) \hat{\varphi}(t(\xi, x_w, u')) = |t|^n \hat{\varphi}(t(\tilde{t}\xi, \tilde{x}_w, \tilde{t}u')).
\]

(iii) For \( \hat{\varphi}(n', a) = (a_0, 0, \cdots, a_{-1}) \in M' \subset Sp(n, \mathbb{R}) \) with \( a \in GL_n^+(\mathbb{R}) \),
\[
(n'(a) \hat{\varphi})(t(\xi, x_w, u')) = (\det a)^{\frac{1}{2}} \hat{\varphi}(n(a^*, x_w a, u'))
\]

(iv) For \( \hat{h} \hat{\varphi}(n(\xi, x_w, u')) = \hat{\varphi}(n(h^{-1}x_w, u')). \)

(v) For \( n'(a) = (1^t, 1) \in N' \subset Sp(n, \mathbb{R}) \) with \( b \in Sym_n(\mathbb{R}) \),
\[
(n'(b) \hat{\varphi})(t(\xi, x_w, u')) = e \left( \text{tr} (b(x_w, x_w)) \right) \hat{\varphi}(t(\xi + \frac{1}{2} u'b, x_w, u')).
\]

Proof. This is straightforward. \( \Box \)

For \( n \in N \) such that \( n^{-1} = N(n') n_w(w) z(b) \) with \( N(n') \in N'_p \) as in (2.32) (so \( n' \in N_p \subset \text{SL}(E) \)) and \( n_w(w) \) and \( z(b) \) as above, we set
\[
\phi(n, \xi, x_w, u') = e \left( -2 N(n')^{-1}(n_w(w)x_w + n_w(w)z(b)u'), \xi \right)
\]
and obtain

Lemma 4.3. Let \( \varphi \in \mathcal{S}(V^n) \) and let \( P \) a standard parabolic subgroup of \( G \). Let \( E = E_\ell \) be the biggest totally isotropic subspace of \( V \) in the flag stabilized by \( P \) with Witt decomposition \( V = E \oplus W \oplus E' \). Let \( p = na(t)m(g, h) \) be the Langlands decomposition of \( p \in P \). Put \( \varphi_p(x) = \varphi(p^{-1}x) \). Then
\[
\hat{\varphi}_p \left( \begin{array}{c} \xi \\ x_w \\ u' \end{array} \right) = \Phi(n, \xi, x_w, u') |t|^n \hat{\varphi} \left( \begin{array}{c} \hat{g}tN(n') \xi \\ h^{-1}(x_w + n_w(w)u') \\ \hat{g}tN(n') u' \end{array} \right)
\]

Moreover, we have

Proposition 4.4. Let \( \varphi \in \mathcal{S}(V^n) \). Then the restriction of \( \hat{\varphi} \) to \( W^n \),
\[
\varphi \mapsto \hat{\varphi}|_{W^n},
\]
defines an \( g' \times MN \) intertwiner from \( \mathcal{S}(V^n) \) to \( \mathcal{S}(W^n) \). Here, we identify \( W \) with \( E^\perp / E \) to define the action of \( MN \) on \( W \). In particular, \( N \) and \( M_{P'} \) (see 2.28) act trivially on \( \mathcal{S}(W^n) \).
4.2.2. Weil representation restriction.

Definition 4.5. Let \( \varphi \in \mathcal{S}(V^n) \) and let \( P \) a standard parabolic of \( G \), and let \( E = E_\ell \) be the biggest totally isotropic subspace of \( V \) in the flag stabilized by \( P \) with Witt decomposition \( V = E \oplus W \oplus E' \). We then define the "local" restriction \( r_P^W(\varphi) \in \mathcal{S}(W^n) \) with respect to \( P \) for the Schrödinger model of the Weil representation \( \mathcal{W} \) by
\[
r_P^W(\varphi) = \tilde{\varphi}|_{W^n}.
\]

In the following, we will describe this restriction on a certain class of Schwartz functions on \( V^n \).

For \( x = (x_1, \ldots, x_n) \in V^n \), we write \( \left( \begin{array}{c} x_{ij} \\ x_{mj} \end{array} \right) \) for the standard coordinates of \( x_j \). We define a family of commuting differential operators on \( \mathcal{S}(V^n) \) by
\[
(4.12) \quad \mathcal{H}_{rj} = \left( x_{rj} - \frac{1}{2\pi} \frac{\partial}{\partial x_{rj}} \right),
\]
where \( 1 \leq r \leq m \) and \( 1 \leq j \leq n \). Then there exists a polynomial \( \tilde{H}_k \) such that
\[
(4.13) \quad \mathcal{H}_{rj}^k \varphi_0(x) = \tilde{H}_k(x_{rj})\varphi_0(x),
\]
where \( \varphi_0(x) \) is the standard Gaussian, see (4.7).

In fact, it is easy to see that \( \tilde{H}_k \) is essentially the \( k \)-th Hermite polynomial \( H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2} \):
\[
(4.14) \quad \tilde{H}_k(x) = (2\pi)^{-k/2} H_k\left( \sqrt{2\pi}x \right).
\]

We let \( \Delta \in M_{m \times n}(\mathbb{Z}) = (\delta_{rj}) \) be an integral matrix with non-negative entries and split \( \Delta \) into \( \Delta_+ \in M_{p \times n}(\mathbb{Z}) \) and \( \Delta_- \in M_{q \times n}(\mathbb{Z}) \) into its "positive" and "negative" part, where \( \Delta_+ \) consists of the first \( p \) rows of \( \Delta \) and \( \Delta_- \) of the last \( q \). (Recall \( m = p + q \).) We define operators
\[
\mathcal{H}_\Delta = \prod_{1 \leq r \leq m} \mathcal{H}_{rj}^{\delta_{rj}}, \quad \mathcal{H}_{\Delta+} = \prod_{1 \leq n \leq p} \mathcal{H}_{\alpha j}^{\delta_{\alpha j}}, \quad \mathcal{H}_{\Delta-} = \prod_{p+1 \leq n \leq m} \mathcal{H}_{\mu j}^{\delta_{\mu j}}
\]
so that \( \mathcal{H}_\Delta = \mathcal{H}_{\Delta+} \mathcal{H}_{\Delta-} \). Here again we make use of our convention to use early Greek letters for the "positive" indices of \( V \) and late ones for the "negative" indices.

Definition 4.6. For \( \Delta \) as above, we define the Schwartz function \( \varphi_\Delta \) by
\[
\varphi_\Delta(x) = \mathcal{H}_\Delta \varphi_0(x) = \prod_{1 \leq n \leq p} \tilde{H}_{\delta_{\alpha j}}(x_{\alpha j}) \tilde{H}_{\delta_{\mu j}}(x_{\mu j}) \varphi_0(x).
\]

We now describe \( \varphi_\Delta^V \) in the mixed model. The superscript \( V \) emphasizes that the Schwartz function is associated to the space \( V \). We begin with some auxiliary considerations. The following little fact will be crucial for us.

Lemma 4.7. For a Schwartz function \( f \in \mathcal{S}(\mathbb{R}) \), let \( \hat{f}(\xi) = \int_{\mathbb{R}} f(y) e^{-2\pi i y \xi} dy \) be its Fourier transform. Let \( g_k(y) = \tilde{H}_k(-y^2/2) e^{-\pi y^2/2} \). Then
\[
\hat{g}_k(\xi) = \sqrt{2}(-2i\xi)^k e^{-2\pi \xi^2}.
\]
Proof. We use induction and differentiate the equation \( \widetilde{\gamma}(y) = H_k(y) e^{-\pi y^2 / 2} \). The assertion follows from the recursion \( \widetilde{H}_{k+1}(y) = 2y \widetilde{H}_k(y) - \frac{1}{2\pi} \widetilde{H}_k'(y) \), which is immediate from the definition of \( \widetilde{H}_k \). The claim also follows easily from [15], (4.11.4). \( \Box \)

Remark 4.8. Recall that on the other hand that \( \widetilde{H}_k(y) e^{-\pi y^2} \) is an eigenfunction under the Fourier transform with eigenvalue \(-i^k\), see [15], (4.12.3). This fact is underlying the automorphic properties of the theta series associated to the special Schwartz forms \( \varphi_{nq, \lambda} \).

The Gaussian is given in standard coordinates by \( \varphi_0^V(x) = \exp(-\pi \sum_{j=1}^n \sum_{i=1}^m x_{ij}^2) \). On the other hand, in Witt coordinates, we have \( x_{rj} = \frac{1}{2}(y_{rj} - y_{(m-r)j}) \) and \( x_{(m-r)j} = \frac{1}{2}(y_{rj} + y_{(m-r)j}) \) for \( r \leq \ell \). Thus

\[
\varphi_0^V(x_W, u') = \exp \left( -\frac{1}{2} \pi \sum_{j=1}^n \sum_{r=1}^\ell (y_{rj}^2 + y_{(m-r)j}^2) \right) \varphi_0^W(x_W).
\]

We write slightly abusing

\[
\varphi_0^E(u, u') := \varphi_0^V \begin{pmatrix} u \\ 0 \\ u' \end{pmatrix} = \exp \left( -\frac{1}{2} \pi \sum_{j=1}^n \sum_{r=1}^\ell (y_{rj}^2 + y_{(m-r)j}^2) \right)
\]

We let \( \Delta' \) be the truncated matrix of size \((m - 2\ell) \times n\) given by eliminating the first and the last \(\ell\) rows from \(\Delta\). We let \(\Delta''\) be the matrix of these eliminated rows. Note that \( \mathcal{H}_{\Delta'} \) now defines an operator on \( S(W^n) \) and \( \mathcal{H}_{\Delta''} \) on \( S((E \oplus E^n)^n) \). We also obtain matrices \( \Delta'_+ \) of size \((p - \ell) \times n\) and \(\Delta'_-\) of size \((q - \ell) \times n\) by eliminating the first \(\ell\) and the last \(\ell\) rows from \(\Delta_+\) and \(\Delta_-\) respectively. Similarly we obtain \(\Delta''_+\) and \(\Delta''_-\). With these notations we obtain

Lemma 4.9. (i)

\[
\varphi_0^V \begin{pmatrix} \xi \\ x_W \end{pmatrix} = \varphi_0^W(x_W) \varphi_0^{E'}(\xi, u').
\]

(ii)

\[
\varphi_0^W (\varphi_0^V(x_W)) = \varphi_0^W(x_W) \varphi_0^{E'}(0, 0).
\]

In our applications all entries of \(\Delta_-\) will be zero, so \(\Delta = \Delta_+\) (by abuse of notation).

We first note

Lemma 4.10.

\[
\varphi_0^{E'}(\xi, 0) = 2^{n\ell / 2} \left( \prod_{j=1}^n \prod_{a=1}^{\ell} (-2i\xi_{aj})^{\delta_{ja}} \right) \varphi_0^E(\xi, 0).
\]

In particular, if in addition all entries of \(\Delta''_+\) vanish, then

\[
\varphi_0^V \begin{pmatrix} \xi \\ x_W \end{pmatrix} = 2^{n\ell / 2} \varphi_0^W(x_W) \varphi_0^E(\xi, 0).
\]
Proof. This follows from applying Lemma 4.7. \hfill \Box

We conclude

**Proposition 4.11.**  
(i) Assume that one of the entries of $\Delta''_+$ is nonzero, then

$$ r_P^{\mathcal{W}}(\varphi^V_{\Delta_+}) = 0. $$

(ii) If all of the entries of $\Delta''_+$ vanish, then

$$ r_P^{\mathcal{W}}(\varphi^V_{\Delta_+}) = 2^{n\ell/2} \varphi^W_{\Delta_+}. $$

**Remark 4.12.** Of course the analogous result holds for $r_P^{\mathcal{W}}(\varphi^V_{\Delta_-})$. However, a general formula for the restriction of $r_P^{\mathcal{W}}(\varphi^V_{\Delta_-})$, i.e., for $\varphi^F_{\Delta_0}(0,0)$, is more complicated (and is not needed in this paper).

4.3. **The Fock model.** It will be also convenient to consider the Fock model $\mathcal{F} = \mathcal{F}_{n,V}$ of the Weil representation. For more details for what follows, see the appendix of [10].

There is an intertwining map $\iota : S(V^n) \rightarrow \mathcal{P}(\mathbb{C}^{n(p+q)})$ from the $K'$-finite Schwartz functions to the infinitesimal Fock model of the Weil representation acting on the space of complex polynomials $\mathcal{P}(\mathbb{C}^{n(p+q)})$ in $n(p+q)$ variables such that $\iota(\varphi_0) = 1$. We denote the variables in $\mathcal{P}(\mathbb{C}^{n(p+q)})$ by $z_{\alpha i}$ ($1 \leq \alpha \leq p$) and $z_{\mu j}$ ($p+1 \leq \mu \leq p+q$) with $i = 1, \ldots, n$. Moreover, the intertwining map $\iota$ satisfies

$$ \iota \left( x_{\alpha i} - \frac{1}{2\pi i} \frac{\partial}{\partial x_{\alpha i}} \right) t^{-1} = \frac{1}{2\pi i} z_{\alpha i}, $$

$$ \iota \left( x_{\mu j} - \frac{1}{2\pi i} \frac{\partial}{\partial x_{\mu j}} \right) t^{-1} = -\frac{1}{2\pi i} z_{\mu j}. $$

By slight abuse of notation, we use the same symbol for corresponding objects in the Schrödinger and Fock model.

In the Fock model, $\varphi^V_{\Delta}$ looks as follows.

**Lemma 4.13.**

$$ \varphi^V_{\Delta} = \prod_{\substack{1 \leq \alpha \leq p \\ p+1 \leq \mu \leq m \\ 1 \leq j \leq n}} \left( \frac{1}{2\pi i} z_{\alpha j} \right)^{\delta_{\alpha j}} \left( -\frac{1}{2\pi i} z_{\mu j} \right)^{\delta_{\mu j}}. $$

Proposition 4.11 translates to

**Proposition 4.14.** If one of the entries of $\Delta''_+$ is nonzero, then

$$ r_P^{\mathcal{W}}(\varphi^V_{\Delta_+}) = 0. $$

If all of the entries of $\Delta''_+$ vanish, then

$$ r_P^{\mathcal{W}}(\varphi^V_{\Delta_+}) = 2^{n\ell/2} \prod_{\substack{1 \leq \alpha \leq p \\ 1 \leq j \leq n}} \left( \frac{1}{2\pi i} z_{\alpha j} \right)^{\delta_{\alpha j}}. $$
5. Differential graded algebras associated to the Weil representation

In this section, we construct the differential graded algebras $C^*_V$, $A^*_P$, and $B^*_P$, which were introduced in the introduction and define the map $i_P$ from $C^*_W$ to $B^*_P$. Again $V$ will denote a nondegenerate real quadratic space of dimension $m$ and signature $(p, q)$.

5.1. Relative Lie algebra complexes. For the convenience of the reader, we very briefly review some basic facts about relative Lie algebra complexes, see e.g., [5]. For this subsection, we deviate from the notation of the paper and let $g$ be any real Lie algebra and let $k$ be any subalgebra. We let $(U, \pi)$ be a representation of $g$. We set

$$ C^q(g, k; U) = \left[ \text{Hom} \left( \bigwedge^q (g/k), U \right) \right]^k \simeq \left[ \bigwedge^q (g/k)^* \otimes U \right]^k, $$

where the action of $k$ on $\bigwedge^q (g/k)$ is induced by the adjoint representation. Thus $C^q(g, k; U)$ consists of those $\varphi \in \text{Hom} \left( \bigwedge^q (g/k), U \right)$ such that

$$ \sum_{i=1}^q \varphi(Y_1, \ldots, [X, Y_i], \ldots, X_q) = X \cdot \varphi(Y_1, \ldots, Y_q) \quad (X \in k). $$

The differential $d : C^q \to C^{q+1}$ is defined by

$$ d\varphi(Y_0, Y_1, \ldots, Y_q) = \sum_{i=0}^q (-1)^i Y_i \cdot \varphi(Y_0, \ldots, \hat{Y}_i, \ldots, Y_q) $$

$$ + \sum_{i<j} (-1)^{i+j} \varphi([Y_i, Y_j], Y_0, \ldots, \hat{Y}_i, \ldots, \hat{Y}_j, \ldots, Y_q) $$

for $Y_0, \ldots, Y_q \in g/k$. We let $\{X_i\}$ be a basis of $g/k$ and let $\{\omega_i\}$ be the dual basis. Then in the setting of $\left[ \bigwedge^q (g/k)^* \otimes U \right]^k$, the differential $d$ is given by

$$ d = \sum_i A(\omega_i) \otimes \pi(X_i) + \frac{1}{2} \sum_i A(\omega_i) \text{ad}^*(X_i) \otimes 1. $$

Here $A(\omega_i)$ denotes the left multiplication with $\omega_i$ in $\bigwedge^*(g/k)^*$, and $\text{ad}^*(X_i)$ is the dual of the adjoint action on $\bigwedge^*(g/k)^*$, that is, $(\text{ad}^*(X_i)(\alpha))(Y_1, \cdots, Y_q) = \sum_{i=1}^q \alpha(Y_1, \ldots, [Y_i, X], \ldots, X_q)$. We easily see

**Lemma 5.1.** Consider two relative Lie algebra complexes $C^*(g, k; U)$ and $C^*(g', k'; U')$. Then the following datum,

(i) $\rho : g \to g'$, a Lie algebra homomorphism such that $\rho(k) \subseteq k'$,  
(ii) $T : U' \to U$, an intertwining map with respect to $\rho$ (i.e., $T(\rho(X) \cdot u') = X \cdot T(u')$ for $X \in g'$),

induces a natural map of complexes

$$ C^*(g', k'; U') \to C^*(g, k; U) $$

When realizing $\varphi$ as an element $\left[\bigwedge^q \left( (g'/\mathfrak{k}')^* \otimes U' \right) \right]^p$, then the map is given by
\[ \rho^* \otimes T, \]
where $\rho^* : (g'/\mathfrak{k}')^* \rightarrow (g/\mathfrak{k})^*$ is the dual map.

Now we let $G$ be any real Lie group with Lie algebra $\mathfrak{g}$ and let $K$ be a closed connected subgroup of $G$ (not necessarily compact) with Lie algebra $\mathfrak{k}$. For $U$ a smooth $G$-module, we let $\mathcal{A}^q(G/K; U)$ be the $U$-valued differential $q$-forms on $G/K$ (with the usual exterior differentiation). The $G$-action on $\mathcal{A}^q(G/K; U)$ is given by
\[ (g \circ w)_x(Y) = g(\omega_{g^{-1}}(g^{-1} \cdot Y)), \]
for $\omega \in \mathcal{A}^q(G/K; U)$, $x \in G/K$, and $Y \in T^*_x(G/K)$. Then evaluation at the base point of $G/K$ gives rise to an isomorphism of complexes
\[ \mathcal{A}^q(G/K; U) \simeq C^q(\mathfrak{g}, \mathfrak{k}; U) \]
of the $G$-invariant forms on $G/K$ with $C^q(\mathfrak{g}, \mathfrak{k}; U)$.

5.2. The differential graded algebra $C_V^\bullet$. We begin this section by defining a differential graded (but not graded-commutative) algebra $C_V^\bullet$. We first define the underlying complex.

The complex $C_V^\bullet$ is essentially the relative Lie algebra complex for $O(V)$ with values in $W_{n,v}$ tensored with the tensor algebra of $V_C$ and twisted by some factors associated to $C^n$. Precisely, it is the complex given by
\[ C_{V}^{j,r,k} = \left[ T^j(U)[\frac{p-2}{2}] \otimes T^k(C^n)^* \otimes W_{n,v} \otimes \bigwedge^r p_C^* \otimes T^k(V_C) \right]^{K' \times K \times S_k} \]
\[ \simeq \left[ T^j(U)[\frac{p-2}{2}] \otimes T^k(C^n)^* \otimes W_{n,v} \otimes \mathcal{A}'(X) \otimes T^k(V_C) \right]^{K' \times G \times S_k}. \]
Here $j, r, k$ are nonnegative integers and $\mathcal{A}'(X)$ denotes the space of complex-valued differential $r$-forms on $X$. We let $U = \bigwedge^n(C^n)^*$, and we define the action of $K'$ on $T^j(U)[\frac{p-2}{2}]$ by requiring $K'$ to act by the character $\det^{-j-\frac{p-2}{2}}$ on $T^j(U)$. Thus $K'$ acts by algebra homomorphisms shifted by the character $\det^{-\frac{p-2}{2}}$. We will usually drop the $[\frac{p-2}{2}]$ in what follows. Also note that all tensor products are over $C$. The differential is the usual relative Lie algebra differential for the action of $O(V)$. The group $K'$ acts on the first three factors, while the maximal compact subgroup $K_V = K$ of $SO_0(V)$ fixing the basepoint $z_0$ acts on the last three factors. Finally, the symmetric group $S_k$ acts on the second and the last factor.

We now give the complex $C_V^\bullet$ an associative multiplication. In order to give the complex the structure of a graded algebra we choose as a model for the Weil representation that has an algebra structure, the Fock model $\mathcal{F}_{n,V}$, the multiplication law is multiplication of polynomials. However, it is important to observe that $K'$ does not act on $\mathcal{F}_{n,V}$ by algebra homomorphisms (but rather by homomorphisms twisted by
the character $\det^{\frac{n-r}{2}}$). Now the vector space underlying $C^*_{V}$ is a subspace (of invariants under a group action) of a tensor product of graded algebras. Thus it remains to prove that the group acts by homomorphisms of the product multiplication.

**Lemma 5.2.** The group $K' \times K \times S_k$ acts by algebra homomorphisms on the tensor product of algebras $T^*(U) \otimes T^*(\mathbb{C}^n)^* \otimes W_{n,V} \otimes \bigwedge^r p_C^* \otimes T^*(V_C)$.

**Proof.** The statement is obvious except possibly for the action of the group $K'$. The group $K' \times K \times S_k$ acts on the algebra $F_{n,V}$ by algebra homomorphisms twisted by the character $\det^{\frac{n-r}{2}}$. It acts on the tensor product $T^*(U)$ by algebra homomorphisms twisted by the inverse character $det^{-\frac{n-r}{2}}$. The two twists cancel on the tensor product and we find that $K'$ acts by algebra homomorphisms. □

Sometimes it is more convenient to view an element $\phi \in C_{V}^{j,r,k}$ as an element in

$$\left[ Hom \left( T^k(\mathbb{C}^n); T^j(U) \otimes W_{n,V} \otimes \bigwedge^r p_C^* \otimes T^k(V_C) \right) \right] ^{K' \times K \times S_k}.$$  

For $w \in T^k(\mathbb{C}^n)$, we write $\phi(w)$ for its value in $T^j(U) \otimes W_{n,V} \otimes \bigwedge^r p_C^* \otimes T^k(V_C)$.

By Schur-Weyl theory, see [7], Lecture 6, we have the decomposition

$$T^k(\mathbb{C}^n)^* \simeq \bigoplus_{\lambda} s(t(\lambda))(T^k(\mathbb{C}^n)^*) \otimes V_{\lambda}^*.$$  

Here the sum is over the Young diagrams $\lambda$ with $k$ boxes and no more than $n$ rows, $t(\lambda)$ is a chosen standard filling of $\lambda$ for each $\lambda$ and $V_{\lambda}$ is the irreducible representation of $S_k$ corresponding to $\lambda$. We also have the corresponding decomposition

$$T^k(V_C) \simeq \bigoplus_{\mu} s(t'(\mu))(T^k(V_C)) \otimes V_{\mu}.$$  

Combining the two decompositions we obtain

$$C_{V}^{j,r,k} \simeq \bigoplus_{\lambda,\mu} \left[ T^j(U) \otimes S_{t(\lambda)}(\mathbb{C}^n)^* \otimes W_{n,V} \otimes \bigwedge^r p_C^* \otimes S_{t'(\mu)}(V_C) \otimes V_{\mu} \right] ^{K' \times K \times S_k}.$$  

Noting that

$$(V_{\lambda}^* \otimes V_{\mu})^{S_k} \simeq \begin{cases} 0 & \text{if } \lambda \neq \mu \\ \mathbb{C} & \text{if } \lambda = \mu, \end{cases}$$

we obtain

**Lemma 5.3.**

$$C_{V}^{j,r,k} \simeq \bigoplus_{\lambda} \left[ T^j(U) \otimes S_{t(\lambda)}(\mathbb{C}^n)^* \otimes W_{n,V} \otimes \bigwedge^r p_C^* \otimes S_{t(\lambda)}(V_C) \right] ^{K' \times K}.$$  

We have assumed (as we may do) that the fillings $t(\lambda)$ and $t'(\lambda)$ are the same. For the summands in the lemma we write $C_{V}^{j,r,t(\lambda)}$ (or just $C_{V}^{j,r,\lambda}$ if we do not want specify the filling) and obtain the complex $C^*_{V,\lambda}$ introduced in the introduction. The
application of the Schur functor $S_{t(\lambda)}(\cdot)$ on $T^k(\mathbb{C}^n)^*$ or equivalently applying $S_{t(\lambda)}(\cdot)$ on $T^k(V_C)$, gives rise to a projection map

$$(5.13) \quad \pi_{t(\lambda)} : C^{j,r,k}_V \longrightarrow C^{t_j,r,t(\lambda)}_V.$$  

That is,

$$(5.14) \quad \pi_{t(\lambda)} = 1_U \otimes s(t(\lambda))([\mathbb{C}^n])^* \otimes 1_{W_{n,V}} \otimes 1_{P^*} \otimes 1_V$$

$$= 1_U \otimes 1_{C^n} \otimes 1_{W_{n,V}} \otimes 1_{P^*} \otimes s(t(\lambda))_V.$$  

Here we have used subscripts to indicate which spaces the respective identity transformations 1 operate on. We will do this henceforth. We apply the harmonic projection $\mathcal{H}_V$, see (3.11), on the last factor to obtain $S_{t(\lambda)}(V_C)$, and we obtain a complex $C^{t_j,r,t(\lambda)}_V$ (or $C^{t_j,r,t(\lambda)}_V$) and a projection map

$$(5.15) \quad \pi_{[t(\lambda)]]} : C^{j,r,k}_V \longrightarrow C^{j,r,[t(\lambda)]}_V.$$  

That is,

$$(5.16) \quad \pi_{[t(\lambda)]]} = 1_U \otimes 1_{C^n} \otimes 1_{W_{n,V}} \otimes 1_{P^*} \otimes s([t(\lambda)])_V$$

$$= (1_U \otimes 1_{C^n} \otimes 1_{W_{n,V}} \otimes 1_{P^*} \otimes \mathcal{H}_V) \circ \pi_{t(\lambda)}.$$  

Remark 5.4. We can interpret an element $\varphi \in C^{j,r,k}_V$ as a $K' \times K \times S_k$-invariant homomorphism from $T^k(\mathbb{C}^n)$ to $T^j(U) \otimes W_{n,V} \otimes \bigwedge^r p^*_C \otimes T^k(V_C)$, see (5.8). In this setting, we can interpret $\pi_{t(\lambda)} \varphi$ as the restriction of the homomorphism $\varphi$ to the $S_{t(\lambda)}(\mathbb{C}^n)$. From this point of view, Lemma 5.3 states that the homomorphism $\pi_{t(\lambda)} \varphi$ on $S_{t(\lambda)}(\mathbb{C}^n)$ automatically takes values in $W_{n,V} \otimes \bigwedge^r p^*_C \otimes S_{t(\lambda)}(V_C)$.

5.3. The face differential graded algebra $A^*_P$ and the map $r_P$. In this section we assume $P$ is the stabilizer of a standard flag $E_{i_1} \subset E_{i_2} \subset \cdots \subset E_{i_k} = E_t$ and $N_P$ is the unipotent radical of $P$. We will abbreviate $E_{i_k}$ to $E$ and let $Q$ be the stabilizer of $E$ (a maximal parabolic subgroup). We will now construct a differential graded algebra $A^*_P$, which is the relative Lie algebra version of a differential graded subalgebra of the de Rham complex of the face $e(P) = N_P \times X_P$ of the Borel-Serre enlargement of $X$. We will continue with the notation of section 2.

We define the differential graded algebra $A^*_P$ associated to the face $e(P)$ of the Borel-Serre boundary corresponding to $P$ by

$$(5.17) \quad A^{j,r,k}_P = \left[ T^j(U) \otimes T^k(\mathbb{C}^n)^* \otimes W_{n,W} \otimes \bigwedge^r (n \oplus p_M)^*_C \otimes T^k(V_C) \right]^{K' \times K_P \times S_k}$$

$$\simeq \left[ T^j(U) \otimes T^k(\mathbb{C}^n)^* \otimes W_{n,W} \otimes A^r(e(P)) \otimes T^k(V_C) \right]^{K' \times N \times S_k}.$$  

Furthermore, we define $A^{*,\lambda}_P$ and $A^{*,[\lambda]}_P$ as for $C^{*,\lambda}_V$.

Definition 5.5. The “local” restriction map of de Rham algebras with coefficients

$$r_P : C^*_V \to A^*_P$$

of de Rham algebras with coefficients is given by

$$1 \otimes 1 \otimes r^W_P \otimes t^* \otimes 1,$$
where
\[ \iota : n \oplus m \hookrightarrow g \]
is the underlying Lie algebra homomorphism, and the map from the coefficients of
\[ C^*_V \] to the coefficients of \( A^{*,\Lambda}_P \) is given by the tensor product
\[ 1 \otimes 1 \otimes r^W_P \otimes 1, \]
where \( r^W_P : W_{n,V} \to W_{n,W} \) is the restriction map of the Weil representation (see Definition 4.5). By Lemma 5.1 we therefore see that \( r_P \) is a map of complexes. We note that \( r^W_P \) is not a ring homomorphism so \( r_P \) is not a map of algebras. Since \( r_P \) commutes with the action of the symmetric group \( S_k \), we obtain maps \( C^*_V \Lambda \to A^{*,\Lambda}_P \) and \( C^{*,[\Lambda]}_V \to A^{*,[\Lambda]}_P \) as well, which we also denote by \( r_P \).

Note that the induced map \( \iota^* : (g/k)^* \simeq p^* \to ((n \oplus m)/t_M)^* \simeq (n \oplus p_M)^* \) is the composition of the isomorphism \( \sigma^* : p^* \to (n \oplus a \oplus p_M)^* \), see (2.47), with the inclusion \( (n \oplus a \oplus p_M)^* \to (n \oplus p_M)^* \).

Finally observe that on the level of homogeneous spaces, the map \( r_P \) arises by realizing \( e(P) \) as the orbit of the basepoint \( z_0 \) under the group \( NM \). So in this setting, we are no longer thinking of \( e(P) \) as being at the boundary of \( X \); we have pushed \( e(P) \) far inside \( X \).

5.4. The distinguished subcomplex \( B^*_P \) of \( A^*_P \). We now introduce a subcomplex \( B^*_P \) of \( A^*_P \) associated to the subspace \( W \) of \( V \). We define a subgroup \( H \) of \( NM \) by
\[ (5.18) \quad H = (N_{p^*} \times Z_Q) \rtimes (K_W \times M_{p^*}) \]
and let
\[ (5.19) \quad e_W(P) = NM/H. \]
We have a fiber bundle
\[ (5.20) \quad \pi_P : e(P) \to e_W(P). \]
By the discussion of section 2.2, in particular (2.35), we immediately see that
\[ (5.21) \quad e_W(P) = X_W \times N_Q/Z_Q. \]
We let \( h = n_{p^*} \oplus z_Q \oplus t_W \oplus m_E \) be the Lie algebra of \( H \). We then see by (2.30) and Lemma 2.1
\[ (5.22) \quad (n \oplus m)/h \simeq n_W \oplus p_W, \]
where as in section 2.2, we have \( n_W \simeq W \otimes E \) as \( o(W) \times gl(E) \) representation. We define \( B^*_P \) as the relative (to \( h \)) version of \( A^*_P \). Precisely, we define
\[ (5.23) \quad B^*_{P,r^1,r^2,k} = \left[ T^j(U) \otimes T^k(C^n)^* \otimes W_{n,W} \otimes \bigwedge^{r_1}(p_W)^* \otimes \bigwedge^{r_2}(n_W)^* \otimes T^k(V_C) \right]^{K' \times H \times S_k} \]
\[ \simeq \left[ T^j(U) \otimes T^k(C^n)^* \otimes W_{n,W} \otimes A^{r_1}(X_W) \otimes A^{r_2}(N_Q/Z_Q) \otimes T^k(V_C) \right]^{K' \times MN \times S_k} \]
and define subspaces $B_P^{\bullet \lambda}$ and $B_P^{\bullet [\lambda]}$ as above. Then using Lemma 5.1 we see that

The identity map on $\mathfrak{n} \oplus \mathfrak{m}$ (since $\mathfrak{f}_M \subset \mathfrak{h}$) and on the coefficients gives rise to a natural inclusion of complexes

$$B_P^{\bullet, r_1, r_2, k} \hookrightarrow A_P^{\bullet, r_1 + r_2, k}.$$  

For the differential $d$ on $B_P^\bullet$, we have

**Lemma 5.6.** Recall that $\{X_{\alpha p}; \ell + 1 \leq \alpha \leq p, p + 1 \leq \mu \leq m - \ell\}$ is a basis of $\mathfrak{p}_W$ and $\{\omega_{\alpha p}\}$ denotes their dual. Furthermore $\{Y_{\alpha i}, Y_{\mu j}; \ell + 1 \leq \alpha \leq p, p + 1 \leq \mu \leq m - \ell, 1 \leq i \leq \ell\}$ is a basis for $\mathfrak{n}_W$ with dual $\{\nu_{\alpha i}, \nu_{\mu j}\}$. Then the differential $d$ on $B_P^\bullet$ is given by

$$(5.25) \quad d = \sum_{\alpha, \mu} \left( 1 \otimes 1 \otimes \omega(X_{\alpha p}) \right) \otimes A(\omega_{\alpha p}) \otimes 1 \otimes 1$$

$$+ 1 \otimes 1 \otimes 1 \otimes A(\omega_{\alpha p}) \otimes 1 \otimes \rho(X_{\alpha p})$$

$$+ 1 \otimes 1 \otimes 1 \otimes A(\omega_{\alpha p}) \otimes \mathrm{ad}^*(X_{\alpha p}) \otimes 1$$

$$+ \sum_{\ell} 1 \otimes 1 \otimes 1 \otimes (\sum_{\alpha} A(\nu_{\alpha i}) \otimes \rho(Y_{\alpha i}) + \sum_{\mu} A(\nu_{\mu j}) \otimes \rho(Y_{\mu j})).$$

Here $\omega(X_{\alpha p})$ is the Weil representation action on $\mathcal{W}_{\alpha, \omega}$, $\rho(X_{\alpha p})$ is the standard action on $T^k(W_C)$, and $\mathrm{ad}^*(X_{\alpha p})$ is the dual of the adjoint action of $\mathfrak{p}_W$ on $\mathfrak{n}_W$.

In the following, we write $d'$ for the action given by (5.26).

**Proof.** The first two terms of (5.25) arise from the action of $\mathfrak{p}_W$ on the coefficient system, while (5.26) comes from the $\mathfrak{n}_W$-action on $T^k(V_C)$ (note that $\mathfrak{n}_W$ acts trivially on $\mathcal{W}_{\alpha, \omega}$). This covers the first term of (5.5). For the second term of (5.5), we first note that since $[\mathfrak{n}_W, \mathfrak{p}_W] \subset \mathfrak{n}_W$, we have $\mathrm{ad}^*(X)(\varphi) = 0$ for $X \in \mathfrak{p}_W \oplus \mathfrak{n}_W$ and $\varphi \in \bigwedge^{r_1}(\mathfrak{p}_W^*)^\mathbb{C}$. Hence the action of $A(\omega_{\alpha p}) \otimes \mathrm{ad}^*(X_{\alpha p})$ on $\bigwedge^{r_1}(\mathfrak{p}_W^*)^\mathbb{C} \otimes \bigwedge^{r_2}(\mathfrak{n}_W^*)^\mathbb{C}$ is given by $A(\omega_{\alpha p}) \otimes \mathrm{ad}^*(X_{\alpha p})$. Thus the sum over $\mathfrak{p}_W$ in the second term of (5.5) accounts for half of the third line in (5.25). For the other half, we will show that it is given by the sum over $\mathfrak{n}_W$ in the second term of (5.5), that is,

$$\frac{1}{2} \sum_{\alpha, \mu} \left( A(\nu_{\alpha i}) \otimes \mu \otimes \omega(Y_{\alpha i}) \right) \otimes A(\omega_{\alpha p}) \otimes \mathrm{ad}^*(X_{\alpha p})$$

as operators on $\bigwedge^{r_1}(\mathfrak{p}_W^*)^\mathbb{C} \otimes \bigwedge^{r_2}(\mathfrak{n}_W^*)^\mathbb{C}$. For the left hand side of (5.27), we note $A(\nu_{\alpha i}) \otimes \mu \otimes \omega(Y_{\alpha i}) = (-1)^{r_1} (1 \otimes A(\nu_{\alpha i}) \otimes \omega(Y_{\alpha i}))$ similarly as above. This implies that it suffices to establish (5.27) on $\bigwedge^{r_2}(\mathfrak{n}_W^*)^\mathbb{C}$ (which is mapped to $\bigwedge^{r_2}(\mathfrak{n}_W^*)^\mathbb{C}$ by both sides of (5.27)). Since both sides act as derivations on $\bigwedge^{r_2}(\mathfrak{n}_W^*)^\mathbb{C}$, it suffices to check the identity on $\mathfrak{n}_W$, thus on the basis elements $\nu_{\beta j}, \nu_{\delta j} \in \mathfrak{n}_W$. But now, since $[X_{\alpha p}, Y_{\beta j}] = \delta_{\alpha j} Y_{\beta j}$ and $[X_{\alpha p}, Y_{\delta j}] = \delta_{\alpha j} Y_{\delta j}$, we see $\mathrm{ad}^*(Y_{\alpha i})(\nu_{\beta j}) = \delta_{ij} \omega_{\alpha i}$ and $\mathrm{ad}^*(Y_{\alpha i})(\nu_{\delta j}) = \delta_{ij} \omega_{\alpha i}$, while $\mathrm{ad}^*(Y_{\alpha i})(\nu_{\beta j}) = \delta_{ij} \omega_{\alpha i}$, $\mathrm{ad}^*(Y_{\alpha i})(\nu_{\delta j}) = \delta_{ij} \omega_{\alpha i}$, and $\mathrm{ad}^*(Y_{\alpha i})(\nu_{\beta j}) = \delta_{ij} \omega_{\alpha i}$, $\mathrm{ad}^*(Y_{\alpha i})(\nu_{\delta j}) = \delta_{ij} \omega_{\alpha i}$. This gives (5.27).
Remark 5.7. Assume $k = 0$. Then the $\text{SL}(E)$-part, $N_{\rho}M_{\rho'}$, inside $NM$ acts on $B^{r_1,r_2,0}_p$ solely on the fifth tensor factor $\bigwedge^{r_2}(n^*_W)_C$ (by the dual of the adjoint action). Since $N_{\rho}M_{\rho'} \subset H$, we can then replace $\bigwedge^{r_2}(n^*_W)_C$ with $\left(\bigwedge^{r_2}(n^*_W)_C\right)^{N_{\rho}M_{\rho'}}$, $H$ with $Z_Q \times K_W$, and $MN$ with $N_Q SO_0(W)$. That is,

\begin{equation}
\tag{5.28}
B^{r_1,r_2,0}_p \simeq \left[T^j(U) \otimes W_{n,W} \otimes \bigwedge^{r_1}(p^*_W)_C \otimes \left(\bigwedge^{r_2}(n^*_W)_C\right)^{N_{\rho}M_{\rho'}}\right]^{K' \times Z_Q K_W}.
\end{equation}

Note that since $Z_Q$ acts trivially on all factors, we can drop it in the first line of (5.28). But then Lemma 5.6 implies that $B^{r_1,r_2,0}_p$ is isomorphic to a relative Lie algebra complex for the orthogonal group for $W$. More precisely,

\begin{equation}
\tag{5.29}
B^{r_1,r_2,0}_p \simeq \left[T^j(U) \otimes C^*(\mathfrak{o}_W, \mathfrak{t}_W; W_{n,W} \otimes \left(\bigwedge^{r_2}(n^*_W)_C\right)^{N_{\rho}M_{\rho'}})\right]^{K'}.
\end{equation}

with $K'$ acting on $T^j(U)$ and $W_{n,W}$. In the next subsection, we will extend (5.29) for $r_2 = n\ell$ to the case of $k > 0$ and identify a subspace of $B^{r_1,r_2,0}_p$ with a certain complex $C^*_W$ for $O(W)$.

5.5. The map $\iota_B$. We now construct the map $\iota_B : C^*_W \hookrightarrow B^*_p$ of complexes mentioned in the introduction. First recall that we have by Lemma 2.1 $n^*_W \simeq W \otimes E^*$ as $O(W) \times GL(E)$-modules. Furthermore, see for example [7], p. 80,

\begin{equation}
\tag{5.30}
\bigwedge^r(n^*_W)_C \simeq \bigwedge^r(W_C \otimes E^*_C) \simeq \bigoplus_{\mu} S_{\mu}(W_C) \otimes S_{\mu'}(E^*_C),
\end{equation}

as $O(W) \times GL(E)$-modules. Here the sum extends over all partitions $\mu$ of $r$ with at most $\dim W = m - 2\ell$ rows and at most $\dim E = \ell$ columns, and $\mu'$ denotes the conjugate partition of $\mu$. We now identify $E^*$ with $E'$. For $r = n\ell$, we can take $\mu = \ell \varpi_n = (\ell, \ell, \ldots, \ell)$, so that $\mu' = n\varpi_\ell = (n, n, \ldots, n)$ and $S_{\mu'}(E^*_C) = \left(\bigwedge^{\ell} E^*_C\right)^{\otimes n} \simeq \mathbb{C}$ is the trivial $\text{SL}(E)$-module. We obtain

\begin{equation}
\tag{5.31}
S_B(W_C) \simeq \left[\bigwedge^{n\ell}(n^*_W)_C\right]_{\text{SL}(E)}.
\end{equation}

as $O(W)$-modules. Here $B = B_{n,\ell}$ is the filling of the Young diagram associated to $\mu$ described in section 3.2. We make this isomorphism explicit and compose it with the Young projector $s(B) : T^{n\ell}(W_C) \rightarrow S_B(W_C)$ to obtain

Definition 5.8.

\begin{equation*}
g_E : T^{n\ell}(W_C) \rightarrow S_B(W_C) \simeq \left[\bigwedge^{n\ell}(n^*_W)_C\right]_{\text{SL}(E)}.
\end{equation*}

Note that this map only depends on the choice of the generator of $\bigwedge^{\ell} E^*_C$ (and the choice of the filling). Explicitly, we observe that $g_E$ factors through $\text{Sym}^{\ell}(W_C) \otimes \cdots \otimes$
Sym^ℓ(W_C), and we normalize g_E such that for w_1, …, w_n ∈ W_C it is given by

\[ g_E(w_1^ℓ \otimes \cdots \otimes w_n^ℓ) = \frac{(-1)^{nℓ}}{h(B)} \left[ (w_1 \otimes u_1') \wedge \cdots \wedge (w_1 \otimes u') \right] \wedge \cdots \wedge \left[ (w_n \otimes u_1') \wedge \cdots \wedge (w_n \otimes u') \right], \]

where h(B) is given by (3.7). We will also need that on a standard basis element of \((T^d(W_C))^{⊗ n} \simeq T^{nℓ}(W_C)\), \(g_E\) is given by

\[ g_E((e_{i_1,1} \otimes \cdots \otimes e_{i_{ℓ,1}}) \otimes \cdots \otimes (e_{i_1,n} \otimes \cdots \otimes e_{i_{ℓ,n}})) = \frac{1}{h(B)} \left( \sum_{σ \in S_ℓ} ν_{σ(1),1ℓ} \wedge \cdots \wedge ν_{σ(nℓ),nℓ} \right) \wedge \cdots \wedge \left( \sum_{σ \in S_ℓ} ν_{σ(n),nℓ} \wedge \cdots \wedge ν_{σ(ℓ)n,1} \right). \]

Per construction we have

**Lemma 5.9.** The map \(g_E\) is \(O(W) \times S_{nℓ}\)-equivariant. Here the action of the symmetric group on \(\bigwedge^{nℓ}(W_C ⊗ E'_C)\) is given by acting on the tensor factors involving \(W\). Furthermore, \(g_E(s(B)w) = g_E(w)\) for \(w \in T^{nℓ}(W_C)\).

**Definition 5.10.** We define the map \(ι_P\) on \(C_W^{j_ℓ,k}\) as follows. In fact, it is defined on the underlying tensor spaces without taking the group invariants. First we set \(ι_P\) to be zero if \(k < nℓ\). If \(k ≥ nℓ\) we split the two tensor factors

\[ T^k(C^n)^* = T^{nℓ}(C^n)^* ⊗ T^{k-nℓ}(C^n)^* \quad \text{and} \quad T^k(W_C) = T^{nℓ}(W_C) ⊗ T^{k-nℓ}(W_C). \]

We define \(ι_P\) on tensors which are decomposable relative to these two splittings. We let \(u_1 = θ_1 \wedge \cdots \wedge θ_n\) be the standard generator of \(U = \bigwedge^n(C^n)^*\) (with the twisted \(K'\)-action). Let \(u_1^l ⊗ η ⊗ f ⊗ ω ⊗ w\) be a single tensor component of an element in \(C_W^{j_ℓ,k}\) and assume that \(k ≥ nℓ\). Assume that \(η\) and \(w\) are decomposable, that is

\[ η = η_1 ⊗ η_2 ∈ T^{nℓ}(C^n)^* ⊗ T^{k-nℓ}(C^n)^* \quad \text{and} \quad w = w_1 ⊗ w_2 ∈ T^{nℓ}(W_C) ⊗ T^{k-nℓ}(W_C). \]

Then we define

\[ ι_P(u_1^l ⊗ η ⊗ f ⊗ ω ⊗ w) = (-1)^{(q-ℓ)^n(n-1)} 2^{ℓn} h(B) \left( u_1^l ⊗ s(B)^*(η_1) \right) ⊗ ω \otimes g_E(w_1) \otimes w_2) \]

\[ ∈ T^{j+ℓ}(U) ⊗ T^{k-nℓ}(C^n)^* ⊗ W_{n,W} ⊗ \bigwedge^ℓ p_{decor}^*_c ⊗ \bigwedge^n n_{decor}^*_c ⊗ T^{k-nℓ}(W_C). \]

Note here that by Lemma 3.2, we see that \(S_B(C^n)^* = s(B)^* T^{nℓ}(C^n)^* \simeq T^{ℓ}(U)[0]\) and therefore \(u_1^l ⊗ s(B)^*(η_1)\) lies in \(T^{j+ℓ}(U)[−\frac{n-1}{2}]\) and is zero if and only if \(s(B)^*(η_1)\) is
zero. The normalization has no further meaning at this point and is just introduced to make the statement of the main results cleaner.

**Proposition 5.11.** \( \iota_P \) is a map of complexes

\[
\iota_P : C_W^{i,r,k} \to B_P^{i+\ell,r,n\ell,k-n\ell}.
\]

**Proof.** To see that \( \iota_P \) takes indeed values in \( B_P^{i+\ell,r,n\ell,k-n\ell} \) we first check the \( H \)-invariance of the image of \( \iota_P \). But this is clear, since \( \iota_P \) has coefficients in \( T^\ell(W) \) and \( g_E \) takes values in the \( SL(E) \)-invariants of \( \bigwedge_{n\ell} (n_W^*) \). The invariance under \( K' \) is inherited from \( C_W \), so is the \( S_{k-n\ell} \)-invariance; the invariance for \( C_W \) under the bigger symmetric group \( S_k \) implies the invariance under the smaller \( S_{k-n\ell} \) acting on the last \( k-n\ell \) tensor factors, which is preserved by \( \iota_P \).

It remains to show that \( \iota_P \) intertwines the differentials on \( C_W^* \) and \( B_P^* \). By Lemma 5.6, it enough to show that the part of the differential on \( B_P^* \) given by (5.26) acts trivially on the image of \( \iota_P \), i.e., \( d'\iota_P(\varphi) = 0 \). For this, it is enough to assume that \( k-n\ell = 1 \), so (5.26) acts on a single vector \( w \in W_C \). For a fixed \( i \leq \ell \), we let \( d'_i \) be the corresponding summand in (5.26) such that \( d' = \sum_i d'_i \). We identify \( d'_i \) with an operator on \( \bigwedge(n_W^*)_C \otimes V_C \), that is,

\[
d'_i = \left( \sum_{\alpha} A(\nu_{\alpha i}) \otimes \rho(Y_{\alpha i}) + \sum_{\mu} A(\nu_{\mu i}) \otimes \rho(Y_{\mu i}) \right).
\]

For \( w \in W_C \), we easily see

\[
d'_i w = (w \otimes u'_i) \otimes u_i \in (n_W^*)_C \otimes E_C.
\]

Here we identified \( n_W^* \) with \( W \otimes E^* \). This shows together with (5.32) that the composition

\[
d'_i \circ g_E : T^{n\ell+1}(W_C) \to \bigwedge_{n\ell+1} (W_C \otimes E_C^*) \otimes E_C
\]

is given for \( w_1^I \otimes \cdots \otimes w_n^I \otimes w \in (Sym^\ell(W))^{\otimes n} \otimes W \) by

\[
d'_i \circ g_E(w_1^I \otimes \cdots \otimes w_n^I \otimes w) = \frac{(-1)^{n\ell}}{h(B)} \left[ (w_1 \otimes u'_1) \wedge \cdots \wedge (w_1 \otimes u'_1) \right] \wedge \cdots \wedge \left[ (w_n \otimes u'_1) \wedge \cdots \wedge (w_n \otimes u'_1) \right] \wedge (w \otimes u'_1) \otimes u_i.
\]

In particular, \( d'_i \circ g_E \) is \( S_{n\ell+1} \)-equivariant (again with the action on the right hand on the tensor factors occupied by \( W \)). Furthermore, considering the decomposition (5.30) for \( r = n\ell + 1 \), we actually have

\[
d'_i \circ g_E : T^{n\ell+1}(W_C) \to \mathbb{S}_{\tilde{\alpha}}(W_C) \otimes \mathbb{S}_{\tilde{\alpha}'}(E_C^*) \otimes E_C,
\]

where \( \tilde{\alpha} \) denotes the partition \( (\ell, \ell, \ldots, \ell, 1) \) of \( n\ell + 1 \). In fact, for the \( W_C \)-component, \( d'_i \circ g_E \) is explicitly realized by applying the Schur functor \( s(\tilde{B}_i) \) associated to a certain (nonstandard) filling \( \tilde{B}_i \) of the Young diagram associated to \( \tilde{\alpha} \). Namely the (first) column of length \( n+1 \) has entries \( i, i+\ell, \ldots, i+(n-1)\ell, n\ell + 1 \) and the filling of the
other columns of length \( n \) are induced from the filling for \( B \). By the \( S_{n\ell+1} \)-invariance of \( d'_i \circ g_E \) we conclude that \( d'_i \circ g_E \) factors through \( S_{B_i}(W_C) \).

For \( \iota_P \), this discussion implies that for \( \varphi \in C_{W_{\ell'}}^{j,r,B_i} \), we have

\[
d'_i \iota_P(\varphi) = d'_i \iota_P \left( (1_U \otimes 1_{C^n} \otimes 1_{W_{n,W}} \otimes 1_{p_W} \otimes s(\tilde{B}_i)) \varphi \right).
\]

But from this we conclude by the discussion following Lemma 5.3, in particular Remark 5.4, that \( d'_i \circ \iota_P \) factors through \( C_{W_{\ell'}}^{j,r,B_i} \). But \( C_{W_{\ell'}}^{j,r,B_i} = 0 \) by (5.9) since \( \tilde{B}_i \) has \( n + 1 \) rows.

The reader easily checks from the definition that \( \iota_P \) satisfies the following properties.

**Lemma 5.12.** (1) \( \iota_P \) is a \( [T(U) \otimes W_{n,W} \otimes \bigwedge p_W]^{K'_r \times K_W} \)-module homomorphism. That is,

\[
\iota_P(\varphi_{W_{\ell'}}^{j',r',0} \cdot \varphi_{j,r,k}^W) = \varphi_{j',r',0}^W \cdot \iota_P(\varphi_{j,r,k}^W)
\]

for \( \varphi_{j',r',0}^W \in C_{W_{\ell'}}^{j',r',0} \) and \( \varphi_{j,r,k}^W \in C_{W_{\ell'}}^{j,r,k} \).

(2) \( \iota_P(\varphi_{j,r,k}^W) \) is zero if \( k < n\ell \).

(3) Suppose \( \varphi_{j,r,k}^W \in C_{W_{\ell'}}^{j,r,k} \) with \( k \geq n\ell \) and \( \varphi_{j',r',\ell'}^W \in C_{W_{\ell'}}^{j',r',\ell'} \). Then

\[
\iota_P(\varphi_{j,r,k}^W \cdot \varphi_{j',r',\ell'}^W) = \iota_P(\varphi_{j,r,k}^W) \cdot \varphi_{j',r',\ell'}^W.
\]

(4) Let \( \eta \in T^{n\ell}(\mathbb{C}^n)^* \) and \( w \in T^{n\ell}(W_C) \). Then

\[
\iota_P(1_U \otimes \eta \otimes 1_{\mathcal{T}} \otimes 1_{p_W} \otimes w) = \eta(\varepsilon_B)(u_\ell \otimes 1_{C^n} \otimes 1_\mathcal{T} \otimes 1_{p_W} \otimes g_E(w) \otimes 1_{1T(V_C)}).
\]

**Proposition 5.13.** Let \( k = n\ell + \ell' \) as above. Let \( \lambda \) be a dominant weight of \( \text{GL}_n(\mathbb{C}) \), and we let \( A \) be a standard filling of the associated Young diagram \( D(\lambda) \). We let \( B|A \) be the associated filling for the weight \( \ell \pi_n + \lambda \), see section 3. Then \( \iota_P \) descends to a map

\[
\iota_P : C_{W_{\ell'}}^{j,r,B|A} \longrightarrow B_{P_{\ell+\ell',n\ell,A}}^{j+\ell,r,n\ell,A},
\]

i.e.,

\[
\iota_P = \iota_P \circ \pi_{B|A}.
\]

Here \( \pi_{B|A} \) is the projection from \( C_{W_{\ell'}}^{j,r,n\ell+\ell'} \) to \( C_{W_{\ell'}}^{j,r,B|A} \), see (5.13). In particular, if \( \ell' = 0 \), we get \( \iota_P = \iota_P \circ \pi_B \).

**Proof.** We first observe that \( \iota_P \) is invariant under \( s(B) \) in the \( T^{n\ell}(W) \)-factor because \( g_E \) is and also \( s(B^*) \)-invariant in the \( T^{n\ell}(\mathbb{C}^n)^* \)-factor by definition, that is,

\[
\iota_P = \iota_P \circ (1_U \otimes 1_{T^{n\ell}(\mathbb{C}^n)^*} \otimes 1_{T^{n\ell}(\mathbb{C}^n)^*} \otimes 1_{W} \otimes 1_{p_W} \otimes s(B) \otimes 1_{T^{n\ell}(W)}) = \iota_P \circ (1_U \otimes s(B^*) \otimes 1_{T^{n\ell}(\mathbb{C}^n)^*} \otimes 1_{W} \otimes 1_{p_W} \otimes 1_{T^{n\ell}(W)} \otimes 1_{T^{n\ell}(W)}).
\]

Taking the \( S_p \)-invariance into account, we see that \( \iota_P \) maps (5.41)

\[
\left[ T^{j}(U) \otimes S_B(\mathbb{C}^n)^* \otimes S_A(\mathbb{C}^n)^* \otimes W_{n,W} \otimes \bigwedge^r (p_W)_C \otimes S_B(W_C) \otimes S_A(W_C) \right]^{K'_r \times K_W}
\]

to \( B_{j+2\ell,r,n\ell,A}^{j+2\ell,r,n\ell,A} \). But now
Lemma 5.14.

\[(5.42) \quad \left[ T^j(U) \otimes S_B(\mathbb{C}^n)^* \otimes S_A(\mathbb{C}^n)^* \otimes \mathcal{W}_{n,W} \otimes \bigwedge^r(p_W^*) \otimes S_B(W_C) \otimes S_A(W_C) \right]_{K' \times K_W}^{\otimes} = C_{W}^{j,r,B|A} \, .\]

Proof. In (5.42), we first observe that \( S_B(\mathbb{C}^n)^* \otimes S_A(\mathbb{C}^n)^* = S_{B|A}(\mathbb{C}^n)^* \) as subspaces of \( T^{nq,\ell}(\mathbb{C}^n) \), see Corollary 3.4. But then by Schur-Weyl theory, see Lemma 5.3 or Remark 5.4, we can now replace \( S_B(W_C) \otimes S_A(W_C) \) with its subspace \( S_{B|A}(W_C) \) in (5.42), that is, we can replace (5.42) with \( C_{W}^{j,r,B|A} \).

From this we obtain Proposition 5.13. \( \square \)

6. Special Schwartz forms

Again, in this section, \( V \) will denote a real quadratic space of dimension \( m \) and signature \( (p, q) \).

6.1. Construction of the special Schwartz forms. We recall the construction in [10] of the special Schwartz forms \( \varphi_{nq,\ell} \), \( \varphi_{nq,\lambda} \), and \( \varphi_{nq,|\lambda|} \), which define cocycles in \( C_V^\bullet \), \( C_V^{\bullet |\lambda} \), and \( C_V^{|\lambda|} \) respectively. It will be more convenient to use the model consisting of homomorphisms on \( T^\ell(\mathbb{C}^n) \) (and its subspaces \( S_{\ell(\lambda)}(\mathbb{C}^n) \)), see (5.8) and Remark 5.4. We will initially use the Schrödinger model \( S(V_n) \) of the Weil representation.

In [10], we construct for \( n \leq p \) a family of Schwartz forms \( \varphi_{nq,\ell} \) on \( V^n \) such that \( \varphi_{nq,\ell} \in C_V^{nq,nq,\ell} \). So

\[(6.1) \quad \varphi_{nq,\ell} \in \left[ \text{Hom} \left( T^\ell(\mathbb{C}^n), T^q(U) \otimes S(V_n) \otimes A^{nq}(X) \otimes T^\ell(V_C) \right) \right]_{K' \times G \times S_{\ell'}}^{K' \times G \times S_{\ell'}} \, .\]

These Schwartz forms are generalizations of the Schwartz forms considered by Kudla and Millson [12, 13, 14]. Under the isomorphism in (6.1), the standard Gaussian \( \varphi_0(x) = 1 \otimes e^{-\pi r(x,x)} \) is in \( [T^0(U) \otimes S(V^n)]_{K' \times K}^{K' \times G} \) corresponds to \n
\[\varphi_0(x, z) = 1 \otimes e^{-\pi r(x, x)} \in [T^0(U) \otimes S(V^n) \otimes C^\infty(X)]_{K' \times G}^{K' \times G} \, .\]

Definition 6.1. Let \( n \leq p \). The ‘scalar-valued’ form \( \varphi_{nq,0} \) is given by applying the operator

\[\mathcal{D} = \frac{1}{2^{n/2}} A(u_1) \otimes \prod_{i=1}^{n} \prod_{\mu=p+1}^{n+p} \left[ \sum_{\alpha=1}^{p} \left( x_{\alpha i} - \frac{1}{2\pi} \frac{\partial}{\partial x_{\alpha i}} \right) \otimes A(\omega_{\alpha \mu}) \right] \cdot \]

to \( \varphi_0 \):

\[\varphi_{nq,0} = \mathcal{D}(\varphi_0) \in C_V^{nq,nq,0} = \left[ T^q(U) \otimes S(V_n) \otimes \bigwedge^n (p_C^*) \right]_{K' \times K}^{K' \times K} \, .\]

Here as before \( A(\cdot) \) denotes left multiplication and \( u_1 \) is the generator of \( U = \bigwedge^n (\mathbb{C}^n)^* \). Furthermore, Theorem 3.1 of [12] implies that \( \varphi_{nq,0} \) is indeed \( K' \)-invariant.
For $T(V_{\mathbb{C}})$, we define for $1 \leq i \leq n$ another $K$-invariant differential operator $D_i'$ which acts on
\begin{equation}
S(V^n) \otimes \bigwedge^\bullet (p_{\mathbb{C}}^*) \otimes T(V_{\mathbb{C}})
\end{equation}
by
\begin{equation}
D_i' = \frac{1}{2} \sum_{\alpha=1}^p \left( x_{\alpha i} - \frac{1}{2\pi} \frac{\partial}{\partial x_{\alpha i}} \right) \otimes 1 \otimes A(e_{\alpha}).
\end{equation}
Let $I = (i_1, \ldots, i_{\ell'}) \in \{1, \ldots, n\}^{\ell'}$ be a multi-index of length $\ell'$ and write
\begin{equation}
\varepsilon_I = \varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_{\ell'}}
\end{equation}
for the corresponding standard basis element of $T_{\ell'}(\mathbb{C}^n)$. Then for $\varepsilon_I \in T_{\ell'}(\mathbb{C}^n)$, we define an operator
\begin{equation}
T_{\ell'}(\varepsilon_I) = D_{i_1}' \circ \cdots \circ D_{i_{\ell'}}'
\end{equation}
extend $T_{\ell'}$ linearly to $T_{\ell'}(\mathbb{C}^n)$.

**Definition 6.2.** Define
\begin{equation}
\varphi_{nq,\ell'} \in C_{V_{\mathbb{C}}}^{nq,\ell'} = \text{Hom}_{\mathbb{C}} \left( T_{\ell'}(\mathbb{C}^n), T^{q}(U) \otimes S(V^n) \otimes \bigwedge^{nq}(p_{\mathbb{C}}^*) \otimes T_{\ell'}(V_{\mathbb{C}}) \right)^{K' \otimes K \otimes S_{\ell'}}
\end{equation}
by
\begin{equation}
\varphi_{nq,\ell'}(w) = T_{\ell'}(w)\varphi_{nq,0}
\end{equation}
for $w \in T_{\ell'}(\mathbb{C}^n)$. We put $\varphi_{nq,\ell'} = 0$ for $\ell' < 0$. Here the $S_{\ell'}$-invariance of $\varphi_{nq,\ell'}$ is shown in Proposition 5.2 in [10], while the $K'$-invariance is Theorem 5.6 in [10].

Using the projections $\pi_{t(\lambda)}$ and $\pi_{[t(\lambda)]}$, see (5.13) and (5.15), we can therefore make the following definitions.

**Definition 6.3.** Let $i(\lambda) \leq \max(p, \lfloor \frac{\mu}{2} \rfloor)$ and $n \leq p$. (Otherwise, $\varphi_{nq,[t(\lambda)]}$ vanishes). For any standard filling $t(\lambda)$ of $D(\lambda)$, we define
\begin{equation}
\varphi_{nq,t(\lambda)} = \pi_{t(\lambda)} \varphi_{nq,\ell'} \in C_{V_{\mathbb{C}}}^{nq,t(\lambda)}
\end{equation}
\begin{equation}
\varphi_{nq,[t(\lambda)]} = \pi_{[t(\lambda)]} \varphi_{nq,\ell'} \in C_{V_{\mathbb{C}}}^{nq,[t(\lambda)]}
\end{equation}
We write $\varphi_{nq,\lambda}$ and $\varphi_{nq,[\lambda]}$, if we do not want to specify the standard filling.

**Proposition 6.4** (Theorem 5.7 [10]). The form $\varphi_{nq,\ell'}$ is closed. That is, for $w \in T_{\ell'}(\mathbb{C}^n)$ and $x \in V^n$, the differential form
\begin{equation}
\varphi_{nq,\ell'}(w)(x) \in \left[ A^{nq} \left( X; T_{\ell'}(V_{\mathbb{C}}) \right) \right]^G
\end{equation}
is closed.

6.2. **Explicit formulas.** We give more explicit formulas for $\varphi_{nq,\ell'}$ in the various models of the Weil representation.
6.2.1. Schrödinger model. We introduce multi-indices $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{iq})$ of length $q$ (typically) with $1 \leq i \leq n$ and $\beta = (\beta_1, \ldots, \beta_{\ell'})$ of length $\ell'$ (typically) with values in $\{1, \ldots, p\}$ (typically). Note that we suppressed their length from the notation. We also write $\alpha = (\alpha_{ij})$ for the $n \times q$ matrix of indices. With $I$ as above, we then define

$$\omega_{\alpha} = \omega_{\alpha_{1p} + 1} \wedge \cdots \wedge \omega_{\alpha_{q}p + q}$$
$$\mathcal{H}_{\alpha} = \mathcal{H}_{\alpha_{1i}} \circ \cdots \circ \mathcal{H}_{\alpha_{qi}}$$
$$\mathcal{H}_{\beta, I} = \mathcal{H}_{\beta_{1i}} \circ \cdots \circ \mathcal{H}_{\beta_{\ell' i'}}$$
$$e_{\beta} = e_{\beta_1} \otimes \cdots \otimes e_{\beta_{\ell'}}$$

Let $1 \leq \gamma \leq p$ and $1 \leq j \leq n$. For $I$, $\alpha$, and $\beta$ fixed, let

$$\delta_{\gamma j} = \# \{k; \alpha_{kj} = \gamma\} + \# \{k; (\beta_k, i_k) = (\gamma, j)\}.$$ 

This defines a $p \times n$ matrix $\Delta_{\alpha, \beta, I} = \Delta_{\alpha, \beta, I}^+$ and Schwartz functions $\varphi_{\Delta_{\alpha, \beta, I}^+}$ as in Definition 4.6.

Lemma 6.5. The Schwartz form $\varphi_{nq, \ell'}(\varepsilon_I)$ is given by

$$\varphi_{nq, \ell'}(\varepsilon_I) = \frac{1}{2^{nq/2 + \ell'}} \sum_{\alpha, \beta} u_1^q \otimes \varphi_{\Delta_{\alpha, \beta, I}^+} \otimes \omega_{\alpha} \otimes e_{\beta}.$$  

Proof. With the above notation we have

$$\varphi_{nq, \ell'}(\varepsilon_I) = \frac{1}{2^{nq/2 + \ell'}} \sum_{\alpha, \beta} u_1^q \otimes (\mathcal{H}_{\alpha_{1i}} \circ \cdots \circ \mathcal{H}_{\alpha_{qi}}) \mathcal{H}_{\beta_{1i}} \circ \cdots \circ \mathcal{H}_{\beta_{\ell' i'}} \varphi_0 \otimes (\omega_{\alpha_{1i}} \wedge \cdots \wedge \omega_{\alpha_{qi}}) \otimes e_{\beta}$$

$$= \frac{1}{2^{nq/2 + \ell'}} \sum_{\alpha, \beta} u_1^q \otimes (\mathcal{H}_{\alpha} \circ \mathcal{H}_{\beta, I}) \varphi_0 \otimes \omega_{\alpha} \otimes e_{\beta}.$$ 

But now we easily see

$$\mathcal{H}_{\alpha} \circ \mathcal{H}_{\beta, I} \varphi_0(x) = \prod_{\gamma=1}^{p} \prod_{j=1}^{n} \widetilde{H}_{\delta_{\gamma j}}(x_{\gamma j}) \varphi_0(x),$$

which gives the assertion. \qed

6.2.2. Mixed model. We now describe the Schwartz form $\varphi_{nq, \ell'}$ in the mixed model. We describe this in terms of the individual components $\varphi_{\Delta_{\alpha, \beta, I}^+}$ described in the Schrödinger model. From Lemma 4.9, Lemma 4.10, and Proposition 4.11 we see

Lemma 6.6.

$$\varphi_{\Delta_{\alpha, \beta, I}^+}^{\text{V}}(x_{\text{W}}, u') = \varphi_{\Delta_{\alpha, \beta, I}^+}^{\text{W}}(x_{\text{W}}) \varphi_{\Delta_{\alpha, \beta, I}^+}^{\text{E}}(\xi, u').$$
Note that \( \varphi^{W}_{\Delta_{\alpha, \beta}, t} \) only depends on the indices \( \alpha_{ij}, \beta_{j} \) such that \( \alpha_{ij}, \beta_{j} \geq \ell + 1 \), while \( \hat{\varphi}^{E}_{\Delta_{\alpha, \beta}, t} \) only depends on the indices \( \alpha_{ij}, \beta_{j} \) such that \( \alpha_{ij}, \beta_{j} \leq \ell \). In particular, if all \( \alpha_{ij}, \beta_{j} \geq \ell + 1 \), then

\[
\hat{\varphi}^{V}_{\Delta_{\alpha, \beta}, I}(\xi, x^{W}) = 2^{nq/2} \varphi^{W}_{\Delta_{\alpha, \beta}, t}(x^{W}) \varphi^{E}_{0}(\xi, 0).
\]

On the other hand, if one of the \( \alpha_{ij}, \beta_{j} \) is less or equal to \( \ell \), then

\[
\hat{\varphi}^{E}_{\Delta_{\alpha, \beta}, I}(0, 0) = \hat{\varphi}^{V}_{\Delta_{\alpha, \beta}, I}(0, x^{W}) = 0.
\]

### 6.2.3. Fock model

In the Fock model, the form \( \varphi_{nq, \ell'} \) looks particularly simple. We have

**Lemma 6.7.**

\[
\varphi_{nq, \ell'}(\varepsilon_{I}) = \frac{1}{2^{nq/2+\ell'}} \left( \frac{1}{2\pi i} \right)^{nq+\ell'} \sum_{\alpha_{1}, \ldots, \alpha_{n}} u_{\alpha_{1}} \otimes z_{\alpha_{1}, 1} \cdots z_{\alpha_{n}, n} \otimes z_{\beta_{1} I} \otimes \omega_{\alpha_{1}} \wedge \cdots \wedge \omega_{\alpha_{n}} \otimes e_{\beta}.
\]

Here we use the notational conventions in (6.6) and in addition

\[
(6.10) \quad z_{\alpha_{j}} = z_{\alpha_{1}, j} \cdots z_{\alpha_{n}, j}, \quad z_{\beta_{I}} = z_{\beta_{1} I} \cdots z_{\beta_{\ell'} I}.
\]

### 6.3. The forms \( \varphi_{0,k} \)

We now define another class of special forms. We will only do this in the Fock model.

**Definition 6.8.** We define \( \varphi_{0,k} \in \text{Hom}(T^{k}(C^{n}); T^{0}(U) \otimes F_{n,V} \otimes T^{k}(V_{C})) \) by

\[
\varphi_{0,k}(\varepsilon_{I}) = \frac{1}{2^{k}} \left( \frac{1}{2\pi i} \right)^{k} \sum_{\beta} 1 \otimes z_{\beta_{I}} \otimes e_{\beta}.
\]

**Remark 6.9.** The element \( \varphi_{0,k} \) is the image of the operator \( T_{k} \) (see (6.5)) applied to the Gaussian \( \varphi_{0} \) under the intertwiner from the Schrödinger to the Fock model. Also note that \( \varphi_{0,k} \) is not closed, hence they do not define cocycles.

We also leave the proof of the following lemma to the reader. It follows (in large part) from Remark 6.9 and the corresponding properties of \( \varphi_{nq, \ell'} \).

**Lemma 6.10.**

\[
\varphi_{0,k} \in [T^{0}(U) \otimes T^{k}(C^{n})^{*} \otimes F_{n,V} \otimes T^{k}(V_{C})]^{K' \times K \times S_{k}},
\]

i.e.,

\[
\varphi_{0,k} \in C_{V}^{0, 0, k}.
\]

From Lemma 6.7, we immediately see
Lemma 6.11.

\[ \varphi_{nq,\ell'} = \varphi_{nq,0} \cdot \varphi_{0,\ell'} \]

and

\[ \varphi_{0,k_1} \cdot \varphi_{0,k_2} = \varphi_{0,k_1+k_2}, \]

where the multiplication is the one in \( C_V^* \).

Remark 6.12. This kind of product decomposition for \( \varphi_{nq,\ell'} \) and \( \varphi_{0,k} \) in Lemma 6.11 only holds in the Fock model. In the Schrödinger model this only makes sense in terms of the operators \( D \) and \( T_\rho \) of Definition 6.1 and Definition 6.2 respectively.

We apply the projection \( \pi_{t(\lambda)} \), see (5.13), to define \( \varphi_{0,t(\lambda)} \):

Definition 6.13.

\[ \varphi_{0,t(\lambda)} := \pi_{t(\lambda)} \varphi_{0,k} \in C_V^{0,0,t(\lambda)}. \]

The following product formula will be important later.

Proposition 6.14. Let \( A = t(\lambda) \) be a filling of the Young diagram associated to \( \lambda \) and let \( B = B_{n,\ell} \) be the filling of the \( n \times \ell \) rectangular Young diagram introduced in section 3. Then

\[ \varphi_{0,B}^W \cdot \varphi_{0,A}^W = \varphi_{0,B|A}^W. \]

The proposition will follow from the next two lemmas.

Lemma 6.15. Both \( \varphi_{0,B}^W \cdot \varphi_{0,A}^W \) and \( \varphi_{0,B|A}^W \) are elements of

\[ C_W^{0,B|A,0} = \left[ T^0(U) \otimes S_{B|A}(C^n)^* \otimes F_{n,W} \otimes S_{B|A}(W_C) \right]^{K' \times K_W}. \]

Proof. Since \( S_B(C^n)^* \otimes S_A(C^n)^* = S_{B|A}(C^n)^* \) as subspaces of \( T^{n+\ell}(\mathbb{C}^n) \), see Corollary 3.4, the claim follows in the same way as Lemma 5.14.

Lemma 6.16.

\[ (\varphi_{0,B}^W \cdot \varphi_{0,A}^W)(s(B)\varepsilon_B \otimes s(A)\varepsilon_A) = \varphi_{0,B|A}^W(s(B)\varepsilon_B \otimes s(A)\varepsilon_A). \]

Proof. This is a little calculation using Lemma 3.3 and Lemma 6.11. Indeed, we have

\[ (\varphi_{0,B}^W \cdot \varphi_{0,A}^W)(s(B)\varepsilon_B \otimes s(A)\varepsilon_A) = (\varphi_{0,n\ell+\ell'}^W \cdot \varphi_{0,0}^W)(s(B)\varepsilon_B \otimes s(A)\varepsilon_A) \]

\[ = \varphi_{0,n\ell+\ell'}^W(s(B)\varepsilon_B \otimes s(A)\varepsilon_A) = c(A,B)\varphi_{0,n\ell+\ell'}^W(s(B|A)\varepsilon_{B|A}) \]

\[ = c(A,B)\varphi_{0,B|A}^W(s(B|A)\varepsilon_{B|A}) = \varphi_{0,B|A}^W(s(B)\varepsilon_B \otimes s(A)\varepsilon_A). \]

Thus we can prove Proposition 6.14. By Lemma 6.15 we see that \( \varphi_{0,B}^W \cdot \varphi_{0,A}^W \) and \( \varphi_{0,B|A}^W \) are \( U(n) \)-equivariant homomorphisms from \( S_{B|A}(C^n)^* \) to \( T^0(U) \otimes F_{n,W} \otimes S_{B|A}(W_C) \). By Lemma 6.16 they agree on the highest weight vector (see Lemma 3.3), hence coincide.
7. Local Restriction

We retain the notation from the previous sections. In this section, we will give formulas for the restrictions $r^W_P$ and $r_P$ of $\varphi_{nq,\ell'}$. Furthermore, we will show that the restriction of the special classes lies in the boundary subcomplex $B_P$. Finally, we will establish Theorem 1.1, the local restriction formula.

**Proposition 7.1.** We have

$$ (r^W_P \varphi^V_{nq,\ell'}(\varepsilon_I)) = \frac{1}{2^{(n-\ell)p/2+\ell}} \sum_{\alpha', \beta'} u_{\ell'}^q \otimes \varphi^W_{\alpha', \beta', \ell'} \otimes \omega_{\alpha'} \otimes e_{\beta'}. $$

Here $\varepsilon_I = \varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_r} \in T^\ell(\mathbb{C}^n)$, $\alpha'$ and $\beta'$ are the same indices as before with $\ell + 1 \leq \alpha'_{ij}, \beta'_{ij} \leq p$.

Loosely speaking, $r^W_P (\varphi^V_{nq,\ell'})$ is obtained from $\varphi^V_{nq,\ell'}$ by "throwing away" all the indices less or equal to $\ell$. In particular,

$$ r^W_P \varphi^V_{nq,\ell'} = 0 $$

if $n > p - \ell$.

**Proof.** This follows from Lemma 6.5, the formula for $\varphi^V_{nq,\ell'}$ in the Schrödinger model, and from Lemma 6.6. For the last statement, we observe that $\omega_{\alpha'}$ is in the $nq$-exterior power of a $(p - \ell)q$-dimensional space. □

The local restriction looks particularly simple in the Fock model. We have

**Proposition 7.2.**

$$ r^W_P (\varphi^V_{nq,\ell'}(\varepsilon_I)) = \frac{1}{2^{(n-\ell)q/2+\ell}} \left( \frac{1}{2\pi i} \right)^{nq+\ell'} \sum_{\alpha'_{11}, \ldots, \alpha'_{nq}} u_{\ell'}^q \otimes z_{\alpha'_{11}} \cdots z_{\alpha'_{nq}} \cdot z_{\beta'_{ij}} \otimes \left( \omega_{\alpha'_{11}} \wedge \cdots \wedge \omega_{\alpha'_{nq}} \right) \otimes e_{\beta'}. $$

Here $\alpha'_{ij}$ and $\beta'_{ij}$ are as before in Proposition 7.1.

**Proof.** This follows immediately either from Proposition 7.1 and applying the intertwiner to the Fock model or also from Proposition 4.14 and Lemma 6.7. □

**Theorem 7.3.** The image of $\varphi^V_{nq,\ell'}$ under the Weil representation restriction $r^W_P$ lies in $A^*_{\ell'}$ and coincides with the image under the restriction map of complexes $r_P$. More precisely,

$$ r_P \varphi^V_{nq,\ell'} = \left( 1_U \otimes 1_{\mathbb{C}^n} \otimes r^W_P \otimes \sigma^* \otimes 1_V \right) \varphi^V_{nq,\ell'}. $$

Moreover,

$$ r_P \varphi^V_{nq,\ell'} = 0 $$

for $n > p - \ell$. Analogous statements hold for $\varphi^V_{nq,\lambda}$ and $\varphi^V_{nq,[\lambda]}$. In particular,

$$ r_P \varphi^V_{nq,[\lambda]} = 0, $$

if we have in addition $\iota(\lambda) > \min(p, \left[ \frac{m}{2} \right]) - \ell$. 

Proof. By Definition 5.5, the restriction \( r_P : C^*_V \rightarrow A^*_P \) is given by \( 1_U \otimes 1_{C^n} \otimes r^W_P \otimes (\nu^* \otimes \sigma^*) \otimes 1_Y \). Then the theorem follows from Proposition 7.2 and Lemma 2.2, in particular (2.51): The components of \( \sigma^\psi_{nq,\ell}^V \) involving \( a^* \) already become annihilated under \( r^W_P \), so that \( \nu^* \) acts trivially on \( \sigma^\psi_{nq,\ell}^V \). Furthermore, the 'surviving' components \( \omega_{a^*_i} \) map under \( \sigma^* \) to \( \bigwedge^{q-\ell} (p^*) \otimes \bigwedge^\ell (n^*_{1\ell}) \).

We define

\[
(7.1) \quad \varphi_{P,n\ell} = \left( \frac{1}{2\pi i} \right)^{n\ell} \sum_{\gamma_1,\ldots,\gamma_n} u_1^\ell \otimes z_{\gamma_1,1} \cdots z_{\gamma_n,n} \otimes (\nu_{\gamma_1} \wedge \cdots \wedge \nu_{\gamma_n}).
\]

Here \( \gamma_j = (\gamma_{jm-\ell+1}, \ldots, \gamma_{jm}) \) is a multi-index of length \( \ell \) such that \( \ell + 1 \leq \gamma_{ji} \leq p \), and \( z_{\gamma_{ji}} \) as in (6.10). Furthermore, we have set

\[
(7.2) \quad \nu_{\gamma_j} = \nu_{\gamma_{jm-\ell+1}} \wedge \cdots \wedge \nu_{\gamma_{jm}} \in \bigwedge^{\ell} (n^*_{1\ell})
\]

We have

**Lemma 7.4.**

\[ \iota_P(\varphi^W_{P,0,B}) = (-1)^{(q-\ell)\ell\ell(n-1)/2} \varphi_{P,n\ell}. \]

In particular,

\[ \varphi_{P,n\ell} \in B^\ell_{P,n\ell,0}. \]

**Proof.** First note that by Proposition 5.13 we have \( \iota_P(\varphi^W_{P,0,B}) = \iota_P(\varphi^W_{P,0,n\ell}) \). We let \( \beta_1, \ldots, \beta_n \) be \( n \) indices of length \( \ell \) with \( \ell + 1 \leq \beta_{ji} \leq p \). For the corresponding elements \( e_{\beta_j} \in T^\ell (W) \), we easily see

\[
(7.3) \quad \sum_{\beta_1,\ldots,\beta_n} (z_{\beta_1,1} \cdots z_{\beta_n,n}) \otimes g(e_{\beta_1} \otimes \cdots \otimes e_{\beta_n}) = \left( \frac{\ell! \ell^n}{n!} \right) \sum_{\beta_1,\ldots,\beta_n} (z_{\beta_1,1} \cdots z_{\beta_n,n}) \otimes (\nu_{\beta_1} \wedge \cdots \wedge \nu_{\beta_n})
\]

with \( \nu_{\beta_j} \) as in (7.2). With that, we conclude

\[
(7.4) \quad \iota_P(\varphi^W_{P,0,B}) = (-1)^{(q-\ell)\ell\ell(n-1)/2} \left( \frac{1}{2\pi i} \right)^{n\ell} \sum_{\beta_1,\ldots,\beta_n} u_1^\ell \otimes (z_{\beta_1,1} \cdots z_{\beta_n,n}) \otimes (\nu_{\beta_1} \wedge \cdots \wedge \nu_{\beta_n}) = (-1)^{(q-\ell)\ell\ell(n-1)/2} \varphi_{P,n\ell}
\]

by (7.1). \( \square \)

**Remark 7.5.** Lemma 7.4 states that the image of \( \varphi^W_{P,0,B} \) under \( \iota_P \) corresponds to a scalar differential \( n\ell \)-form “in the nil-manifold directions” in the Borel-Serre boundary component corresponding to \( P \).

**Proposition 7.6.** We have

\[ r_P \varphi^V_{nq,\ell} \in B^{n(n-1)}_{P,0} \cdot \varphi_{nq,\ell,0} \cdot \varphi^{W}_{0,\ell}. \]

Moreover,

\[ r_P \varphi^V_{nq,\ell} = (-1)^{(q-\ell)\ell\ell(n-1)/2} \varphi^W_{nq,\ell,0} \cdot \varphi_{P,n\ell} \cdot \varphi^{W}_{0,\ell}. \]
Here we view $\varphi^{W}_{n(q-\ell),0} \in B^p_{\ell,n(q-\ell),0,0}$ and $\varphi^{W}_{0,\ell} \in B^0_{p,0,0,\ell'}$ in the natural fashion. The analogous statements hold for $\varphi^{V}_{nq,\lambda}$ and $\varphi^{V}_{nq,[\lambda]}$.

Proof. This follows immediately from Proposition 7.2 and

\[(7.5) \quad \sigma^* \omega_{\alpha_i^j} = \omega_{\alpha_i^j,p+1} \wedge \cdots \wedge \omega_{\alpha_i^j,m-\ell} \wedge \nu_{\alpha_i^j-\ell+1} \wedge \cdots \wedge \nu_{\alpha_i^j},\]

which follows from Lemma 2.2. The sign arises from 'sorting' $\sigma^* \left(\omega_{\alpha_i^j} \wedge \cdots \wedge \omega_{\alpha_i^k}\right)$ according to (7.5) into $\omega_{\alpha_i^j}$'s (which lie in $p_{W}$) and $\nu_{\alpha_i^j}$'s (which lie in $n_{W}$).

Corollary 7.7.

$$r_P(\varphi^{V}_{nq,\ell'}) = t_P(\varphi^{W}_{n(q-\ell),0} \cdot \varphi^{W}_{0,B} \cdot \varphi^{W}_{0,\ell'})$$

and by $S_{\nu}$-equivariance of $t_P$ also

$$r_P(\varphi^{V}_{nq,A}) = t_P(\varphi^{W}_{n(q-\ell),0} \cdot \varphi^{W}_{0,B} \cdot \varphi^{W}_{0,A}).$$

Proof. By Lemma 5.12 the claim is equivalent to

\[(7.6) \quad r_P(\varphi^{V}_{nq,\ell'}) = \varphi^{W}_{n(q-\ell),0} \cdot t_P(\varphi^{W}_{0,B} \cdot \varphi^{W}_{0,\ell'}),\]

and the assertion follows from Lemma 7.4 and Proposition 7.6.

Now Theorem 1.1 stated in the introduction easily follows.

Theorem 7.8. Let $A$ be a standard filling of Young diagram with $\ell'$ boxes and let $B_{n,\ell}$ be the standard tableau associated to the $n$ by $\ell$ rectangle as in section 3. Then

$$r_P(\varphi^{V}_{nq,A}) = t_P(\varphi^{W}_{n(q-\ell),B|A}).$$

Furthermore, for the form $\varphi^{V}_{nq,[A]}$ with harmonic coefficients, we have

$$r_P(\varphi^{V}_{nq,[A]}) = \pi_{[A]} t_P(\varphi^{W}_{n(q-\ell),B|A}).$$

Here $\pi_{[A]}$ is the projection onto $S_{[A]}(V_{\ell'})$-coefficients in $B^p_{\ell,n,q}$. Proof. This follows from combining Corollary 7.7, $\varphi^{W}_{0,B} \cdot \varphi^{W}_{0,A} = \varphi^{W}_{0,B|A}$ (see Proposition 6.14) and $\varphi_{nq,B|A} = \varphi_{nq,0} \cdot \varphi_{0,B|A}$ (see Lemma 6.11).

8. The Theta Series Associated to $\varphi_{nq,\ell'}$ and Global Restriction

In this section, we return to the global situation and assume that $V,W,E$ etc. are $\mathbb{Q}$-vector spaces. Using the Schrödinger model $\mathcal{S}(V(\mathbb{R})^n)$, we first introduce the theta series $\theta_{\varphi_{nq,\ell'}, \varphi_{nq,(\lambda)}},$ and $\theta_{\varphi_{nq,[(\lambda)]}}$. We show that they restrict to the boundary of the Borel-Serre compactification $\overline{Y}$ of $Y$ and describe the restriction to the series to the individual boundary components $e'(P)$.

We fix $h \in (L^\#)^n$ in the dual lattice once and for all. We set $\mathcal{L} = L^n + h$. For $g' \in G'$, we then define for $z \in X$, the theta series

\[(8.1) \quad \theta_{\varphi_{nq,\lambda}}(g', z) = \sum_{x \in \mathcal{L}} \omega(g') \varphi_{nq,\lambda}(x, z)\]
We define \( \theta_{\varphi_{nq,\lambda}} \) and \( \theta_{\varphi_{nq,[\lambda]}} \) in the same way. We easily see that the series are \( \Gamma \)-invariant, where \( \Gamma \) stabilizes \( L + h \). Thus they descend to closed differential \( nq \)-forms on the locally symmetric space \( Y \). More precisely,

\[
\theta_{\varphi_{nq,\lambda}}(g', z) \in C^\infty(\Gamma' \backslash G'; T^m(U) \otimes S_\lambda(C^n)^*)^{K'} \otimes Z^{nq}(Y, S_\lambda(V)).
\]

Here \( Z^{nq}(Y, S_\lambda(V)) \) denotes the closed differential \( nq \)-forms on \( Y \) with values in the local system \( S_\lambda(V) \) associated to \( S_\lambda(V) \). Furthermore, \( \Gamma' \) is an appropriate arithmetic subgroup of \( \text{Sp}(n,\mathbb{Z}) \), and we can identify \( C^\infty(\Gamma' \backslash G'; T^m(U) \otimes S_\lambda(C^n)^*)^{K'} \) in the usual way with the space of vector-valued \( C^\infty \)-functions on the Siegel upper half space of genus \( n \), transforming like a Siegel modular form of type \( \det^{m/2} \otimes \mathbb{S}_\lambda(C^n) \).

We can write

\[
L^n + h = \bigoplus_k (L^n_{E,k} + h_{E,k}) \oplus (L^n_{W,k} + h_{W,k}) \oplus (L^n_{E',k} + h_{E',k})
\]

for certain lattices \( L_{E,k}, L_{W,k}, L_{E',k} \) in \( E, W, E' \) respectively and vectors \( h_{E,k} \in (L^n_{E,k})^n \).

We define

\[
\theta_{r_P \varphi_{nq,\lambda}}(g', z) = \sum_k \det(L_{E,k})^{-n} \sum_{x \in L_{W,k}^\ast + h_{W,k}} \omega_W(g') r_P \varphi_{nq,\lambda}(x_W, z),
\]

where \( g' \in G' \) and \( z \in e(P) \). Again, it descends to a form on the corresponding corner \( e'(P) \) of the Borel-Serre compactification \( \overline{Y} \). We define \( \theta_{r_P \varphi_{nq,e'}} \) and \( \theta_{r_P \varphi_{nq,[\lambda]}} \) in the same way.

**Theorem 8.1.** The theta series \( \theta_{\varphi_{nq,\lambda}} \) extends to \( \overline{X} \). Moreover, for a standard rational parabolic \( P \), we denote by \( r_P \) be the restriction the boundary component \( e'(P) \) of \( \overline{Y} = \Gamma \backslash \overline{X} \). Then

\[
(r_P \theta_{\varphi_{nq,\lambda}}) = \theta_{r_P \varphi_{nq,\lambda}}.
\]

The same statement holds for the restriction of \( \theta_{r_P \varphi_{nq,e'}} \) and \( \theta_{r_P \varphi_{nq,[\lambda]}} \).

**Proof.** For \( g \in G \) and \( g' \in G' \), we let

\[
\theta^V_{\alpha,\beta,I}(g', g) = \sum_{x \in L^n + h} \omega_V(g') \varphi^V_{\Delta_{\alpha,\beta,I}}(g^{-1}x) \otimes g^* \omega_\alpha \otimes g_{e_\beta}.
\]

be the theta series associated to one fixed component of \( \varphi_{nq,e'} \). For the purposes of studying the restriction to \( e'(P) \), we can assume \( g' = 1 \) (since it intertwines with the restriction) and also \( g = a(t) \in A \) (since \( g \) varies in a Siegel set and by Lemma 4.2 and Lemma 4.3). Then by Poisson summation we have

**Lemma 8.2.** Let \( a(t) \in A \). Then

\[
\theta^V_{\alpha,\beta,I}(a(t)) = \sum_k \det(L_{E,k})^{-n} \sum_{x \in L_{W,k}^\ast + h_{W,k}} \sum_{\xi \in (L_{E,k}^\ast)^n} e(4\pi i(\xi, h_{E,k}))
\]

\[
\times |t|^n \varphi_{\Delta_{\alpha,\beta,I}}(t(\xi + u), \tilde{t}u') \varphi^W_{\Delta_{\alpha,\beta,I}}(x_W) \otimes a(t)^* \sigma^* \omega_\alpha \otimes a(t)e_\beta.
\]

**Proof.** This follows directly from the formulas given in Lemma 4.2. □
Write $\lambda_i = \alpha_i(a(t))$ for the value of the rational root $\alpha_i$ for $a(t)$. Then, if $W \neq 0$,
\begin{equation}
    t_i = \prod_{j=i}^{r} \lambda_j.
\end{equation}

**Lemma 8.3.** Assume that at least one of the $\alpha_{kj}$ and $\beta_k$ is less or equal than $\ell$. Then
\[ r_E^V \theta_{\alpha,\beta,t} = 0. \]

**Proof.** By Lemma 8.2 and (8.6), we clearly see that each term in $\theta_{\alpha,\beta,t}(a(t))$ is rapidly decreasing as $\lambda_i \to \infty$ for a nontrivial root $\alpha_i$ for $P$ unless both $\xi = u = 0$. But by Lemma 6.6, we have
\begin{equation}
    (8.7) \quad \tilde{\varphi}_{\Delta_{\alpha,\beta,t}}^E(0,0) = \tilde{\varphi}_{\Delta_{\alpha,\beta,t}}^V\left(\begin{array}{c} 0 \\ 0 \end{array}\right) = 0.
\end{equation}
in that case. We leave the case $W = 0$ to the reader. \hfill \Box

Now for the remainder of the proof of Theorem 8.1, assume that
\begin{equation}
    (8.8) \quad \alpha_{kj}, \beta_k \geq \ell + 1.
\end{equation}
Again, each term in Lemma 8.2 is rapidly decreasing unless $\xi = u = 0$. So it suffices to consider
\begin{equation}
    (8.9) \quad a(t)\varphi_{\Delta_{\alpha,\beta,t}}^W\left(\begin{array}{c} x_w \\ 0 \end{array}\right) = 2^{n/2} |t| \varphi_{\Delta_{\alpha,\beta,t}}^W(x_W) \otimes a(t)^* \sigma^* \omega_{\alpha} \otimes a(t)e_\beta.
\end{equation}
Now $a(t)e_\beta = e_\beta$ by (8.8). We have
\begin{equation}
    (8.10) \quad \sigma^* \omega_{\alpha_j} = \omega_{\alpha_{jp+1}} \land \cdots \land \omega_{\alpha_{q-\ell+1}} \land \nu_{\alpha_{q-\ell+1}} \land \cdots \land \nu_{\alpha_{q-1}},
\end{equation}
and $A$ acts trivially on the $\omega_*$'s, while for the $\nu_*$'s we have
\begin{equation}
    (8.11) \quad a(t)^* \nu_{ji} = \frac{db_{ji}}{t_i},
\end{equation}
where $1 \leq i \leq \ell$ and $\ell + 1 \leq j \leq m - \ell$. Here $b_{ji}$ is the coordinate of $W \otimes E$ for $e_j \otimes u_i$ and $t_i$ is the parameter in $a(t_1, \ldots, t_i, \ldots, t_r) \in A$. We obtain
\[
    |t| a(t)^* \sigma^* \omega_{\alpha} = |t| \omega_{\alpha_{1p+1}} \land \cdots \land \omega_{\alpha_{q-\ell+1}} \land \frac{db_{\alpha_{q-\ell+1}}}t \land \cdots \land \frac{db_{\alpha_{q}}}t \\
    \land \cdots \\
    \land \omega_{\alpha_{1p+1}} \land \cdots \land \omega_{\alpha_{q-\ell+m-\ell}} \land \frac{db_{\alpha_{q-\ell}}}t \land \cdots \land \frac{db_{\alpha_{q}}}t \\
    = \omega_{\alpha_{1p+1}} \land \cdots \land \omega_{\alpha_{q-\ell+1}} \land \frac{db_{\alpha_{q-\ell+1}}}t \land \cdots \land \frac{db_{\alpha_{q}}}t \\
    \land \cdots \\
    \land \omega_{\alpha_{1p+1}} \land \cdots \land \omega_{\alpha_{q-\ell+m-\ell}} \land \frac{db_{\alpha_{q-\ell+1}}}t \land \cdots \land \frac{db_{\alpha_{q}}}t.
\]
This shows for (8.9) we have
\begin{equation}
    (8.12) \quad a(t)\varphi_{\Delta_{\alpha,\beta,t}}^W\left(\begin{array}{c} x_w \\ 0 \end{array}\right) = r_E^V \varphi_{\Delta_{\alpha,\beta,t}}^W(x_W)
\end{equation}
independent of $t$. This completes the proof of Theorem 8.1. \hfill \Box


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