Great Problems of Mathematics: A Course Based on Original Sources

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Stimulating problems are at the heart of many great advances in mathematics. In fact, whole subjects owe their existence to a single problem which resisted solution. Nevertheless, we tend to present only polished theories, devoid of both the motivating problems and the long road to their solution. As a consequence, we deprive our students of both an example of the process by which mathematics is created and of the central problems which fueled its development.

A more motivating approach could, for example, begin a discussion of infinite sets with Galileo's observation that there are as many integers as there are perfect squares. This observation seems as paradoxical to today's students as it did to Galileo. Its ingeniously simple resolution (through a better definition of "size") is a tremendous educational experience, an example of the kind of education which the German logician Heinrich Scholz characterized as "that which remains after we have forgotten everything we learned".

We have designed a lower division honors course aimed at giving students the "big picture". In the course we examine the evolution of selected great problems from five mathematical subjects. Crucial to achieving this goal is the use of original sources to demonstrate the fundamental ideas developed for solving these problems. Studying original sources allows students to truly appreciate the progress achieved through time in the clarity and sophistication of concepts and techniques, and also reveals how progress is repeatedly stifled by certain ways of thinking until some quantum leap ushers in a new era. In addition to allowing a firsthand look at the mathematical mindscape of the time, no other method would show so clearly the evolution of mathematical rigor and the conception of what constitutes an acceptable proof. Thus most homework assignments focus on gaps and difficult points in the original texts.

Since mathematics is not created in a social vacuum, we supplement the mathematical content with cultural, biographical, and mathematical history, as well as a variety of prose readings, ranging from Plato's dialogue *Socrates and the Slave Boy* to modern writings such as an excerpt on "Mathematics and the End of the World" from [8]. They form the basis of regular class discussions. Two good sources for such readings are [11, 18]. To encourage student involvement, the discussions are led by one or two students, and everybody is expected to contribute. As the finale, each student gives a short presentation of a research paper written on a topic of his or her choice.

Our course serves as an "Introduction to Mathematics" drawing good students to the subject. It attracts students from remarkably diverse disciplines, serving as a general education course for some while acting as a springboard to further mathematics for others.

Here are our mathematical themes and original sources.

Area and the definite integral

Since ancient Greek times, mathematicians have attempted to compute areas and volumes as limits of approximations. The origins of the definite integral can be seen in Proposition 1 of Archimedes' *Measurement of the Circle* [16, pp. 91–93]. In his proof, Archimedes computes the area of a circle from polygonal approximations using a clever double *reductio ad absurdum* argument combined with the "method of exhaustion".

The next major advance is found in a text of Cavalieri's [21, pp. 214–219] illustrating his powerful "method of indivisibles" for computing the definite integral of simple polynomials. Cavalieri's book [6] was a very influential seventeenth century calculus text. While his method lacked rigor in part due to his cavalier attitude toward the infinite, he nevertheless succeeded in correctly computing many definite integrals.

Shortly thereafter, discovery of the inverse relationship between differentiation and integration transformed the definite integral into the most powerful computational tool in the mathematics and science of the time. Leibniz, in 1693, was the first to give a “proof” of the Fundamental Theorem of Calculus [21, pp. 282–284], an intuitive geometric argument based on infinitesimals.

These ideas matured greatly in Cauchy’s definition of the integral as a limit of sums in his series of calculus textbooks [5, vols. III and IV] [11, pp. 566–571], published in 1821–1823, which include his proofs of the most important theorems about the integral. Cauchy’s methods are significant for two reasons: his departure from the traditional use of geometry to treat the definite integral, and his effective use of the developing concept of limit. By replacing a geometric definition by the power of algebra and the limit concept, Cauchy dispensed with the use of infinitesimals, and thus made more rigorous proofs of the basic theorems possible for the first time. Subsequently, Cauchy’s work was put on a firm foundation by Weierstrass and his students, and generalized to apply to larger classes of functions via the Lebesgue integral.

The beginnings of set theory

While the apparent paradoxes associated with infinite sets have been known since the Renaissance, they did not receive serious attention until the nineteenth century, when Bolzano made a more systematic study of them in [1]. The issue arose again when progress in the development of analysis demanded a rigorous definition of the real numbers. Increased standards of rigor and the theory of functions of several variables necessitated a complete arithmetization of the real numbers. In order to improve upon Cauchy’s still partly geometric arguments for many of the central theorems in analysis, Dedekind and Cantor, both students of Weierstrass, gave two (equivalent) definitions of the real numbers not employing any geometric concepts.

Cantor’s definition of the real numbers [11, p. 577] is based on the concept of a Cauchy sequence, a notion which Cauchy had used to give an “internal” criterion for a sequence of numbers to converge, and one which makes no reference to its limit. Once Cantor had a suitable definition for the real numbers, he was in a position to study them as an infinite set.

Bolzano had made it clear in [1] that he considered the property of a one-to-one correspondence between an infinite set and a proper subset fundamental to the nature of infinite sets. After Cantor realized that this property should be used as the very definition of "infinite set", it was an easy task for him to demonstrate both the countability of the rational numbers [3, pp. 110–111] (using a nonstandard order relation on the rationals equivalent to the usual diagonal argument) as well as the uncountability of the real numbers [11, pp. 579–580]. The latter proof can of course immediately be generalized to prove that the power set operation increases cardinality, thus providing the basis for Cantor's system of infinite numbers. Cantor's continuum hypothesis [11, pp. 580–581] (which he considered to be a theorem) became one of the important modern problems in set theory, which was solved only relatively recently.

Solutions of algebraic equations

The search for algorithms to solve algebraic equations has always been one of the important problems of mathematics. Greek mathematics accomplished only the systematic solution of quadratic equations. Despite some progress by Arab mathematicians, most notably Omar Khayyam, nothing resembling a "formula" for higher degree equations emerged until the Renaissance. During that time, Greek mathematics was rediscovered and the old problems were attacked by new methods. Further progress for equations of degree three and four became possible through the introduction of algebraic techniques into Europe.

Cardano and several of his contemporaries discovered methods for solving equations such as $x^3 + ax = b$, published in his Ars Magna (The Great Art) [20, pp. 203–206]. In the Greek spirit, his arguments are geometric, viewing the cubic term as a volume, although the computation is easily translated into algebra.

The significance of his work (or, at least, of the publication of his book [4] in 1545) is twofold: it generated widespread interest in the problem of solving algebraic equations, and it raised the specter of imaginary numbers; even equations whose roots are all real may require imaginary numbers in the evaluation of Cardano's formula. (A selection of his work on imaginary roots can be found in [20, pp. 201–202].) Even by the time Lagrange summarized the state of the art in his lengthy 1770 memoir [17], no real progress had been made for equations of degree five and higher, despite much effort. Then in the early nineteenth century, Galois completely solved this two millenium old problem, using truly revolutionary methods which paved the way towards the development of abstract algebra.

Fermat's Last Theorem

The high point of Greek number theory was the determination of all Pythagorean triples by Euclid [15, Book X, Lemmas 1,2; in v. 3, p. 63f] and Diophantus. The motivation was of course geometric, namely to determine all right triangles with integer sides, via the Pythagorean Theorem. Diophantus' *Arithmetica* [9, 14, 22] inspired Fermat to conjecture in the margin of his copy what is now known as Fermat's Last Theorem, arguably the most famous open problem in all of mathematics. (Fermat's annotation can be found in [10, p. 2] [11, p. 218] [20, p. 213].)

Fermat probably could prove the conjecture for $n < 5$, but it was left to Euler to publish the first explicit proofs (which contained a gap for $n = 3$). Euler's proof for $n = 4$ [21, pp. 36–37] is quite accessible, using Fermat's method of "infinite descent" to reduce the problem to the determination of Pythagorean triples. (See e.g. [10, pp. 5–7] for a rigorous classification of all Pythagorean triples.)

The problem subsequently has had immense impact on the development of algebraic number theory and algebraic geometry. Examples of modern approaches are the use of complex roots of unity to factor the equation in various subfields of the complex numbers, and a reformulation in terms of algebraic geometry by considering rational points of curves. A good reference for modern developments is [10].

The parallel postulate

Since the time Euclid included his parallel postulate as a "self-evident truth", it has been the subject of controversy, and for two thousand years geometers attempted to prove it. It was not until the nineteenth century that these attempts were shown to be futile through the simultaneous development of non-Euclidean geometry by Bolyai, Lobachevsky, and Gauss. Their work demonstrated that geometrical axiomatic systems exist independent of the physical world.

Euclid's *Elements* was the first attempt at an axiomatized mathematical theory, with rigorous proofs based on his definitions, postulates and common notions [15, Book I; in v. 1, pp. 153–155]. A good illustration of their use is the proof of the Pythagorean Theorem [15, Book I, Proposition 47; in v. 1, pp. 349–350], which of course requires the parallel postulate.

Lobachevsky published his exploration of a non-Euclidean geometry in his *Geometrical Researches on the Theory of Parallels*, translated in [2], and his *Pangeometry* [20, pp. 360–374]. The first work presents Lobachevsky's development of the basic theorems of his non-Euclidean geometry and their proofs. The second, written near the end of his life, is more expository, giving a condensed presentation of the final development of his ideas. The consistency, and thus the acceptability, of this non-Euclidean geometry was made beautifully clear later in the century when Euclidean models for it were constructed, such as Poincaré's conformal model in the disk [19, pp. 241–242] [24, p. 2.3f] [7, 12, 13, 23].

These revolutionary ideas were popularized and developed further by Riemann, evolving into differential geometry and forming the mathematical basis for the physical theory of relativity. The shock waves of this revolution also affected the humanities, demolishing Kant's philosophy of space, and raising many fundamental questions in epistemology.

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