Exercise 2.31. (a) By expanding with the binomial theorem and gathering together common powers of y, show that with the substitution $x = d + y$, the polynomial, $p(x) = c_4 x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0$, becomes

$$c_4 y^4 + (4c_4 d + c_3)\, y^3 + \left(6c_4 d^2 + 3c_3 d + c_2\right) y^2 \\ + \left(4c_4 d^3 + 3c_3 d^2 + 2c_2 d + c_1\right) y + \left(c_4 d^4 + c_3 d^3 + c_2 d^2 + c_1 d + c_0\right).$$

(b) Recast algebraically Qin's method up to the first transformation so that we achieve (a). To accomplish this, rewrite each step in the form $c_4 Q + c_3 \to c_3'$ (Tableau 3 to Tableau 4, using primes differently from Qin). Be wary of the word "subtract" after Tableau 4, and be sure to accumulate primes. Compound these substitutions. The final c_4'''', c_3'''', etc., at Tableau 16 should be the coefficients of the powers of y found in (a). But notice how Qin's method, with its clever sequencing of multiplications and additions, implicitly mimics binomial coefficients while explicitly avoiding them.

Exercise 2.32. (a) Substitute $x_1 + 800$ into Qin's polynomial P_0 and thereby obtain P_1.
(b) Expand Qin's polynomial P_0 in a Taylor series about $d = 800$ and obtain P_1.

Exercise 2.33. Estimating the initial or even subsequent digits of a root of a general polynomial is tricky. Suppose, after having computed a new translation, you suspect your choice of a new digit was wrong. How do you know whether to add or subtract 1 from your suspect digit to create a new trial digit? Formulate a rule to determine this by using the signs of the coefficients c_0 and c_1 of the new polynomial. Recall that c_0 is its value and c_1 its derivative at the new estimate of the root.

Exercise 2.34. Find a root of $x^3 - 130x^2 - 250x - 1848$ by Qin's method.

Exercise 2.35. Reflect on the differences between the method of Qin for finding roots of polynomials and modern methods, such as pocket calculators and computer packages, e.g., Maple and Mathematica. In particular, find out in detail how a particular modern method actually does it.

2.3 Newton's Proportional Method

Here is how John Fauvel opens the book *Let Newton Be!* [72].

> In April 1727, the French writer Voltaire viewed with astonishment the preparations for the funeral of Sir Isaac Newton. The late President of the Royal Society lay in state in Westminster Abbey for the week preceding the funeral on 4 April. At the ceremony, his pall was borne in a ceremonious pageant by two dukes, three earls, and the Lord Chancellor. "He was buried," Voltaire observed, "like a king who had done well by his subjects." No scientist before had been so revered. Few since have been interred with such dignity and high honour.

Since there are many fine books about Newton (1642–1727) and his extraordinary contributions to science, let us just note that he lived in tumultuous times. He was born at Woolsthorpe in Lincolnshire, England, when the British Civil War was beginning. He came of age at the restoration of Charles II, who, although his rule was relaxed in many ways, received secret financial help from Louis XIV so that England might become again an absolute monarchy like France. Newton studied at Cambridge University for three years, which then closed because of the bubonic plague then raging. He returned to his family manor from 1664 to 1666, where the magic apple presumably fell and he discovered the universal law of gravitation. At that time he also demonstrated that all colors of the rainbow compose white light, and he began to develop the calculus. His lifetime was the time in England of the writers John Milton, John Dryden and John Bunyan, the composers Henry and Daniel Purcell, and fellow scientists Edmund Halley, Robert Hooke, Robert Boyle and William Harvey.

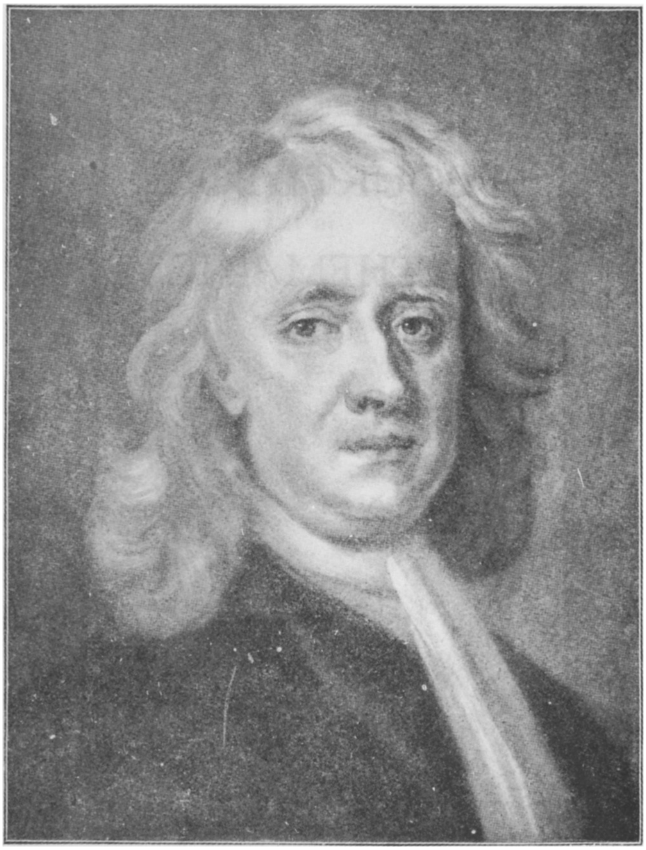

Photo 2.4. Newton.

"Newton's proportional method" [178, pp. 489–491] was probably written in 1665. Not surprisingly, as just a sketch in his "Waste Book," already described in the introduction of this chapter, his informal exposition has several annoying but minor errors, as well as a few inconsistencies, most of which we correct. Newton had read Wallis's (1616–1703) *Arithmetica Infinitorum*, by which time, through the efforts of Simon Steven (1548–1620), Rafael Bombelli (1526–1572) and François Viète (1540–1603), algebraic notation had evolved close to its present form. Consult [93] for details about the history of the Waste Book.

In preparation for reading the selection of Newton in its original language, we make several preliminary remarks about his first paragraph.[5] The comma serves as a decimal point in Europe. Parenthetically, Newton lists various ways of obtaining a first estimate, which is needed to start iterating. "Geometrically by description of lines" would mean something like the method 'Umar used in the introduction; and "an instrument … of numbers made to slide by one another" would be a slide rule [120], invented by William Oughtred and Edmund Wingate about 1630, and superseded by the pocket calculator in the 1970s. Then Newton splits his first estimate 2,2 into $g = 2$ and $y = 0{,}2$. At each step in the algorithm these are updated, although g and y are never mentioned again. Maybe this split is indicated by Newton initially writing 2,2 as $2|\underline{2}$ in the original manuscript. In other words, 2 and 2,2 are the first two estimates for the root. From these he will compute a better third estimate, and a final and significantly better fourth. Beware that his letters are slippery: they are recycled with new values, over and over again.

Newton, from

The resolusion of $y^{\rm e}$ *affected Equation* $x^3 + pxx + qx = r$. *Or* $x^3 + 10x^2 - 7x = 44$.

First having found two or 3 of $\mathrm{y^e}$ first figures of $\mathrm{y^e}$ desired roote viz 2,2 ($\mathrm{w^{ch}}$ may bee done either by rationall or Logarithmical tryalls as $\mathrm{M^r}$ Oughtred hath tought, or Geometrically by descriptions of lines, or by an instrument consisting of 4 or 5 or more lines of numbers made to slide by one another $\mathrm{w^{ch}}$ may be oblong but better circular) this knowne pte of $\mathrm{y^e}$ root I call g, $\mathrm{y^e}$ other unknowne pte I call y then is $g + y = x$. Then I prosecute $\mathrm{y^e}$ Resolution after this manner (making $x + p$ in $x = a$. $a + q$ in $x = b$. $b - r$ in $x = c$. &c.)

[5] As an aside, we explain the origin of the curious word "$\mathrm{y^e}$" occurring here, and meaning simply "the." The "y" was sometimes used by printers to mimic an old runic character called "thorn," which looked somewhat like a "y." But "$\mathrm{y^e}$" should not be confused with the archaic English pronoun "ye," as used in the Christmas carol *God rest ye merry Gentlemen*.

We interrupt the flow to make several points. To follow Newton's sketch, one could rewrite his equation as a polynomial set to zero: $x^3+pxx+qx-r= x^3+10x^2-7x-44=0$.[6] This is helpful in comparing his algorithm to the other algorithms of this chapter. But it seems best for now to follow Newton as he wrote it; then we will be finding out how much the left side deviates from 44.

There are two parts to his algorithm: computing the values that his polynomial x^3+10x^2-7x achieves at the estimates, which he does in the arrays below; and calculating a new estimate from a natural proportion created from four numbers: the previous two estimates and their polynomial values, i.e., by linear extrapolation a new estimate is found.[7]

Values of the polynomial are going to be calculated at 2 and 2,2; the results are both designated by b and these subtracted from r are designated by h and k, respectively, which are the deviations from 44, and which should be approaching 0. Newton evaluates a polynomial, $P(x)=x^3+px^2+qx$, as $((x+p)x+q)x$, a form computer programmers rediscovered as minimizing the number of multiplications (this is what his parenthetical phrase above explains). In the arrays below, $a=(x+p)x$ and $b=(a+q)x$; in short, multiply 12 by 2 to get 24, etc.

$$\left|\begin{array}{ll} 12 = x+p & \times \\ \hline 24 = a & x = 2 \end{array}\right. \cdot \begin{array}{c|l} a+q=17 & \times \\ \hline b = 34 & 2 = x \end{array}.$$

$r-b=10=h$. by supposing $x=2$.

This is essentially synthetic division, as described in the last section, although here the g at which the polynomial is evaluated may be longer than one digit, for example, 2,2. When $g=2$, we have, in the language of that section, that $P(x)=(x-2)Q(x)+P(2)=(x-2)\left(x^2+12x+17\right)+34$, calculated in the synthetic division

$$\begin{array}{rrrrr} 2\;) & 1 & 10 & -7 & 0 \\ & & \underline{2} & \underline{24} & \underline{34} \\ & 1 & 12 & 17 & 34 \end{array}$$

These coefficients are readily read off from Newton's calculations. But Newton does not complete the sequence of synthetic divisions to translate the polynomial. At the next iteration he uses the original polynomial, increasing his

6 Actually Newton starts out his sketch with the equation $x^3+pxx+qx+r=0$ and then shortly shifts his r to the right side without changing its sign! But we have altered his original in order to treat r consistently throughout.

7 This method of finding a real root by approaching it from both sides is called *Regula falsi* or the *method of false position* [123, article 301c]. It is natural, old, and has appeared in many different cultures [133]. Newton's contribution was to find an efficient way to evaluate polynomials.

labors unnecessarily (but Joseph Raphson (1648–1715) did complete the substitution to obtain a new polynomial, although not in the efficient manner of Qin, Horner, et al. [29, pp. 193–194]). Here are more of Newton's calculations, where he distributes the multiplication by 2,2 over 2 and ,2, and adds up the partial products.

Againe supposing $x = 2{,}2$.

$x+p = 12{,}2$	$\times$
2,44	,2
24,4	2,
$26{,}84 = a$	

$a+q = 19{,}84$	$\times$
3,968	,2
39,68	2,
$43{,}648 = b$	

$r - b = 0{,}352 = k$. $h - k = 9{,}648$.

In preparation for creating a proportion, differences in the value of the polynomial are being calculated, corresponding to differences in the argument. In geometrical terms, the heights of similar right triangles are being figured, 9,648 and 0,352; the bases will be 0,2 and y, if the graph of the polynomial is visualized (Exercise 2.36). Corresponding sides of the similar triangles lead to the proportion, as Newton explains next and which is at the heart of the method.

That is y^e^

$$\left.\begin{cases}\text{latter } r-b \text{ substracted from the former } r-b \text{ there remaines} \\ \text{difference twixt this \& y}^{\text{e}} \text{ former valor of } r-b \text{ is}\end{cases}\right\} 9{,}648.$$

& y^e^ difference twixt this & y^e^ former valor of x is 0,2. Therefore make

$$9{,}648 : 0{,}2 :: 0{,}352 : y.$$

Then is $y = \frac{0{,}0704}{9{,}648} = 0{,}00728$ &c. the first figure of w^ch^ being added to y^e^ last valor of x makes $2{,}207 = x$.

Now iterate the process (Exercise 2.37).

Then w^th^ this valor of x prosecuting y^e^ operacon as before tis

$x+p = 12{,}207$	$\times$
0,08544 9	,007
2,44140	,20
24,414	2,
26,94084 9 $= a$	

$a+q = 19{,}94084$	$\times$
0,13958 588	,007
3,98816 80	,20
39,88168	2,
44,00943 388 $= b$	

$r - b = -0{,}00943\,388$. w$^{\text{ch}}$ valor of $r - b$ substracted from y$^{\text{e}}$ precedent valor of $r - b$ y$^{\text{e}}$ diff: is $+0{,}36143\,388$. Also y$^{\text{e}}$ diff twixt this & y$^{\text{e}}$ precedent valor of x is $0{,}007$. Therefore I make

$$0{,}36143\,388 : 0{,}007 :: -0{,}00943\,388 : y.$$

That is

$$y = \frac{-0{,}00005\,90371\,6}{0{,}36143\,388} = -0{,}00016\,33\text{\&c.}$$

2 figures of w$^{\text{ch}}$ (because negative) I substract from y$^{\text{e}}$ former value of x & there rests $x = 2{,}20684$. And so might y$^{\text{e}}$ Resolution be prosecuted.

The final answer is slightly wrong. The leading 59 in the numerator of y should really be 66; thus $y = \frac{-0{,}00006\,603716}{0{,}36143\,388} = -0{,}00018\,27$, and thus $x = 2{,}20682$.

Stimulated by the need to solve equations arising from the birth of modern science, Newton and others created diverse algorithms (Exercises 2.38, 2.39 and 2.40).

Exercise 2.36. Graph the polynomial $x^3 + 10x^2 - 7x$ carefully, perhaps with a pocket calculator, together with the horizontal, $y = 44$. Plot all the points, quantities, and similar triangles that enter into the first iteration of Newton's example. Emphasize the secant. Notice that the first correction is positive, but the next is negative. Why is this? What sign do you expect the next corrections to have?

Exercise 2.37. (a) For Newton's proportional method derive the iterative formula,

$$x_{n+1} = x_n - \frac{y_n}{\frac{y_n - y_{n-1}}{x_n - x_{n-1}}} \qquad (n = 1, 2, 3, \ldots),$$

where x_0 and x_1 are initial estimates, $y_n = f(x_n)$, and f is any continuous function whose root we desire. Why is the fraction in the denominator close to the derivative, if it exists? How close are we to the classical "Newton's" method.

(b) Simplify the formula of (a) to

$$x_{n+1} = \frac{x_{n-1} y_n - x_n y_{n-1}}{y_n - y_{n-1}}.$$

Exercise 2.38. Rather than guessing the new root proportionally, that is, by linear extrapolation, try to speed up convergence by using quadratic extrapolation, i.e., by fitting a quadratic curve to the old estimates. Work out the details of this by resolving the equation, $x^3 + 10x^2 - 7x = 44$, of the selection in this new way. You will need three original guesses x, so add 2.1 to 2.0 and

2.2. Use these to plot three points on the cubic curve, and then solve three linear equations to find the a, b, c that fit $ax^2 + bx + c$ to the cubic polynomial at these three values of x; that is, make the quadratic pass through the three points. Then find where the quadratic crosses the line $y = 44$, and use this for a new estimate of x. How many iterations do you need to match Newton's final accuracy? Edmund Halley (1656?–1743) proposed quadratic extrapolation in 1694. (See [29, pp. 191–195] for the history of this method and many other proposed variants of Newton's and Simpson's methods.)

Exercise 2.39. Newton had another way of finding a root of a polynomial, which should first be read in [179, pp. 328–340], and which is sometimes cited as evidence for Newton discovering "Newton's method." But to do this, one must conjecture what Newton was thinking. He applied this method only to specific polynomials where the coefficients were numbers, with the arithmetical operations leaving no trace of any algebra, and even obscuring the presence of a derivative. So the general form is not obvious. The reader is asked to create the theory from his specific example.
(a) Explain how close it is to Qin's and the Horner–Ruffini methods, excepting that Newton's layout is radically different and he allows the answer to increase more than one decimal place at a time.
(b) When Newton estimates the next few digits, he is in effect computing a derivative. Show numerically that in his polynomial, $f(y) = y^3 - 2y - 5$, when he substitutes $y = 2 + p$ and uses the linear and constant terms to estimate the next few digits, he has computed $-\frac{f(2)}{f'(2)}$.
(c) Algebraically, retrace (b) for an arbitrary polynomial with literals as coefficients: make a linear substitution, use the binomial theorem to expand, throw out all terms higher than the first power, and show that indeed the correction term has the renowned form of Simpson's fluxional method [29, pp. 191–194].

Exercise 2.40. [179, p. 326] Although Newton did not discover the method named after him, he did invent many remarkable techniques. In particular, he was fond of taking strictly numerical methods and applying them algebraically. As an example of this technique, you are to expand $\sqrt{a^2 + x}$ in a power series in x.
(a) By completing powers in complete analogy with the numerical method of calculating square roots at the beginning of Section 2, find $\sqrt{a^2 + x}$ by starting with a trial divisor of a into $a^2 + x$. Continue through the fourth power of x.
(b) Find $\left(a^2 + x\right)^{1/2}$ by the binomial theorem, and compare with (a). They should be the same.
(c) Find $\sqrt{5}$ by taking $a = 2$ and $x = 1$, using the series of (b) through the fourth power. How accurate is the answer? (Hint: Partial sums of an alternating series with the terms decreasing in size bracket the answer.)