Mathematical Masterpieces: Teaching with Original Sources

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Our upper-level university honors course, entitled *Great Theorems: The Art of Mathematics*, views mathematics as art and examines selected mathematical masterpieces from antiquity to the present. Following a common practice in the humanities, for example in Chicago's Great Books program and St. John's College curriculum, we have students read original texts without any modern writer or instructor as intermediary or interpreter. As with any unmediated learning experience, a special excitement comes from reading a first-hand account of a new discovery. Original texts can also enrich understanding of the roles played by cultural and mathematical surroundings in the invention of new mathematics. Through an appropriate selection and ordering of sources, students can appreciate immediate and long-term advances in the clarity, elegance, and sophistication of concepts, techniques, and notation, seeing progress impeded by fettered thinking or old paradigms until a major breakthrough helps usher in a new era. No other method shows so clearly the evolution of mathematical rigor and abstraction.

The end result is a perception of mathematics dramatically different from the one students get from traditional courses. Mathematics is now seen as an evolving human endeavor, its theorems the result of genius struggling with the mysteries of the mathematical universe, rather than an unmotivated, ossified edifice of axioms and theorems handed down without human intervention. For instance, after reading Cayley's paper (see below) introducing abstract group theory, the student is much less bewildered upon seeing the axiomatic version so devoid of any motivation. Furthermore, Cayley makes the connection to the theory of algebraic equations, which a student might otherwise never become aware of. An additional feature of the method is that suddenly value judgments need to be made: there is good and bad mathematics, there are elegant proofs and clumsy ones, and of course plenty of mistakes and unsubstantiated assertions which need to be examined critically. Later follows the natural realization that new mathematics is being created even today, quite a surprise to many students.

To achieve our aims we have selected mathematical masterpieces meeting the following criteria. First, sources must be original in the sense that new mathematics is captured in the words and notation of the inventor. Thus we assemble original works or English translations. When English translations are not available, we and our students read certain works in their original French, German, or Latin. In the case of ancient sources, we must often depend upon restored originals and probe the process of restoration. Texts selected also encompass a breadth of mathematical subjects from antiquity to the twentieth century, and include the work of men and women and of Western and non-Western mathematicians. Finally, our selection provides a broad view of mathematics building upon our students' background, and aims, in some cases, to reveal the development over time of strands of mathematical thought. At present the masterpieces are selected from the following.

Archimedes: The Greek method of exhaustion for computing areas and volumes, pioneered by Eudoxus, reached its pinnacle in the work of Archimedes during the third century BC. A beautiful illustration of this method is Archimedes's determination of the area inside a spiral. [10] An important ingredient is his summation of the squares of the terms in certain arithmetic progressions. As in all of Greek mathematics, even this computation is phrased in the language of geometry. Further advance toward the definite integral did not come until the Renaissance.

Omar Khayyam: The search for algorithms to solve algebraic equations has long been important in mathematics. After Babylonian and ancient Greek mathematicians systematically solved quadratic equations, progress passed to the medieval Arab world. The work of Arab mathematicians began to close the gap between the numerical algebra of the Indians and the geometrical algebra of the Greeks. Notable is Omar Khayyam's *Algebra* of the late 11th or early 12th century. Here he undertakes the first systematic study of solutions to cubics and writes: "Whoever thinks algebra is a trick in obtaining unknowns has thought it in vain. No attention should be paid to the fact that algebra and geometry are different in appearance. Algebras are geometric facts which are proved." In addition to a general discussion of his view of algebra, an excellent selection to read is his treatment of the cubic $x^3 + cx = d$, which is solved geometrically via the intersection of a parabola and a circle. [20]

GIROLAMO CARDANO: The next major advance toward solving algebraic equations did not come until the sixteenth century with the work of Cardano and his contemporaries in Europe. During that time, ancient Greek mathematics was rediscovered, often via Islamic sources, and old problems were attacked with new methods and symbols. The general arithmetic solution for equations of degree three and four essentially awaited Cardano's seminal *Ars Magna* (1545), in which Khayyam's equation $x^3 + cx = d$ now receives a virtually algebraic treatment. [16] It is instructive to compare the two texts. The final chapter in the search for general solutions for the quintic or higher-degree equations was later written by Niels Abel and Evariste Galois.

EVANGELISTA TORRICELLI: By the early seventeenth century the Greek method of exhaustion was being transformed into Cavalieri's method of indivisibles, the precursor of Leibniz's infinitesimals and of Newton's fluxions. One of the most astonishing results of this period was the discovery that an infinite solid can have finite volume. Torricelli, a pupil of Galileo, demonstrated by the method of indivisibles that the solid obtained by revolving a portion of a hyperbola about its axis has finite volume. [19]

BLAISE PASCAL: Closed formulae for sums of powers of consecutive integers such as $\sum_{i=1}^{n} i^m$ were already of interest to Greek mathematicians. For instance, Archimedes used them to determine areas, such as for the spiral above. After much effort over the centuries, Fermat in the early seventeenth century first recognized the existence of a general rule, and called this "perhaps the most beautiful problem of all arithmetic." [2] Shortly thereafter Pascal provided a recursive description of the formulae for sums of powers in an arithmetic progression in *Potestatum Numericarum Summa (Sommation des Puissances Numériques).* [15] As Pascal mentions, these formulae are connected to the continuing development of integration techniques at the time.

Jacques Bernoulli: Improving on the work of Pascal (which he apparently was not aware of), Bernoulli, in the late seventeenth and early eighteenth centuries, provided the first general analysis of the polynomial expressions giving the sums of powers. His *Ars Conjectandi* (1713) noticed surprising patterns in the coefficients, involving a sequence of numbers now known as the Bernoulli numbers. [18] Today these Bernoulli numbers are important in many areas of mathematics, such as analysis, number theory, and algebraic topology.

Leonhard Euler: The eighteenth century was dominated by applications of the calculus, many of them provided by Euler, who was a master in working with infinite series. His *De Summis Serierum Reciprocarum* contains a variety of results on sums of reciprocal powers, including a recursive analysis of $\sum_{i=1}^{\infty} \frac{1}{i^{2m}}$. [8] Euler's computations are examples of the general formula

$$\sum_{i=1}^{\infty} \frac{1}{i^{2m}} = \frac{(-1)^{m+1} B_{2m} 2^{2m-1} \pi^{2m}}{(2m)!},$$

which involves Bernoulli numbers and can be derived directly from the above text of Bernoulli.

Sophie Germain: The early nineteenth century saw the beginnings of modern number theory with the publication of Gauss' *Disquisitiones Arithmeticae* in 1801. Efforts to prove Fermat's Last Theorem contributed to the development of sophisticated techniques by mid-century. Before then, however, the only progress toward a general solution, beyond confirmation of the conjecture for exponents five and seven (three and four were confirmed by Fermat and Euler), was provided by Sophie Germain. She developed a general strategy toward a complete proof, and used the theorems she proved along the way to resolve Case I of Fermat's Last Theorem for all exponents less than 100. Sophie Germain never published her work. Instead, a part of it appeared in 1825, in a supplement to the second edition of A. M. Legendre's *Théorie des Nombres*, where he credits her in a footnote[1]. [11]

NICOLAI LOBACHEVSKY: From its beginning, Euclid's parallel postulate [9] was controversial. Attempts to prove it from the others led to the nineteenth-century discovery that it is independent of the rest, allowing for other geometries. Lobachevsky, the co-discoverer of non-Euclidean geometries along with Gauss and Bolyai, made several attempts at gaining the attention of the mathematical world with his ideas. In 1840 he published the very readable book *Geometrische Untersuchungen zur Theorie der Parallellinien*, laying out the foundations of hyperbolic geometry. [1]

GOTTHOLD EISENSTEIN: In addition to Fermat's Last Theorem, another driving force in the development of number theory was the Quadratic Reciprocity Theorem and the study of higher reciprocity laws. The theorem, discovered by Euler and restated by Legendre in terms of the symbol now bearing his name, was first proven by Gauss. The eight different proofs Gauss published, for what he called the Fundamental Theorem, were followed by dozens more before the end of the century, including four given by Gotthold Eisenstein in the years 1844–45. His article *Geometrischer Beweis des Fundamentaltheorems für die quadratischen Reste* [7] gives a particularly elegant and illuminating geometric variation on Gauss' third proof. [12, 13]

WILLIAM ROWAN HAMILTON: After extended efforts, Hamilton's attempts to define a multiplication on three-dimensional vectors led to his flash of insight in 1843 that this was possible if one allowed vectors of dimension four. Selections from his book *Elements of Quaternions* give an interesting account of his geometric view of the quaternions. [17] They provided one of the first important examples of a non-commutative number system, thus spurring the development of abstract algebra.

[1] A more comprehensive evaluation of Germain's work on Fermat's Last Theorem, based on her original manuscripts, can be found in [14].

ARTHUR CAYLEY: By the mid-nineteenth century, group structures had emerged implicitly in several branches of mathematics, for instance in modular arithmetic, the theory of quadratic forms, as permutations in the work of Galois on algebraic equations, in the work of Hamilton on the quaternions, and in the theory of matrices. In his paper *On the Theory of Groups, as Depending on the Symbolic Equation* $\theta^n = 1$ Cayley was the first to investigate the abstract concept of a group and began the classification of groups of a given order in a purely abstract way. [4]

RICHARD DEDEKIND: In an attempt to explain calculus better to his students, Dedekind constructed the real numbers through what is now known as Dedekind cuts, from which their continuity can be deduced rigorously. Together with Cantor's equivalent construction, this work represents the culmination of the century-long effort to arithmetize analysis. In 1872 he published these ideas in the celebrated *Stetigkeit und die Irrationalzahlen*. [6]

GEORG CANTOR: Mathematics was changed forever toward the end of the nineteenth century by Cantor's bold embrace of the infinite. Selections from his readable *Beiträge zur Begründung der Transfiniten Mengenlehre* develop the foundations of his theory of transfinite numbers, or what we today call ordinals and cardinals. [3]

JOHN CONWAY: A vast generalization of a Dedekind cut, combined with ideas from game theory, led Conway in the 1970s to create, with a single construction, the so-called surreal numbers, an enormous number system containing both the real numbers and Cantor's ordinals. Chapter 0 of his book *On Numbers and Games* provides a delightful introduction to his surreal world. [5] Conway's work is one of the rare examples of very recent mathematics that is deep but can be read with minimal background.

In our experience students find the study of original sources fascinating, especially when combined with readings in the history of mathematics. The benefits for instructors and students alike are a deepened appreciation for the origins and nature of modern mathematics, as well as the lively and stimulating class discussions engendered by the interpretation of original sources.

As part of their assignment students complete a research project on a topic of their choice, with the only constraint that it be part mathematical and part historical. Other assignments focus for the most part on mathematical points in the sources and related topics.

After using the traditional lecture approach for some time, we discovered the amazing effectiveness of a combination of two pedagogical devices: the "discovery approach"; and extensive writing. The discovery method assumes that students should discover the mathematics for themselves. Hence, for each source we briefly provide the historical and mathematical context, alert the students to any difficult points in the text, and then stand by to answer questions while they work through the source in pairs. A wrap-up discussion lets everyone share his or her understanding of the material, and any remaining difficulties are resolved. This method generates tremendous enthusiasm and a genuine sense of discovery. Strikingly, we see that this method also leads to a deeper understanding of the sources than the lecture approach achieves.

The students write frequently and about every aspect of the course: the mathematical details of the sources, their historical context, lecture notes, thoughts jotted in the throes of problem solving, and their own ideas about the process that creates mathematics. This writing experience leads to a more comprehensive view of the great theorems we study as well as a much better grasp of the mathematical details in their proofs.

For students in the sciences, engineering, and mathematics education, our course provides both a broad and humanistic view of mathematics, and for many students it is a breath of fresh air within the traditional mathematics curriculum. For mathematics majors the course is an enriching capstone for their entire undergraduate experience.

References

[1] Roberto Bonola, *Non-Euclidean Geometry*, Dover, New York, 1955.

[2] Carl Boyer, Pascal's formula for the sums of powers of the integers, *Scripta Mathematica*, **9** (1943), 237–244.

[3] Georg Cantor, *Contributions to the Foundations of the Theory of Transfinite Numbers, Article I*, Dover, New York, 1895.

[4] Arthur Cayley, *The Collected Mathematical Papers of Arthur Cayley*, vol. 1, 423–424; vol. 2, 123–132, Cambridge University Press, Cambridge, 1889.

[5] John Conway, *On Numbers and Games*, Academic Press, London, 1976, pp. 3–14.

[6] Richard Dedekind, *Essays on the Theory of Numbers*, Dover, New York, 1963, pp. 1–27.

[7] F. Gotthold Eisenstein, Geometrischer Beweis des Fundamentaltheorems für die quadratischen Reste, *Journal für die Reine und Angewandte Mathematik (Crelle's Journal)*, **28** (1844), 246–248 and Taf. II: Figs. 1,2.

[8] Leonhard Euler, *Opera Omnia*, Teubner, Leipzig & Berlin, 1924–25, Series I, vol. 14, 73–86.

[9] T. L. Heath (ed.), *The Elements*, Dover, New York, 1956, vol. I, pp. 153–155.

[10] T. L. Heath, *The Works of Archimedes*, Dover, New York, pp. 107–109, 176–182.

[11] Adrien M. Legendre, *Sur quelques objets d'analyse indeterminée et particulièrement sur le théoreme de Fermat*, Second Supplément (Sept. 1825) to *Théorie des Nombres*, Second Edition, 1808.

[12] Reinhard Laubenbacher and David Pengelley, Eisenstein's Misunderstood Geometric Proof of the Quadratic Reciprocity Theorem, *College Mathematics Journal*, **25** (1994), 29–34.

[13] Reinhard Laubenbacher and David Pengelley, Gauß, Eisenstein, and the 'Third' Proof of the Quadratic Reciprocity Theorem: Ein Kleines Schauspiel, *Mathematical Intelligencer*, **16** (1994), 67–72.

[14] Reinhard Laubenbacher and David Pengelley, "Here is What I Have Found": Sophie Germain's Forgotten Number Theory Manuscripts, in preparation.

[15] Blaise Pascal, *Oeuvres de Blaise Pascal*, Kraus Reprint, Vaduz, Liechtenstein, 1976, vol. 3, 341–367.

[16] David E. Smith, *Source Book in Mathematics*, Dover, New York, 1959, pp. 203–206.

[17] Ibid., pp. 677–683.

[18] Ibid., pp. 85–90.

[19] Dirk J. Struik, *A Source Book in Mathematics, 1200–1800*, Princeton Univ. Press, Princeton, 1986, pp. 227–230.

[20] J. J. Winter and W. ‘Arafat, The Algebra of ‘Umar Khayyam, *Journal of the Royal Asiatic Society of Bengal*, **41** (1950), 27–78.