1.5 Poincaré's Euclidean Model for Non-Euclidean Geometry

Jules Henri Poincaré was born in Nancy, in the French region of Lorraine near the German border, in an upper-middle-class family. He was a brilliant student, and entered the elite Ecole Polytechnique in Paris in 1873. After graduating, he worked briefly as an engineer while finishing his doctoral thesis in mathematics, completed in 1879. He taught for a short time at the University of Caen, became a professor at the University of Paris in 1881, and taught there until his untimely death in 1912, by which time he had received innumerable prizes and honors [42].

Poincaré was well known in his era as the most gifted expositor of mathematics and science for the layperson. His essay *Mathematical Creation* remains unsurpassed to this day as an insightful and provocative description of the mental process of mathematical discovery, exploring the interplay between the conscious and subconscious mind [130, pp. 2041–2050].

Poincaré was one of the last universal mathematicians. With a grasp of the breadth of mathematics, he contributed his genius to many of its branches, and single-handedly created several new fields. Altogether he wrote nearly five hundred research papers, along with many books and lecture notes. Throughout his work, the idea of continuity was a leitmotiv. When he attacked a problem, he would investigate what happens when the

PHOTO 1.8. Poincaré.

conditions of the problem vary continuously, and this brought him not only new discoveries but the inauguration of entire new areas of mathematics.

In analysis he began the theory of "automorphic functions" (certain types of these are called Fuchsian or Kleinian by him), generalizations of the trigonometric functions, and these have played a critical role throughout twentieth-century mathematics. We shall see in this section how these discoveries led him to a beautiful Euclidean model explaining Lobachevsky's non-Euclidean hyperbolic geometry, which then itself further stimulated his work on automorphic functions.

At the center of Poincaré's thought was the theory of differential equations and its applications to subjects like celestial mechanics, and he published in this area almost annually for over thirty years. A differential equation is an equation involving functions and their derivatives (see the analysis chapter), which expresses fundamental properties of the functions, often arising from physical constraints such as the motion of celestial bodies according to Newton's law of gravitation. Poincaré initiated the qualitative theory of differential equations. One of the big questions, unresolved in general to this day, is the long-term stability of something like our solar system, i.e., whether the motions under gravity of the sun and planets will cause them to remain in periodic (repeating) orbits, or orbits remaining "close" to such stable orbits, or to fly off in some way. Poincaré used his qualitative theory of differential equations to study this problem, particularly his method of varying the conditions continuously to see what happens with small changes in the initial conditions, and was able to prove that such stable solutions do exist in certain situations.

Poincaré also made great contributions to the theory of partial differential equations for functions of several variables, which has pervasive applications throughout mathematical physics. He played a critical role in the discovery of radioactivity via insights about the connection between x-rays and phosphorescence, and was involved in the invention of special relativity theory, with many physicists considering him a coinventor with Lorentz and Einstein.

In number theory and algebra Poincaré introduced important new ideas. He wrote the first paper on what we today call "algebraic geometry over the rational numbers," which addresses the problem of finding solutions to Diophantine equations. The central theme of the number theory chapter, the equation of Fermat, is an example of a Diophantine equation, and many questions raised by Poincaré remain important, and some unanswered, a century later.

Finally, as if this weren't enough, Poincaré began the mathematical subject called algebraic topology. It emerged from his interest, described above, in whether and how varying something continuously will change its qualitative features. The behavior of these qualitative features under continuous change is what we call topology, and Poincaré began to apply ideas from algebra (e.g., the groups discovered by Galois and others earlier in the nineteenth century; see the algebra chapter) to study the topology of surfaces, including in particular the phenomena he was investigating in celestial mechanics. This marriage of algebra and topology has been one of the most potent forces throughout twentieth-century mathematics.

Poincaré's role in the story of this chapter comes via a connection he discovered between Euclidean and hyperbolic geometry while studying two other branches of mathematics. Although Lobachevsky, Bolyai, and Gauss had developed the new theory of hyperbolic geometry with confidence, still mathematicians did not feel as sure of its validity as they felt of Euclid's geometry. They worried that there could be a contradiction in this new theory, since it seemed so strange and was inconsistent with Euclidean geometry. However, a few decades after the work of Lobachevsky, an extraordinary development occurred that gave his geometry equal standing with Euclid's. Mathematicians began to find "models" (realizations) for hyperbolic geometry inside Euclidean geometry. The inescapable conclusion was that it is just as mathematically correct as Euclid's. We will explain what we mean by all this in the context of the model provided by Poincaré. While his model was not the first one discovered, it is particularly beautiful, intuitively satisfying, and easy to work with in many ways. Certain aspects of Poincaré's model had been described by Riemann and Beltrami in the 1850s and 1860s, but entirely in terms of formulas using coordinates, without geometric interpretations. Further information about this and other models for non-Euclidean geometries can be found in [78, 144, 162, 164].

Poincaré was led to his model by his work on new types of functions in analysis [162, pp. 251–254]. He was working on generalizations of the trigonometric functions and their inverse functions. The reader familiar with calculus will know that the inverse trigonometric functions are integrals (antiderivatives) of expressions involving square roots of quadratic polynomials, such as

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C,$$

and will recall that the trigonometric functions also have periodicity properties, e.g., $\sin(x + 2\pi) = \sin(x)$. For some time mathematicians had tried to find a theory of new types of functions with analogous properties, generalizations of the inverse trigonometric functions that would solve more general integration problems involving higher roots of polynomials of any degree. The first such functions, arising early in the nineteenth century, were called elliptic functions because they came from integrals for calculating the lengths of ellipses, which involve square roots of cubic polynomials. It was discovered that the elliptic functions had double periodicity when considered as functions of a complex variable. By representing complex numbers as points in the plane, the double periodicity of the elliptic functions was exhibited geometrically as the symmetry of the periodic repetition in parallelogram tiling patterns in the Euclidean plane. For instance, Figure

FIGURE 1.22. Double periodicity of elliptic functions.

1.22 illustrates the idea that an elliptic function defined on the Euclidean plane should have the same value at corresponding points in all the parallelograms, for instance at all the points marked *. By the mid–nineteenth century, generalizations of elliptic functions were being studied that provided solutions to the more general integrals and that were central to the thriving study of differential equations. However, the repetitive symmetry patterns of these new functions did not seem to arise from geometric tiling patterns in the plane, as they had for the elliptic functions. Here is where Poincaré made a breakthrough in 1881, and we can read his very own description of it. He called the new functions he was seeking to describe and understand Fuchsian functions, after I. Fuchs, whose work had inspired his own. In his famous essay *Mathematical Creation*, which still makes delightful reading today, Poincaré speaks of the psychology of mathematics in the context of his discovery relating hyperbolic geometry to Fuchsian functions:

> Just at this time I left Caen, where I was then living, to go on a geological excursion under the auspices of the school of mines. The changes of travel made me forget my mathematical work. Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations I had used to define the Fuchsian functions were identical with those of non-Euclidean geometry. I did not verify the idea; I should not have had time, as, upon taking my seat in the omnibus, I went on with a conversation already commenced, but I felt a perfect certainty. On my return to Caen, for conscience' sake I verified the result at my leisure [130, pp. 2041–2050].

In the resulting research paper *Sur les Fonctions Fuchsiennes* (On Fuchsian Functions), completed on February 14, 1881, Poincaré makes only brief mention of this important discovery:

> It was first necessary to form all the Fuchsian groups; I attained this with the assistance of non-Euclidean geometry, of which I will not speak here [133, vol. II, pp. 1–2].

Poincaré had realized that certain types of symmetries possible in the repetitive tiling patterns of the hyperbolic plane, which illustrate the "Fuchsian groups" he speaks of, would enable him to describe and study the Fuchsian functions he was seeking, particularly in light of a new model he had found for Lobachevsky's non-Euclidean geometry. Thus hyperbolic geometry served the role for his new Fuchsian functions that Euclidean geometry had served for elliptic functions.

PHOTO 1.9. *Sur les Fonctions Fuchsiennes.*

Poincaré actually describes this new model in some detail in a slightly later paper on a completely different topic, which also turned out to have an amazing connection with hyperbolic geometry, as he says in continuing his essay on creativity:

> Then I turned my attention to the study of some arithmetic questions apparently without much success and without a suspicion of any connection with my previous researches. Disgusted with my failure, I went to spend a few days at the seaside, and thought of something else. One morning, walking along the bluff, the idea came to me, with just the same characteristics of brevity, suddenness and immediate certainty, the arithmetic transformations of ternary quadratic forms were identical with those of non-Euclidean geometry.

Indeed, just two months after Poincaré first remarked on the role of hyperbolic geometry in his paper on Fuchsian functions, he presented a paper on this second connection to non-Euclidean geometry, at the April 16 meeting of the French Association for the Advancement of the Sciences, in Algiers. His paper there was entitled *Sur les Applications de la Géométrie Non Euclidienne a la Théorie des Formes Quadratiques* (On the Applications of Non-Euclidean Geometry to the Theory of Quadratic Forms) [133, vol. V, pp. 270–271] (also translated in [164]). Here he explains the precise nature of his Euclidean model for hyperbolic geometry. The model resides entirely in the unit disk, constituting the inside of the circle C of radius one with center at the origin in the Euclidean plane.

Poincaré, from

On the Applications of Non-Euclidean Geometry to the Theory of Quadratic Forms

Here I will appeal to non-Euclidean geometry or pseudogeometry. I will write, in short, *ps* and *psly*, for pseudogeometry and pseudogeometrically.

I will designate, as a *ps* line, any circumference which cuts the circle C perpendicularly; as the *ps* distance between two points, half the logarithm of the anharmonic ratio of these two points with the two points of intersection of the circle C with the *ps* line joining them. The *ps* angle between two intersecting curves is their geometric angle. A *ps* polygon will be a portion of the plane bounded by *ps* lines....

... one finds out that *ps* distances, *ps* angles, *ps* lines, etc., satisfy the theorems of non-Euclidean geometry, that is, all the theorems of ordinary geometry, except those which are a consequence of Euclid's *postulate*.

To explain the details of his model, let us first make a dictionary (Figure 1.23) for Poincaré's terms, which translates geometric features in the hyperbolic plane into corresponding features in the ordinary Euclidean plane

Poincaré's disk model of the hyperbolic plane	Euclidean plane
a *ps* point	a Euclidean point inside the unit disk
a *ps* line	the portion inside the unit disk of any Euclidean circle meeting C perpendicularly, or a diameter of the circle C
a *ps* angle between *ps* lines	the Euclidean angle between the two Euclidean curves that are the *ps* lines
a *ps* distance between *ps* points	given by a particular formula that involves the distance of the points from the circle C

FIGURE 1.23. Dictionary for Poincaré's disk model.

in which the model is embedded. Then we may use the dictionary to explore properties of non-Euclidean geometry by working simply with their Euclidean counterparts.

With this dictionary we can consider Figure 1.24, which illustrates some of the features of Poincaré's model. It shows five *ps* lines. Three of them intersect in three *ps* points, thus forming a *ps* triangle, and the other two lines do not intersect any of the other lines shown. *Ps* points, *ps* lines, and *ps* angles are extremely easy to conceive of and work with in this model, and the reader will easily be able to explore each of the fundamental questions addressed by Euclid, Saccheri, Lambert, Legendre, and Lobachevsky.

Before suggesting how to do this, we should first note that *ps* distance is not quite as straightforward, but that many of the features we are interested in can be observed without worrying about distance, or by knowing just a little about it. Distance is given by a formula that, roughly speaking, amounts to stretching the Euclidean distance in the disk by the factor $\frac{1}{1-x^2-y^2}$ at the point with coordinates (x, y). Thus there is no stretching at the center of the disk (i.e., a stretching factor of 1), and the stretching increases as we move outwards. Halfway towards the edge of the disk along a radius, the stretching factor is $\frac{1}{1-(1/2)^2} = \frac{4}{3}$. As we move closer to the edge, the stretching increases dramatically, approaching infinite stretching as we approach the edge of the disk. The nature of the stretching in the distance formula actually causes all non-meeting *ps* lines to grow farther apart in the *ps* geometry as they approach the edge of the disk in the Euclidean plane, except for those that meet in the Euclidean sense at a point on the boundary circle C (such as a pair shown in Figure 1.24). These latter will actually become closer and closer as they approach C, but of course they

FIGURE 1.24. Poincaré's unit disk model.

do not actually meet in the *ps* world of the model: the point they appear to be approaching on the boundary C is not a *ps* point, since it is not inside the disk. According to the distance stretching formula, each of the *ps* lines has infinite length inside the disk. (The reader can prove this from the stretching factor formula using some calculus.)

Note that diameters of the circle C are also *ps* lines; it is often useful to choose one or two diameters as *ps* lines in a construction, if possible. Using the model one can now verify that all of Euclid's postulates except the parallel postulate are satisfied (Exercise 1.21). Moreover, the parallel postulate does not hold, but is replaced by Lobachevsky's alternative (Exercise 1.22). We then know that our model exhibits Lobachevsky's geometry as a completely valid system, provided that we accept the validity of Euclidean geometry, since we used it as our framework for the model.[6] Exercises 1.23–1.26 use the unit disk model to explore the questions studied by Saccheri, Legendre, and Lobachevsky about lengths and angles for hyperbolic quadrilaterals, triangles, and parallels.

Let us end with three illustrations of the repetitive tiling patterns possible in the non-Euclidean plane, analogous to the tiling of the Euclidean plane created by the collection of Euclidean lines in Figure 1.22. These will reveal some of the more curious features of non-Euclidean geometry. Poincaré himself remarked in 1882, when discussing the historical origins of his own work on Fuchsian functions and the corresponding Fuchsian groups [133, vol. II, pp. 168–169], that one of his examples of a Fuchsian group already existed in geometric form as a tiling pattern published by H. Schwarz ten years earlier [154]. Figure 1.25 comes from this paper, and we see immediately that although it is a priori a pattern of curved segments in the unit disk in the Euclidean plane, it is actually a collection of *ps* lines when viewed using Poincaré's unit disk model of hyperbolic geometry.

Whereas the Euclidean lines in Figure 1.22 divide the Euclidean plane up into infinitely many congruent parallelograms, Schwarz's *ps* lines divide the hyperbolic plane into infinitely many triangles. And although it may not seem so at first sight, in fact all the triangles are *ps* congruent (recall that distances are stretched as one moves outwards in the disk). It is not hard to see that each triangle has *ps* angles 36°, 45°, 90°, for a total angle

[6] The great early-twentieth-century mathematician David Hilbert provided a firm foundation for both Euclidean and non-Euclidean geometry that shows that they are consistent, provided that the theory of the real numbers is consistent. On the other hand, Kurt Gödel (see the analysis chapter) proved in 1931 that no proof of the absolute consistency of either Euclidean or Lobachevskian non-Euclidean geometry is possible [164, p. 65]! We will forever live not knowing for certain that there is no internal contradiction in these geometries, or the real numbers. However, Poincaré's model does at least convince us that our non-Euclidean geometry is as free of contradictions as Euclidean geometry or the real numbers. In fact, Euclidean geometry is also contained in higher-dimensional extensions of Lobachevsky's geometry, so the different geometries are actually equally consistent [164, p. 38].

FIGURE 1.25. Schwarz's tiling.

sum of 171°. They all occur in clusters of ten fitted together around a central point to form regular right-angled pentagons. Recalling that Lambert had discovered a fixed relationship between changing *ps* side lengths and changing *ps* angles under shrinkage or expansion of triangles, so that similar *ps* triangles must actually always be *ps* congruent, we conclude that all Schwarz's triangles must be congruent, and so are the pentagons.

Removing the *ps* lines passing through the centers of the pentagons, keeping only the lines that form the edges of the pentagons, we obtain a tiling of the hyperbolic plane by regular right-angled pentagons, meeting four to a corner, shown in Figure 1.26.

Thus a hyperbolic bathroom floor can be tiled with pentagons, unlike the Euclidean floor, which can be tiled only with triangles, squares, and hexagons. A new challenge, though, emerges from the fixed relationship between angles and lengths: there is only one possible size for regular right-angled pentagons, and it might be very large or very small, creating tile-laying problems in any particular bathroom.[7] Imagine what happens if we try to simultaneously expand all the right-angled pentagonal tiles on the hyperbolic floor. Unlike Euclidean geometry, where the entire pattern

[7]In fact, there are many different hyperbolic plane geometries, each with its own particular size relationship between angles and lengths, so the practical prospects of tiling the bathroom floor with right-angled pentagons will depend on which hyperbolic world one happens to live in. In fact, our own physical universe may well be one of these hyperbolic types, but with the right-angled pentagon rather large (so large that in the portion of the universe we have examined everything appears Euclidean to us for all practical purposes).

FIGURE 1.26. Pentagonal tiling.

can be expanded or shrunk at will with no fundamental change in its nature, our pattern will crack and break apart as it changes size, since as the regular pentagons expand uniformly, their angles will lessen in size. Cracks will appear between the pentagons, and four will no longer fit together at a point. However, after a certain amount of expansion, their angles will have decreased to 72°, and then we can try fitting them together again, meeting five to a corner. Later on in the expansion, their angles will decrease to 60°, and we can place them together six to a corner, and so on ad infinitum. It is a remarkable fact that for any of these possibilities they can be fitted together without gaps or overlaps to create perfect tilings of the plane. We see that the hyperbolic designer has a wealth of new aesthetic tiling options that the Euclidean designer lacks, provided that the relationship between hyperbolic angle and distance is such that the scale of each particular pattern suits the designer's constraints, since each pattern comes in only one size! By enlarging the size of each item, new patterns arise, but on the other hand, shrinking the size of the items makes the possibilities disappear as the situation becomes closer and closer to the Euclidean one.

Tilings of the Poincaré model of the non-Euclidean plane have served as great inspiration in art, particularly in the work of twentieth-century Dutch graphic artist Maurits C. Escher. In Figure 1.27, his pattern of angels and devils is overlaid on a tiling with right-angled non-Euclidean hexagons, meeting four to a corner, completely analogous to the pentagonal tiling. If we divide the hexagonal tiling up into a triangular tiling analogous to that of Schwarz, then each pair of matching mutually reflecting triangles represents one of Escher's bilaterally symmetrical angels or devils (Exercise 1.27). The reader may explore all these phenomena further in [78, 162, 164].

FIGURE 1.27. Escher's tiling.

The viewpoint opened by models like Poincaré's has greatly expanded our perception of what geometry is, and there are many other completely different types of geometries known and studied today. Moreover, their connection to other branches of mathematics and to applications is manifold, and research in various geometries is a thriving part of the mathematical enterprise today.

Exercise 1.21: Show that all of Euclid's postulates except the parallel postulate are satisfied in Poincaré's unit disk model.

Exercise 1.22: Does Euclid's parallel postulate hold in Poincaré's unit disk model? Does Lobachevsky's alternative hold?

Exercise 1.23: Which of Saccheri's three hypotheses about angles in Saccheri quadrilaterals holds in Poincaré's unit disk model?

Exercise 1.24: In Poincaré's unit disk model, how is the angle sum of a triangle related to its size? How large or small can the angle sum be? Are there similar triangles?

Exercise 1.25: In Poincaré's unit disk model, is the angle sum of a triangle always equal to, greater than, or less than two right angles, as Legendre studied? Discuss Legendre's hypothesis.

Exercise 1.26: Describe the behavior of Lobachevsky's angle of parallelism for various *ps* lines and points. How big or small can it be? Describe which *ps* lines are parallel according to Lobachevsky's definition.

Exercise 1.27: Find and describe some tilings of the hyperbolic plane in the work of M.C. Escher or other artists.