spherical surface on top of the ice-cream cone. You may use knowledge of the surface area of the entire sphere, which Archimedes had determined.

Exercise 3.24: Imagine boring a round hole through the center of a sphere, leaving a spherical ring. Use the result of Exercise 3.23 to find the volume of the ring. Hint: Amazingly, your answer should depend only on the height of the ring, not the size of the original sphere. Also obtain your result directly from Cavalieri's Principle by comparing the ring with a sphere of diameter the height of the ring.

3.5 Leibniz's Fundamental Theorem of Calculus

Gottfried Wilhelm Leibniz and Isaac Newton were geniuses who lived quite different lives and invented quite different versions of the infinitesimal calculus, each to suit his own interests and purposes.

Newton discovered his fundamental ideas in 1664–1666, while a student at Cambridge University. During a good part of these years the University was closed due to the plague, and Newton worked at his family home in Woolsthorpe, Lincolnshire. However, his ideas were not published until 1687. Leibniz, in France and Germany, on the other hand, began his own breakthroughs in 1675, publishing in 1684. The importance of publication is illustrated by the fact that scientific communication was still sufficiently uncoordinated that it was possible for the work of Newton and Leibniz to proceed independently for many years without reciprocal knowledge and input. Disputes about the priority of their discoveries raged for centuries, fed by nationalistic tendencies in England and Germany.

Leibniz was born and schooled in Leipzig, studying law at the university there. Although he loved mathematics, he received relatively little formal encouragement. Later, after completing his doctorate at Altdorf, the university town of Nürenberg, he declined a professorship there, considering universities "monkish" places with learning, but little common sense, engaged mostly in empty trivialities. Instead, Leibniz entered public life in the service of princes, electors, and dukes, whom he served in legal and diplomatic realms, and in genealogical research trying to prove their royal claims. His work provided great opportunities for travel, and he interacted personally and by correspondence with philosophers and scientists throughout Europe, pursuing mathematics, the sciences, history, philosophy, logic, theology, and metaphysics. He was truly a genius of universal interests and contributions, leading him to concentrate on methodological questions, and to embark on a lifelong project to reduce all knowledge and reasoning to a "universal characteristic." Although today we recognize his contributions to be of outstanding importance, he died essentially neglected, and only his secretary attended his burial [42, 157].

PHOTO 3.5. Leibniz.

In 1672 Leibniz was sent to Paris on a diplomatic mission, beginning a crucially formative four-year period there. Christian Huygens (1629–1695), from Holland, then the leading mathematician and natural philosopher in Europe, guided Leibniz in educating himself in higher mathematics, and Leibniz's progress was extraordinary.

Leibniz's discovery of the calculus emerged from at least three important interests [8, 21, 77]. First, as a philosopher his main goal was a general symbolic language, enabling all processes of reason and argument to be written in symbols and formulas obeying certain rules. His mathematical investigations were thus merely part of a truly grand plan, and this explains his focus on developing useful new notation and theoretical methods, rather than specific results. Indeed, it is his notation and language for the calculus that we use today, rather than Newton's. He sought and found a "calculus" for infinitesimal geometry based on new symbols and rules.

Second, Leibniz studied the relationship between difference sequences and sums, and then an infinitesimal version helped suggest to him the essential features of the calculus. This can be illustrated via a concrete problem

FIGURE 3.9. Triangular numbers.

that Huygens gave Leibniz in 1672: Consider the "triangular numbers" 1, 3, 6, 10, 15,..., the numbers of dots in triangular arrangements (Figure 3.9). These also occur in "Pascal's triangle" of binomial coefficients, and their successive differences are 2, 3, 4, 5,.... The triangular numbers are given by the formula $i(i+1)/2$ for $i = 1, 2, \ldots$. (Can you verify this? Hint: Successive differences.) Huygens challenged Leibniz to calculate the infinite sum of their reciprocals,

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \cdots + \frac{2}{n(n+1)} + \cdots ,$$

and Leibniz proceeded as follows [77, pp. 60–61]. Each term $2/(i(i+1))$ in the sum equals the difference $2/i - 2/(i+1)$; i.e., the sum can be rewritten as

$$\left(\frac{2}{1} - \frac{2}{2}\right) + \left(\frac{2}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{4}\right) + \left(\frac{2}{4} - \frac{2}{5}\right) + \cdots + \left(\frac{2}{n} - \frac{2}{n+1}\right) + \cdots .$$

The terms here can be regrouped and mostly canceled, so the partial sum of the first n original terms, displayed as a sum of differences, collapses, leaving $2 - 2/(n+1)$ as its sum, which leads to 2 as the sum of the original terms all the way to infinity. Viewed more generally, since the terms of the original series $2/(i(i+1))$ were recognized as being the successive differences of the terms in a new pattern (namely $2/i$), the nth partial sum of the original series can be computed, via the collapsing trick, as simply the difference between the first and nth terms in the new pattern. This observation expresses, in a discrete, rather than continuous, way, the essence of the Fundamental Theorem of Calculus, and Leibniz slowly came to realize this.

Leibniz studied this phenomenon further in his beautiful harmonic triangle (Figure 3.10 and Exercise 3.25), making him acutely aware that forming difference sequences and sums of sequences are mutually inverse operations. He used an analogy to think of the problem of area as a summation of infinitesimal differences, leading him to the connection between area and tangent.

The third crucial thread contributing to Leibniz's creation of the calculus was his conception of a "characteristic triangle" with infinitesimal sides at each point along a curve (see GLC or $(C)EC$ in Figure 3.11). The two legs of the right triangle represent infinitesimal elements of change (successive differences) in the horizontal and vertical coordinates between the chosen point and an infinitesimally nearby point along the curve, and their ratio is thus the slope of the tangent line to the curve at the point. Leibniz wrote that these phenomena all came together as "a great light" bursting upon him when he was studying Pascal's *Treatise on the Sines of a Quadrant of a Circle* [21, p. 203].

FIGURE 3.10. Leibniz's harmonic triangle.

Combining Leibniz's connection between sums and successive differences with his connection between infinitesimal differences and tangent lines, we can begin to see a possible connection between area and tangent problems. Several other mathematicians had already been developing methods for finding tangent lines to curves, providing stimulus to Leibniz's ideas. Pierre de Fermat, for instance, illustrated his approach based on infinitesimals by calculating the tangent line to a parabola (Exercise 3.26).

In his early work with characteristic triangles and their infinitesimal sides, Leibniz derived relationships between areas that we today recognize as important general calculation tools (e.g., "integration by parts"), and while studying the quadrature of the circle, he discovered a strikingly beautiful result about an infinite sum, today named Leibniz's series:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \, .$$

He began introducing and refining powerful notation for his ideas of sums and differences involving infinitesimals, ultimately settling on $d(x)$ for the infinitesimal differences between values of x, and on $\int$ (an elongated form of "s," as in Latin "summa") for the sum, or "integration," of infinitesimals. This $\int$ is analogous to Cavalieri's "all lines," so $\int y\,dx$ denotes the summation of the areas of all rectangles with length y and infinitesimal width dx. (Can you see why $d(xy) = x\,dy + y\,dx$? Hint: Think, like Leibniz, of x and y as successive partial sums, with dx and dy the differences between successive sums.)

Using the calculus he developed with these new symbols, Leibniz easily rederived many earlier results, such as Cavalieri's quadrature of the higher parabolas, and put in place the initial concepts, calculational tools, and notation for the enormous modern subject of analysis.

Although many of these seminal ideas are in Leibniz's manuscripts of 1675–1677, publication was slow. We will examine how he brought all these ideas together in his resolution of the problem of quadratures, in a 1693 paper *Supplementum geometriae dimensoriae, seu generalissima omnium tetragonismorum effectio per motum: similiterque multiplex constructio lineae ex data tangentium conditione* (More on geometric measurement, or most generally of all practicing of quadrilateralization through motion: likewise many ways to construct a curve from a given condition on its tangents) published in the first scientific journal *Acta Eruditorum* [110, pp. 294–301], [166, pp. 282–284]. Today we recognize his result as of such paramount importance that we call it the Fundamental Theorem of Calculus.

Leibniz, from

More on geometric measurement, or most generally of all practicing of quadrilateralization through motion: likewise many ways to construct a curve

FIGURE 3.11. Leibniz's Fundamental Theorem of Calculus.

from a given condition on its tangents.

I shall now show that the general problem of quadratures can be reduced to the finding of a line that has a given law of tangency (declivitas), that is, for which the sides of the characteristic triangle have a given mutual relation. Then I shall show how this line can be described by a motion that I have invented. For this purpose [Figure 3.11] I assume for every curve $C(C')$ a double characteristic triangle,[9] one, TBC, that is assignable, and one, GLC, that is inassignable, and these two are similar. The inassignable triangle consists of the parts GL, LC, with the elements of the coordinates CF, CB as sides, and GC, the element of arc, as the base or hypotenuse. But the assignable triangle TBC consists of the axis, the ordinate, and the tangent, and therefore contains the angle between the direction of the curve (or its tangent) and the axis or base, that is, the inclination of the curve at the given point C. Now let $F(H)$, the region of which the area has to be squared, be enclosed between the curve $H(H)$, the parallel lines FH and $(F)(H)$, and the axis $F(F)$; on that axis let A be a fixed point, and let a line AB, the conjugate axis, be drawn through A perpendicular to AF. We assume that point C lies on HF (continued if necessary); this gives a new curve $C(C')$ with the property that, if from point C to the conjugate axis AB (continued if necessary) both its ordinate CB (equal to AF) and tangent CT are drawn, the part TB of the axis between them is to BC as HF to a constant [segment] a, or a times BT is equal to the rectangle AFH (circumscribed about the trilinear figure $AFHA$). This being established, I claim that the rectangle on a and $E(C)$ (we must discriminate between the ordinates FC and $(F)(C)$ of the curve) is equal to the region $F(H)$. When therefore I continue line $H(H)$ to A, the trilinear figure $AFHA$ of the figure to be squared is equal to the rectangle with the constant a and the ordinate FC of the squaring curve as sides. This follows immediately from our calculus. Let $AF = y$, $FH = z$, $BT = t$, and $FC = x$; then $t = zy : a$, according to our assumption; on the other hand, $t = y\ dx : dy$ because of the property of the tangents expressed in our calculus. Hence $a\ dx = z\ dy$ and therefore $ax = \int z\ dy = AFHA$. Hence the curve $C(C')$ is the quadratrix with respect to the curve $H(H)$, while the ordinate FC of $C(C')$, multiplied by the constant a, makes the rectangle equal to the area, or the sum of the ordinates $H(H)$ corresponding to the corresponding abscissas AF. Therefore, since $BT : AF = FH : a$ (by assumption), and the relation of this FH to AF

[9] In the figure Leibniz assigns the symbol (C) to two points, which we denote by (C) and (C'). If, with Leibniz, we write $CF = x$, $BC = y$, $HF = z$, then $E(C) = dx$, $CE = F(F) = dy$, and $H(H)(F)F = z\ dy$. First Leibniz introduces curve $C(C')$ with its characteristic triangle, and then later reintroduces it as the squareing curve [*curva quadratrix*] of curve $AH(H)$.

(which expresses the nature of the figure to be squared) is given, the relation of BT to FH or to BC , as well as that of BT to TC, will be given, that is, the relation between the sides of triangle TBC. Hence, all that is needed to be able to perform the quadratures and measurements is to be able to describe the curve $C(C')$ (which, as we have shown, is the quadratrix), when the relation between the sides of the assignable characteristic triangle TBC (that is, the law of inclination of the curve) is given.

We make only a few remarks on the details of the text, confident that the reader can fill in the necessary connections. First, "inassignable" refers to infinitesimal characteristic triangles, such as GLC and $(C)EC$. Second, the mysterious constant segment a is present to ensure dimensional propriety; i.e., an area should be equated only with another area, not the length of a line, and a ratio of two lengths should be dimensionless. Even in Leibniz's era this view was still carried from ancient Greek traditions. Today we could choose to view a as simply a choice of unit length for measurement. (From this point of view, why does a not really affect the final answer?) Third, note that Leibniz is not using the Cartesian coordinates pioneered earlier in his century quite as we do, but he is nevertheless measuring all locations along two perpendicular axes from a common origin A. Interestingly, he has no qualms about depicting the vertical coordinate for the quadratrix as increasing downwards, while for the curve to be squared it increases upwards (Why do you think he does this?). Finally, although Leibniz shows the quadratrix (squaring curve) only near the point C, in fact it must originate at A (Why?).

Leibniz never explains exactly what the meaning of his inassignables is, and on this he vacillated in his writings. To him, the most important criterion was that his rules for applying the new language worked, and he stated that applying them as if they were the rules of algebra would dispense with the "necessity of imagination" [21, p. 208].

Indeed, while the Fundamental Theorem of Calculus does not actually dispense with the need for imagination, it reduces every quadrature problem to finding a curve with a "given law of tangency," i.e., an inverse tangent problem.

While Fermat and others had shown that finding the tangent to a given curve is often possible, the inverse problem of finding a curve, given only its law of tangency, is generally much harder. It is not always possible to accomplish algebraically, even when the law of tangency is given algebraically.

Nonetheless, let us see some examples of how the Fundamental Theorem can be applied. Suppose (adhering to Leibniz's choices for coordinates, x and y) that we wish to square the curve $x = y^2$ (quadrature of the parabola again). According to his theorem, we must find a curve with y^2 as its law of tangency. This essentially involves guessing an answer based on experience, $y^3/3$ in this case, and then verifying that it works.

Let us imagine how Leibniz might have done this calculation. The law of tangency for $y^3/3$ will be obtained from its infinitesimal (inassignable) characteristic triangle as the ratio of the respective increments in x and y, *i.e.*, $dx : dy$. Since dy is the increment (difference) between successive values (of the sums) y and $y + dy$, with corresponding values $x = y^3/3$ and $x + dx = (y + dy)^3/3$, we calculate

$$\begin{aligned} dx : dy = \frac{dx}{dy} &= \frac{(x+dx)-x}{dy} = \frac{(y+dy)^3 - y^3}{3 \cdot dy} \\ &= \frac{y^3 + 3y^2 \cdot dy + 3y \cdot (dy)^2 + (dy)^3 - y^3}{3 \cdot dy} \\ &= y^2 + y \cdot dy + \frac{(dy)^2}{3} = y^2, \end{aligned}$$

as claimed. Leibniz explains that the final equality holds by dropping remaining infinitesimal terms or, going back one step, because $(dy)^2$ is infinitesimally small in comparison with dy.

Thus, according to the Fundamental Theorem of Calculus, the area under the parabola but above the y-axis, and from the origin to a specific value y for the ordinate, is given by $y^3/3$. Other examples are in the exercises.

Both Leibniz and Newton had their calculus attacked by others for their use of infinitesimals. One of the most eloquent and stinging criticisms came from Bishop George Berkeley's (1685–1753) polemic comparing the validity of science and mathematics with that of religion, entitled *The Analyst: Or a Discourse Addressed to an Infidel Mathematician. Wherein It Is Examined Whether the Object, Principles, and Inferences of the Modern Analysis Are More Distinctly Conceived, or More Evidently Deduced, than Religious Mysteries and Points of Faith. "First Cast the Beam Out of Thine Own Eye; and Then Shalt Thou See Clearly to Cast Out the Mote Out of Thy Brother's Eye,"* addressed to Edmund Halley (discoverer of the comet), a friend and defender of Newton in the early eighteenth century [21, pp. 224–225].

Berkeley writes:

> Whereas then it is supposed that you apprehend more distinctly, consider more closely, infer more justly, and conclude more accurately than other men, and that you are therefore less religious because more judicious, I shall claim the privilege of a Freethinker; and take the liberty to inquire into the object, principles, and method of demonstration admitted by the mathematicians of the present age, with the same freedom that you presume to treat the principles and mysteries of Religion [160, pp. 627–634].

Berkeley then proceeds to ridicule the ideas and terminology of the calculus, which involved not only infinitesimals like Leibniz's dx and dy above, but infinitesimal differences of these differences (i.e., second differences), etc. Berkeley concludes therefore:

> All these points, I say, are supposed and believed by certain rigorous exactors of evidence in religion, men who pretend to believe no further than they can see. That men who have been conversant only about clear points should with difficulty admit obscure ones might not seem altogether unaccountable. But he who can digest...a second or third difference, need not, methinks, be squeamish about any point of divinity...

Berkeley explicitly rips apart the type of tangent calculation we have just seen for $x = y^3$. He says of the differences (increments) dx and dy:

> For when it is said, let the increments vanish, i.e., let the increments be nothing, or let there be no increments, the former supposition that the increments were something, or that there were increments, is destroyed, and yet a consequence of that supposition, i.e., an expression got by virtue thereof, is retained. Which...is a false way of reasoning. Certainly when we suppose the increments to vanish, we must suppose their proportions, their expressions, and everything else derived from the supposition of their existence, to vanish with them....

Berkeley admits that the calculus produces correct answers, but for no solid reasons:

> I have no controversy about your conclusions, but only about your logic and method: how you demonstrate? what objects you are conversant with, and whether you conceive them clearly? what principles you proceed upon; how sound they may be; and how you apply them? ...
>
> Now, I observe, in the first place, that the conclusion comes out right, not because the rejected square of dy was infinitely small, but because this error was compensated for by another contrary and equal error....
>
> And what are these...evanescent increments? They are neither finite quantities, nor quantities infinitely small nor yet nothing. May we not call them the ghosts of departed quantities?

And finally, in one of a series of "questions" issued as a challenge to mathematicians who criticize religion:

> Question 64. Whether mathematicians, who are so delicate in religious points, are strictly scrupulous in their own science? Whether they do not submit to authority, take things upon trust, and believe

in points inconceivable? Whether they have not *their* mysteries, and what is more, their repugnances and contradictions?

While Berkeley's mathematical criticisms were largely valid, it is clear he had primarily a religious axe to grind. By arguing that their calculus was no more scientific than theology, and that it too was also built only on faith, he wanted to shame mathematicians into refraining from criticizing religion. Berkeley admitted that the calculus led to correct answers, and claimed that this resulted from a "compensation of errors," in which the multiple errors implicit in the rules of calculus somehow cancel each other's effects, thus arriving "though not at Science, yet at Truth, For Science it cannot be called, when you proceed blindfold, and arrive at the Truth not knowing how or by what means" [77, pp. 88–89]. We will see how some of his mathematical criticisms slowly began to be resolved.

Exercise 3.25: Study Leibniz's harmonic triangle of successive differences. Determine formulas for all the terms, and show how to find the sums of various infinite series of successive differences.

Exercise 3.26: Read and explain Fermat's method of finding the tangent line to a parabola via infinitesimals. Compare the notation of the French and English translations [59, III, pp. 121–23][58, pp. 358–359][166, pp. 223–24] with the Latin original [59, I, pp. 133–35]. Is Fermat's result already in Proposition 2 from Archimedes' *Quadrature of the Parabola* [3]?

Exercise 3.27: Verify that our quadrature of the parabola with Leibniz's Fundamental Theorem of Calculus effectively yields the same result as those of Archimedes and Cavalieri.

Exercise 3.28: Calculate the area under $x = y^3$, and in general $x = y^n$, using Leibniz's Fundamental Theorem of Calculus and his notation.

Exercise 3.29: Can you extend the results of Exercise 3.28 to exponents n other than 1, 2, 3,..., e.g., fractional or negative powers?

Exercise 3.30: Can you calculate $d\left(\frac{y}{x}\right)$ as Leibniz might have?

Exercise 3.31: Can you calculate $d(\sin(x))$ or $d(10^x)$ or $d(e^x)$ à la Leibniz? Can you use these to find areas?

3.6 Cauchy's Rigorization of Calculus

Augustin-Louis Cauchy was born in 1789, the year the French Revolution began. He was the eldest of six children, and his father, a student of classics and barrister in Normandy, ensured him an excellent education, leading at age 16 to admittance at the elite Ecole Polytechnique. He subsequently entered the Ecole des Ponts et Chaussées (College of Bridges and