CHAPTER 1

# Geometry: The Parallel Postulate

## 1.1 Introduction

The first half of the nineteenth century was a time of tremendous change and upheavals all over the world. First the American and then the French revolution had eroded old power structures and political and philosophical belief systems, making way for new paradigms of social organization. The Industrial Revolution drastically changed the lives of most people in Europe and the recently formed United States of America, with the newly perfected steam locomotive as its most visible symbol of progress. The modern era began to take shape during this time (Exercise 1.1). No wonder that mathematics experienced a major revolution of its own, which also laid the foundations for the modern mathematical era. For twenty centuries one distinguished mathematician after another attempted to prove that the geometry laid out by Euclid around 300 B.C.E. in his *Elements* was the "true" and only one, and provided a description of the physical universe we live in. Not until the end of the eighteenth century did it occur to somebody that the reason for two-thousand years' worth of spectacular failure might be that it was simply *not true*. After the admission of this possibility, proof of its reality was not long in coming. However, in the end this "negative" answer left mathematics a much richer subject. Instead of one geometry, there now was a rich variety of possible geometries, which found applications in many different areas and ultimately provided the mathematical language for Einstein's relativity theory.

To understand what is meant by this statement we need to begin by taking a look at the structure and content of Euclid's *Elements*. Just like other authors before him, Euclid had produced a compendium of geometric results known at the time. What made his *Elements* different from those of his predecessors was a much higher standard of mathematical rigor, not to be surpassed until a few centuries ago, and the logical structure of the work. Beginning with a list of postulates, which we might consider as fundamental truths accepted without proof, Euclid builds up his geometrical edifice as a very beautiful and economical succession of theorems and proofs, each depending on the previous ones, with little that is superfluous. This structure was greatly influenced by the teachings of Aristotle. Naturally, much depends on one's choice of "fundamental truths" that one is willing to accept without demonstration as foundation of the whole theory.

Among the ten postulates, or axioms, as they would be called today, the five most important ones are of two types [51, vol. I, pp. 195 ff]. The first three postulates assert the possibility of certain geometric constructions.

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and radius.

The next one states that

4. All right angles are equal to one another.

Thus, a right angle is a *determinate magnitude*, by which other angles can be measured. A rather subtle consequence of this postulate is that space must be homogeneous, so that no distortion occurs as we move a right angle around to match it with other right angles. We will have more to say about this later.

Finally, the last and most important postulate concerns parallel lines. Again, faithful to Aristotelian doctrine, Euclid precedes his postulates by definitions of the concepts to be used. He defines two parallel straight lines to be "straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction" [51, vol. I, p. 190]. The fifth, or "parallel," postulate, as it is known, states:

5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, will meet on that side on which are the angles less than two right angles.

It is a witness to Euclid's genius that he chose this particular statement as the basis of his geometry and viewed it as undemonstrable, as history was to show. (For more information on Euclid and his works see the chapter on number theory. A detailed description of the *Elements* can be found in [42].)

Much of what we know about the role of the *Elements* in antiquity comes from an extensive commentary by the philosopher Proclus (410–485), head of the Platonic Academy in Athens and one of the last representatives of classical Greek thought. According to him, the parallel postulate was questioned from the very beginning, and attempts were made either to prove it using the other postulates or to replace it by a more fundamental truth, possibly based on a different definition of parallelism. Proclus himself says:

> This ought even to be struck out of the Postulates altogether; for it is a theorem involving many difficulties, which Ptolemy, in a certain book, set himself to solve, and it requires for the demonstration of it a number of definitions as well as theorems. And the converse of it is actually proved by Euclid himself as a theorem. It may be that some would be deceived and would think it proper to place even the assumption in question among the postulates as affording, in the lessening of the two right angles, ground for an instantaneous belief that the straight lines converge and meet. To such as these Geminus[1] correctly replied that we have learned from the very pioneers of this science not to have any regard to mere plausible imaginings when it is a question of the reasonings to be included in our geometrical doctrine. For Aristotle says that it is as justifiable to ask scientific proofs of a rhetorician as to accept mere plausibilities from a geometer; and Simmias is made by Plato to say that he recognizes as quacks those who fashion for themselves proofs from probabilities. So in this case the fact that, when the right angles are lessened, the straight lines converge is true and necessary; but the statement that, since they converge more and more as they are produced, they will sometime meet is plausible but not necessary, in the absence of some argument showing that this is true in the case of straight lines. For the fact that some lines exist which approach indefinitely, but yet remain non-secant, although it seems improbable and paradoxical, is nevertheless true and fully ascertained with regard to other species of lines. May not then the same thing be possible in the case of straight lines which happens in the case of the lines referred to? Indeed, until the statement in the Postulate is clinched by proof, the facts shown in the case of other lines may direct our imagination the opposite way. And, though the controversial arguments against the meeting of the straight lines should contain much that is surprising, is there not all the more reason why we should expel from our body of doctrine this merely plausible and unreasoned hypothesis? [51, pp. 202 f.]

This objection to the parallel postulate, so aptly described here by Proclus, was shared by mathematicians for the next two thousand years and produced a vast amount of literature filled with attempts to furnish the proof Proclus calls for. His description also gives a glimpse of the age-old debate about what constitutes mathematical rigor, which was to play an important role in the subsequent history of the problem.

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[1] Greek mathematician, approx. 70 B.C.E.

FIGURE 1.1. Transport of angles.

Why would Euclid choose to include such an odd and unintuitive statement among his postulates? Parallels play a central role in Euclidean geometry, because they allow us to transport angles around, the central tool for proving even the most basic facts. Thus, given an angle with sides $l$ and $l'$, we want to draw the same angle with side $l'$, through a point $P$ not on $l$. In order to do this, we need to be able to draw a line through $P$ that is parallel to $l$ (Figure 1.1). This raises the question whether such parallels always exist. But to be useful for constructions there must also be a unique parallel to $l$ through $P$. Euclid proves the existence of parallels in Proposition 31 of Book I, without the use of the parallel postulate. Uniqueness follows from Proposition 30 [51, Vol. I, p. 316], for which the parallel postulate is necessary (Exercise 1.2).

The first source in this chapter is Proposition 32 from Book I of the *Elements*, which asserts that the angle sum in a triangle is equal to two right angles. Book I is arranged in such a way that the first 28 propositions can all be proved without using the parallel postulate. It is used, however, in Euclid's proof of Proposition 32. Much subsequent effort was focused on understanding the precise relationship between this result and the parallel postulate, as we will see later in the chapter.

The other central consequence of the parallel postulate in Euclidean geometry is the Pythagorean Theorem,[2] perhaps the best-known mathematical result in the world. Babylonian civilizations knew and used it at least 1,000 years before Pythagoras, and from 800–600 B.C.E. the *Sulbasutras* of Indian Vedic mathematics and religion told how to use it to construct perfect religious altars [91, pp. 105, 228–229]. The earliest Chinese mathematical text in existence shows a diagrammatic proof of the theorem, based on intuitive ideas of how squares fit together (equivalent to assuming the parallel postulate). While it is difficult to date this text, it is certainly a development concurrent with, and completely separate from, classical Greek mathematics [91, pp. 132, 180][116, pp. 124, 126].

Greek mathematics continued to explore the question of the validity of the parallel postulate. Shortly after Euclid, Archimedes wrote a treatise *On Parallel Lines*, in which he replaced Euclid's definition by the property that parallel lines are those equidistant to each other everywhere [144, pp. 41 f]. The parallel postulate can then be proven, provided that one accepts as true that, for instance, a "line" equidistant to a straight line is itself a straight line. The issue was taken up again by a number of distinguished Islamic mathematicians, beginning in the ninth century. A very detailed account of all these efforts is given in [144, Ch. 2]. As knowledge of Greek and

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[2]The sum of the squares of the legs of a right triangle equals the square of the hypotenuse.

PHOTO 1.1. Pythagorean Theorem in Thabit Ibn Qurra's ninth-century translation of Euclid's *Elements.*

Islamic mathematics spread into Western Europe during the Renaissance, so did the desire to prove the parallel postulate. An interesting approach was proposed by the Englishman John Wallis (1616–1703). Much of the plane geometry in the *Elements* deals with similarity of triangles and other figures. Wallis gives a proof of the parallel postulate based on the assumption that triangles similar to a given one *exist.* Now, it is debatable whether this assumption is any more obvious than the parallel postulate itself, but Wallis's argument shows that the validity of the parallel postulate is equivalent to the possibility of shrinking or expanding figures without changing their shape [144, p. 97].

It was the Italian Jesuit Girolamo Saccheri (1667–1733) who proposed a radically new approach to the problem. It was to bring him to the brink of a revolutionary discovery, which ironically and tragically neither he nor his contemporaries realized. Rather than trying to deduce the parallel postulate from the other four, he assumed that it was false and then tried to derive a contradiction from this assumption. And so he proceeded to prove theorem after theorem of a geometry in which the parallel postulate was false, looking in vain for the hoped-for contradiction. All he used was the first 28 propositions of the *Elements*, whose proof depended only on the first four postulates. Finally, using invalid reasoning, he convinced himself that he had found the elusive contradiction, and concluded that the parallel postulate was valid after all. In 1733, he published his collection of theorems and his unfortunate conclusion in the book *Euclid Freed of All Blemish or A Geometric Endeavor in Which Are Established the Foundation Principles of Universal Geometry* [150]. After receiving quite a bit of

FIGURE 1.2. Saccheri's quadrilateral.

attention upon its publication, the book was promptly forgotten for 150 years. Without realizing it, Saccheri had developed a body of theorems about a new geometry, which was free of contradictions and in which the parallel postulate was false. After two thousand years, the quest to solve the parallel problem could have come to a most surprising and wonderful resolution with the discovery that the world of geometry was much richer than humankind had realized.

Saccheri's work centers on the nature of a special type of quadrilateral [18] (Figure 1.2). A Saccheri quadrilateral has two consecutive right angles $A$ and $B$, and two equal sides $AD$ and $BC$, from which one deduces (without assuming the parallel postulate) that angles $C$ and $D$ are equal (Exercise 1.3). Now, assuming Euclid's parallel postulate, one can prove that $C$ and $D$ are also right angles (Exercise 1.4). So if we assume that $C$ and $D$ are not right an gles, we are implicitly denying the parallel postulate and replacing it by an alternative. Saccheri considered the following three possibilities:

**1.** The Hypothesis of the Right Angle (HRA): $C$ and $D$ are right angles.
**2.** The Hypothesis of the Obtuse Angle (HOA): $C$ and $D$ are obtuse angles.
**3.** The Hypothesis of the Acute Angle (HAA): $C$ and $D$ are acute angles.

Saccheri then proved several interesting and useful theorems. First, under HRA, HOA, or HAA, $AB$ is respectively equal to, greater than, or less than $CD$. Second, if HRA, HOA, or HAA holds for just one Saccheri quadrilateral, then the same hypothesis holds for any Saccheri quadrilateral. Third, and most importantly for us here, under HRA, HOA, or HAA, the sum of the angles in any triangle is respectively equal to, greater than, or less than two right angles (Exercise 1.5).

Saccheri then tried to establish the parallel postulate by refuting both the HOA and HAA. (Doing so is enough to establish the parallel postulate, but we will not show the details here.) Both proofs are based on obtaining a contradiction arising from the assumed hypothesis, and while his refutation of the HOA was correct, his proof of the refutation of the HAA is based on assumptions about how lines meet at infinity and was highly questionable. While Saccheri was not the first to consider many of these connections, his work went considerably further than all before him and shows clearly that the question of the parallel postulate and its two alternatives is equivalent to the question of each of the three possibilities for the angle sum of any triangle.

Of course, Saccheri realizes that his arguments leave something to be desired.

> It is well to consider here a notable difference between the foregoing refutations of the two hypotheses. For in regard to the hypothesis of obtuse angle the thing is clearer than midday light...
>
> But on the contrary I do not attain to proving the falsity of the other hypothesis, that of acute angle, without previously proving that the line, all of whose points are equidistant from an assumed straight line lying in the same plane with it, is equal to [a] straight line [144, pp. 99–101].

And so he admits that to refute the HAA he had to make use of another result that he does not consider completely clear and beyond reproach. We will see that this phenomenon of recourse to other questionable and unproven assumptions is a pattern in purported proofs of the refutation of the HAA.

Not long after Saccheri's book appeared, a similar investigation was undertaken by the Swiss mathematician Johann Lambert (1728–1777), whose interest in foundational questions naturally led him to consider the parallel postulate. His treatise *Theory of Parallels* (reproduced in [50]) was not published during his lifetime, however, possibly because he felt unsatified with its inconclusiveness. His work follows that of Saccheri in its approach.

Lambert introduces his treatise with:

> This work deals with the difficulty encountered in the very beginnings of geometry and which, from the time of Euclid, has been a source of discomfort for those who do not just blindly follow the teachings of others but look for a basis for their convictions and do not wish to give up the least bit of rigor found in most proofs. This difficulty immediately confronts every reader of Euclid's *Elements*, for it is concealed not in his propositions but in the axioms with which he prefaced the first book [144, pp. 99–101].

Specifically regarding the parallel postulate (which Lambert calls the "11th axiom"), he says:

> Undoubtedly, this basic assertion is far less clear and obvious than the others. Not only does it naturally give the impression that it should be proved, but to some extent it makes the reader feel that he is capable of giving a proof, or that he should give it.
>
> However, to the extent to which I understand this matter, this is just a *first* impression. He who reads Euclid further is bound to be amazed not only at the thoroughness and rigor of his proofs but also at the well-known delightful simplicity of his exposition. This being so, he will marvel all the more at the position of the 11th axiom when he finds out that Euclid proved propositions that could far more easily be left unproved.

After providing a proof that refutes the HOA, as had Saccheri, Lambert turns to the HAA. It was typical, in trying to refute the HAA, to derive consequences from it that would lead to a contradiction, thereby refuting it. In doing so, Lambert and others were in fact deriving results that would have the status of theorems in a brand new geometry in which the Hypothesis of the Acute Angle holds, replacing the parallel postulate. Lambert's point of view leans just slightly in the direction of this new geometry, rather than simply rejecting it as impossible, when he says:

> It is easy to see that under the [Hypothesis of the Acute Angle] one can go even further and that analogous, but diametrically opposite, consequences can be found under the [Hypothesis of the Obtuse Angle]. But, for the most part, I looked for such consequences under the [Hypothesis of the Acute Angle] in order to see if contradictions might not come to light. From all this it is clear that it is no easy matter to refute this hypothesis ...
>
> The most striking of these consequences is that *under the [Hypothesis of the Acute Angle] we would have an absolute measure of length for every line, of area for every surface and of volume for every physical space*. This refutes an assertion that some unwisely hold to be an axiom of geometry, for until now no one has doubted that there is no absolute measure whatsoever. There is something exquisite about this consequence, something that makes one wish that the Hypothesis of the Acute Angle were true.
>
> In spite of this gain I would not want it to be so, for this would result in countless inconveniences. Trigonometric tables would be infinitely large, similarity and proportionality of figures would be entirely absent, no figure could be imagined in any but its absolute magnitude, astronomers would have a hard time, and so on.

Lambert had discovered that with the HAA, length was no longer relative as it is in Euclid's geometry. Specifically, one could not simply enlarge or shrink geometric figures at will, always creating similar figures with the same shape. To give a concrete example, in the new world Saccheri and Lambert are exploring while trying to refute the HAA, one finds that lengthening the sides of an equilateral triangle, say by doubling the length of each side, will reduce the size of its angles, so that the larger equilateral triangle is not similar to the smaller one. The length of its sides will determine the size of its angles, and vice versa, so that by picking a particular size (say half a right angle) for its angles, one is forcing its sides to have a certain length (which one could choose as the absolute unit of measurement in the new geometry). It is in this sense that Lambert says we would have an absolute measure of length.

How did Lambert react to this strange new world? On the one hand he found it truly enticing, on the other frightening because it seems it would be so much more complex. He recognizes, however, that his desires should not play a role:

> But all these are arguments dictated by love and hate, which must have no place either in geometry or in science as a whole.
>
> To come back to the [Hypothesis of the Acute Angle]. As we have just seen, under this hypothesis the sum of the three angles in every triangle is less than 180 degrees, or two right angles. But the difference up to 180 degrees increases like the area of the triangle; this can be expressed thus: if one of two triangles has an area greater than the other then the first has an angle sum smaller than the second . . .
>
> I will add just the following remark. Entirely analogous theorems hold under the [Hypothesis of the Obtuse Angle] except that under it the angle sum in every triangle is greater than 180 degrees. The excess is always proportional to the area of the triangle.
>
> I think it remarkable that the [Hypothesis of the Obtuse Angle] holds if instead of a plane triangle we take a spherical one, for its angle sum is greater than 180 degrees and the excess is proportional to the area of the triangle.
>
> What strikes me as even more remarkable is that what I have said here about spherical triangles can be proved independently of the difficulty posed by parallel lines....

Lambert is observing that although the HOA cannot hold in plane geometry, it does in fact hold in “spherical geometry,” namely the geometry of the surface of a sphere, in which the “lines” are great circles on the sphere (i.e., circles whose center is the center of the sphere; these great circles provide the shortest distance between two points on the sphere and are therefore the paths preferred by airplanes when flying over the surface of the earth). Exercise 1.6 explores Lambert’s claims about area and angle sum for spherical triangles. In spherical geometry some of Euclid’s other postulates do not hold (for instance, there is more than one line joining pairs of diametrically opposite points, and one cannot extend a line indefinitely in length, in the sense that a great circle joins up with itself). This explains why the proofs of Saccheri and Lambert refuting the HOA do not also refute it in spherical geometry, since they rely on all of Euclid’s other postulates, some of which are missing in spherical geometry.

Lambert speculates that:

> From this I should almost conclude that the [Hypothesis of the Acute Angle] holds on some imaginary sphere. At least there must be some-

> thing that accounts for the fact that, unlike the [Hypothesis of the Obtuse Angle], it has for so long resisted refutation on planes.

Lambert had reason to think that if the HOA holds on an ordinary sphere, then the HAA, which is its opposite, might hold on a sphere of imaginary radius. This idea was actually not so far-fetched, and would be made more precise later by others. Despite his entrancement with the possibilities of this new geometry on an imaginary sphere, based on the HAA, Lambert still felt he should, and could, disprove the HAA for plane geometry, and he provided his own proof, which, like all those before him, was spurious in its own way.

The last serious attempt to prove the parallel postulate was made by the French school, which dominated mathematics at the end of the eighteenth and beginning of the nineteenth century. Here, too, no thought was given to the possibility that the parallel postulate may be an assumption independent of the rest of geometry. An important argument that would have stifled any such doubts came from physics, put forward by Laplace (1749–1827), and described in [18, pp. 53 f.]:

> Laplace observes that Newton's Law of Gravitation, by its simplicity, by its generality and by the confirmation which it finds in the phenomena of nature, must be regarded as rigorous. He then points out that one of its most remarkable properties is that, if the dimensions of all the bodies of the universe, their distances from each other, and their velocities, were to decrease proportionally, the heavenly bodies would describe curves exactly similar to those which they now describe, so that the universe, reduced step by step to the smallest imaginable space, would always present the same phenomena to its observers. These phenomena, he continues, are independent of the dimensions of the universe, so that the simplicity of the laws of nature only allows the observer to recognize their ratios. Referring again to this astronomical conception of space, he adds in a Note: "The attempts of geometers to prove Euclid's Postulate on Parallels have been up till now futile. However, no one can doubt this postulate and the theorems which Euclid deduced from it. Thus the notion of space includes a special property, self-evident, without which the properties of parallels cannot be rigorously established. The idea of a bounded region, e.g., a circle, contains nothing which depends on its absolute magnitude. But if we imagine its radius to diminish, we are brought without fail to the diminution in the same ratio of its circumference and the sides of all the inscribed figures. This proportionality appears to me a more natural postulate than that of Euclid, and it is worthy of note that it is discovered afresh in the results of the theory of universal gravitation."

Laplace, like Wallis, is observing that if we allow for the existence of similarity, then Euclid's theory of parallels follows. But Laplace believed that this similarity is inherent in the physical laws of space.

A particularly clear and very instructive proof of the parallel postulate was given by Adrien-Marie Legendre (1752–1833), an important and influential member of the French Academy of Sciences, in Paris. We include it as the second source of this chapter, representing the very end of the long string of such proofs. It is taken from Legendre's textbook *Eléments de Géométrie* (Elements of Geometry) [108], first published in 1794. The book went through many editions and was an influential geometry text all through the nineteenth century. Just like Saccheri and Lambert before him, he first refutes HOA, and then proposes a proof to refute HAA. In it he uses an interesting, apparently completely obvious assumption, which, however, turns out also to be equivalent to assuming the parallel postulate.

In Laplace's argument above we see one of the most important reasons why so many brilliant mathematicians over so many centuries stubbornly clung to the belief that Euclid's parallel postulate should follow from the others. Geometry was inextricably tied to space, our physical universe. And space was considered infinite, homogeneous, and the basis for all our experience. Nothing other than Euclidean geometry was *thinkable*. Another, more subtle, reason is suggested in the preface to a modern reprint of Saccheri's book.

> At the present day, we have an abundance of organized knowledge, which offers explanations—in which we have the fullest confidence—of many aspects of our physical universe. We need only refer to thermodynamics, geophysics, fluid dynamics, paleontology—the list is endless, and no one can possibly master all the knowledge that is available. But none of this knowledge extends back more than two hundred years. A group of educated eighteenth-century men, for example, sitting before an open fire, could no more understand or explain the nature of that fire than could their Neanderthal ancestors. For, the nature of light, the nature of heat, the nature of chemical combination and, in particular, of combustion, even the existence of oxygen, were yet to be discovered. Nor did Science, in the eighteenth century, at all inspire confidence, as it does today . . .
>
> Let us therefore take a brief inventory of what existed in the eighteenth century to satisfy man's craving for certainty, for organized knowledge.
>
> Physical science, as we have just mentioned, was not yet ready to satisfy this need for certainty. The teachings of the Church were indeed unquestioned Truths, for the Faithful. But the Faithful had to be aware that these Truths were ignored by much of mankind and were under constant attack by heretics. Philosophy seemed to offer

> certainty, but the existence of competing, and contradictory, schools of philosophy betrayed an underlying uncertainty.
>
> Contrast all of this with Geometry, which for two thousand years had been accepted as being the Science of the space in which we live . . .
>
> If a valid geometry, alternative to Euclid's, were to exist, then Euclidean geometry would not necessarily be the science of space, and in fact there would no longer *be* a science of space. And with that science gone, there would be nothing—no science at all. Thus, in addition to the many reasons for not doubting Euclidean geometry to be the one and only geometry (after all, in our day, it is still the geometry of architecture, engineering, and most branches of the sciences), we have another and powerful *silent* motive—a motive which does not reach consciousness and which for that reason is all the more powerful, the sort of motive which, under the right circumstances, makes an idea *unthinkable*. [150, pp. ix–x]

What makes this seemingly stubborn pursuit of the parallel postulate all the more puzzling is that during this entire time a perfectly good non-Euclidean geometry was sitting right under people's noses: the geometry of the sphere. But since Euclidean geometry was tied so strongly to the nature of space itself, the step of viewing plane and spherical geometry as just two examples of geometrical systems of equal status never suggested itself.

But the time was finally ripe for a breakthrough. As in so many other branches of mathematics it was left to Carl Friedrich Gauss (1777–1855), the mathematical titan of the nineteenth century, to make the first step. Ironically, for fear of getting embroiled in controversy, he kept his insights secret for almost fifty years, until others had taken the courageous step to proclaim the existence of a geometry independent of the parallel postulate.

Beginning in 1792, Gauss at first also tried to prove the postulate, like his predecessors, proceeding by assuming it to be false, hoping to reach a contradiction. In a letter to his fellow student Wolfgang Bolyai (1775–1856), who was also working on this problem and had convinced himself of success, he expresses his frustration:

> As for me, I have already made some progress in my work. However, the path I have chosen does not lead at all to the goal which we seek, and which you assure me you have reached. It seems rather to compel me to doubt the truth of geometry itself.
>
> It is true that I have come upon much which by most people would be held to constitute a proof: but in my eyes it proves as good as *nothing*. For example, if one could show that a rectilinear triangle is possible, whose area would be greater than any given area, then I would be ready to prove the whole of geometry absolutely rigorously.

PHOTO 1.2. Gauss.

> Most people would certainly let this stand as an Axiom; but I, no! It would, indeed, be possible that the area might always remain below a certain limit, however far apart the three angular points of the triangle were taken [18, pp. 65 f.].

Some time later Gauss finally convinced himself that the right path was in fact to give up this age-old attempt and instead develop a new geometry, which he called *Anti-Euclidean* and later *Non-Euclidean.* In a letter to his friend and colleague Heinrich Olbers (1758–1840), Gauss writes:

> I am ever more convinced that the necessity of our geometry cannot be proved—at least not by *human* reason for human reason. It is possible that in another lifetime we will arrive at other conclusions on the nature of space that we now have no access to. In the meantime we must not put geometry on a par with arithmetic that exists purely a priori but rather with mechanics [144, p. 215].

Thus, Gauss too had been hampered by the dominant philosophy of space and its geometry, expressed earlier by Laplace, and championed by Immanuel Kant (1724–1804), the most influential philosopher of the eighteenth century. (See [144] for a detailed discussion of the influence of philosophies of space on the parallel problem.)

FIGURE 1.3. Gauss's definition.

Gauss bases everything on the following definition of parallel lines (Figure 1.3). *If the coplanar straight lines AM, BN do not intersect each other, while on the other hand, every straight line through A between AM and AB cuts BN, then AM is said to be parallel to BN* [18, pp. 67 f.].

The lines beginning at $A$ and extending to the right are divided into two classes, those that intersect $BN$ and those that do not. If we think of these lines as generated by taking the line $AB$ extended upwards and rotating it around $A$ clockwise, then the first line that does not intersect $BN$ is called *parallel* to $BN$ (Exercise 1.7).

Without the assumption of the parallel postulate, there could be more than one line that does not intersect *BN*. From this definition Gauss proceeds to prove the fundamental theorems of a new geometry. But he chose to keep his discoveries to himself, except for some close friends. When Wolfgang Bolyai sent him a paper that Bolyai's son János had written, in which he proposes just such a new geometry, Gauss replies:

> If I commenced by saying that I must not praise this work you would certainly be surprised for a moment. But I cannot say otherwise. To praise it, would be to praise myself. Indeed the whole contents of the work, the path taken by your son, the results to which he is led, coincide almost entirely with my meditations, which have occupied my mind partly for the last thirty or thirty-five years. So I remained quite stupefied. So far as my own work is concerned, of which up till now I have put little on paper, my intention was not to let it be published during my lifetime. Indeed the majority of people have no clear ideas upon the questions of which we are speaking, and I have found very few people who could regard with any special interest what I communicated to them on this subject. To be able to take such an interest it is first of all necessary to have developed careful thought to the real nature of what is wanted and upon this matter almost all are most uncertain. On the other hand, it was my idea to write down all this later so that at least it should not perish with me. It is therefore a pleasant surprise for me that I am spared this trouble, and I am very glad that it is just the son of my old friend who takes the precedence of me in such a remarkable manner [144, pp. 215–217].

János Bolyai (1802–1860) was a Hungarian officer in the Austrian army and inherited his interest in mathematics in general and in the parallel postulate in particular from his father. In 1823, he wrote to his father:

> I have now resolved to publish a work on parallels ... I have not yet completed the work, but the road that I have followed has made it almost certain that the goal will be attained, if that is at all possible:

> the goal is not yet reached, but I have made such wonderful discoveries that I have been almost overwhelmed by them, and it would be the cause of continual regret if they were lost. When you see them, you too will recognize them. In the meantime I can say only this: *I have created a new universe from nothing.* All that I have sent you till now is but a house of cards compared to a tower. I am as fully persuaded that it will bring me honour, as if I had already completed the discovery [78, p. 107].

His father replied with excitement:

> [I]f you have really succeeded in the question, it is right that no time be lost in making it public, for two reasons: first, because ideas pass easily from one to another, who can anticipate its publication; and secondly, there is some truth in this, that many things have an epoch, in which they are found in several places, just as violets appear on every side in the Spring. Also every scientific struggle is just a serious war, in which I cannot say when peace will arrive. Thus we ought to conquer when we are able, since the advantage is always to the first comer [78, p. 107].

Wolfgang Bolyai agreed to publish his son's manuscript as an appendix to his own book *Tentamen*, on the foundations of several mathematical subjects including geometry, which appeared in 1831. After sending a copy to Gauss, he received the above-mentioned reply. Gauss's letter dealt a devastating blow to János, crippling his whole subsequent career. While he continued to work on mathematics, he never again published anything. The lack of attention that his manuscript received in subsequent years caused only further discouragement.

There were indeed violets appearing in other places. At Kazan University, in Russia, the mathematics professor Nikolai Lobachevsky (1792–1856) wrote his first major work on geometry in 1823. Subsequent research led him down the exact same path as Gauss and Bolyai had followed. His new geometry, which was very similar to theirs, appeared first in his article *On the Principles of Geometry*, published in 1829–30 in the journal *Kazan Messenger*, produced by Kazan University. In 1835 he published a longer article on his new geometry, which also appeared in French translation in *Journal für die reine und angewandte Mathematik*, one of the foremost European mathematical journals. Then, in 1842 he published the book *Geometrische Untersuchungen zur Theorie der Parallellinien* (Geometrical Researches on the Theory of Parallels), in German, which finally brought recognition to his accomplishment. On the recommendation of Gauss, he was elected to the Göttingen Science Society, a considerable honor. The next and major source in this chapter is a part of Lobachevsky's book. He begins with a definition of parallel lines quite similar to that of Gauss, described earlier, and proceeds to derive all the fundamental theorems of his geometry without the assumption of the parallel postulate. Most importantly, he derives all the basic trigonometric formulas valid in Lobachevskian geometry, including those for triangles.

Thus, the credit for the discovery of hyperbolic geometry, as it is now known, fell in large part to Lobachevsky, who was the first to publish his account. But ultimately he did not fare much better than Bolyai. Lobachevsky being relatively unknown, his work did not receive the instant attention it deserved, and acceptance of Lobachevskian geometry was rather slow in coming. Eventually, three factors led to widespread acceptance of the brave new world of non-Euclidean geometry that was opened up by the pioneers Gauss, Bolyai, and Lobachevsky.

First of all, after Gauss's death, his correspondence with his colleague H.K. Schumacher was published between 1860 and 1863. It makes abundantly clear that Gauss thought very highly of the work of both Bolyai and Lobachevsky. His approval of these two until then unknown mathematicians did much to lend credibility to their work [18, pp. 122 ff.].

Secondly, the visionary work of Bernhard Riemann (1826–1866), one of the most imaginative mathematicians ever, proposed an entirely new paradigm for the concept of space and geometry, with a natural place for hyperbolic geometry. In his legendary paper *On the Hypotheses Which Lie at the Foundations of Geometry* [139] Riemann proposed the study of curved surfaces and higher-dimensional spaces, such as the plane or the sphere. His profound insight was that the geometry that is valid on the su rface depends on its "curvature," measured by a certain constant $K$ that is intrinsic to the surface. This idea was initially proposed by Gauss and represented a radical departure from the conventional idea that the curvature of a surface made sense on ly when viewed within an ambient space. Given two points on the surface, the shortest curve that connects the two points will then play the role of a straight line between two points in the plane. Such curves are now called geodesics. For instance, for the plane, the curvature constant $K$ is zero, and the resulting geometry is just the Euclidean one. The same is true for a surface that can be deformed, without stretching, into the plane, such as a part of a cylinder. For a sphere, on the other hand, we obtain that $K$ is positive; that is, the sphere has positive curvature. The resulting geometry is, of course, spherical geometry, and the HOA is valid. Likewise, a surface for which any part can be deformed into part of a sphere supports the same type of geometry. This leaves surfaces with negative curvature. For t hose that have so-called constant negative curvature, the geometry that is valid for them is precisely hyperbolic geometry and the HAA [18, pp. 130 ff.][78, Ch. 12].

As Riemann's ideas became accepted, so did hyperbolic geometry. The result was a whole new mathematical theory: differential geometry and topology. When Albert Einstein was struggling with the theory of special and general relativity, it was this theory that provided the natural language

PHOTO 1.3. Riemann.

for it. Ultimately, it turned out that the physical universe we live in looks a lot more like Lobachevsky's world than Euclid's.

Thirdly, and most importantly, the issue that needed to be settled before hyperbolic geometry was put on a firm foundation was the question of true independence of the parallel postulate from the other four. Lobachevsky had proven many theorems based on the assumption that the parallel postulate was false, without encountering a contradiction. But that did not necessarily imply that there wasn't one to be found someplace else. What was needed was a rigorous proof that the existence of more than one line through a given point parallel to a given line led to a consistent geometric theory. The ingenious solution to this problem was to produce a Euclidean model of hyperbolic geometry, a sort of faithful projection of the hyperbolic plane onto part of a Euclidean plane, in such a way that parallels, triangles, etc. in hyperbolic geometry corresponded to some type of figure in Euclidean geometry. Then, if a contradiction existed in hyperbolic geometry, it would also have to exist in Euclidean geometry. Several such models were proposed, beginning with those of the Italian mathematician Eugenio Beltrami (1835–1900) [164, p. 35][78, Ch. 13]. As the last source in this chapter we study a model given by the great French mathematician Henri Poincaré (1854–1912). He was led to this model almost incidentally through his work in the analysis of functions. In his model, the hyperbolic plane is represented as a Euclidean disk. That is, points in the hyperbolic plane correspond to Euclidean points inside the disk, and lines in the hyperbolic plane correspond to arcs of Euclidean circles meeting the boundary of the disk perpendicularly, or to diameters of the disk.

Thus, the issue of the consistency of hyperbolic geometry was now squarely in the court of Euclidean geometry. Two thousand years of scrutiny had revealed many cracks in Euclid's foundations and many weaknesses of proofs in the *Elements*. For one, Euclid's definitions were for the most part wholly inadequate. For instance, to define a point as "that which has no part" [51, p. 153] immediately calls out for a definition of "part" and so on, leading to an infinite chain of definitions. In 1899, the German mathematician David Hilbert (1862–1943), on his way to becoming the dominating figure in mathematics during the first quarter of the twentieth century, presented a new system of axioms and definitions for Euclidean geometry [88]. Euclid's five axioms are replaced by a much longer list, and notions like "point" and "line" remained undefined. Only their mutual relationships were specified.

The transition had been made from viewing geometry as the science of the space we live in to geometry as a system of axioms, which is valid as long as the axioms do not lead to contradictory results. All over mathematics the so-called axiomatic method took hold, as evidenced, for instance, in the set theory chapter of this book, as the distinctive mark of twentieth-century mathematics. The failed attempt to prove that Euclid's was the one and only geometry led to a vast new mathematical universe, which is today continuing to enrich mathematics as well as its connections to the other sciences.

**Exercise 1.1:** Read about world history from 1750 until 1850.

**Exercise 1.2:** Look up the proof of Proposition 30 in Book I of the *Elements* [51, vol. I, pp. 316–317] and identify the step(s) that requires the parallel postulate.

**Exercise 1.3:** Prove that angles $C$ and $D$ are equal in Saccheri's quadrilateral.

**Exercise 1.4:** Prove that with the parallel postulate, $C$ and $D$ are right angles in Saccheri's quadrilateral.

**Exercise 1.5:** Prove Saccheri's theorems on his quadrilaterals.

**Exercise 1.6:** Explore what triangles look like in spherical geometry (draw pictures). How long can lines be? How small or large can the angle sum of a triangle be? Give examples. Explore how the area of a spherical triangle appears to be related to its angle sum.

**Exercise 1.7:** Convince yourself that Gauss's definition of parallel lines does not depend on the choice of the points $A$ and $B$ on the given lines.