Exercise 5.1: Use the Babylonian method described in this section to solve the system

$$2xy + x - 4y = 1$$
$$x + 4y = 2.$$

What is the general principle behind the substitution to transform a given system into standard form?

Exercise 5.2: The Babylonians also solved systems of linear equations with a method similar to the one discussed in this section. Solve the following problem (see [128, p. 66]): From one field I have harvested 4 gur of grain per bùr (surface unit). From a second field I have harvested 3 gur of grain per bùr. The yield of the first field was 8,20 more than that of the second. The areas of the two fields were together 30,0 bùr. How large were the fields?

Exercise 5.3: Use the Pythagorean Theorem and the Fundamental Theorem of Arithmetic to show that the diagonal in a square of side length one cannot be a rational number.

Exercise 5.4: Use del Ferro's method to solve the following problem from Fiore's list of challenge problems that he gave to Tartaglia: "There is a tree, 12 braccia high, which was broken into two parts at such a point that the height of the part which was left standing was the cube root of the length of the part that was cut away. What was the height of the part that was left standing?"

Exercise 5.5: Use del Ferro's formula for the roots of a cubic to solve the equation $x^3 = 15x + 4$. (Note that this equation has three real roots.)

Exercise 5.6: Work out the details in Ferrari's method of solving an equation of degree four. For hints check [169, Sect. 3.2].

Exercise 5.7: Find all roots of the polynomial $x^4 - 1$.

5.2 Euclid's Application of Areas and Quadratic Equations

As mentioned in the introduction, Book II of Euclid's *Elements* is rather short, with only fourteen propositions. It contains a collection of results from "geometric algebra." And indeed a number of the results translate into well-known formulas in modern algebra. The original source we discuss in this section is Proposition 5. It turns out to be the essential ingredient in the solution of certain types of quadratic equations, aside from its use in

PHOTO 5.5. First English edition of Euclid, 1570.

application of areas problems, which we also discussed in the Introduction. For a more detailed discussion of Book II see [20, Sect. 7.6].

We begin with several definitions. (For a general introduction to Euclid and the *Elements* see the Euclid sections in the geometry and number theory chapters.)

Euclid, from *Elements*

BOOK II.
DEFINITIONS.

FIGURE 5.2. Gnomon.

1. Any rectangular parallelogram is said to be **contained** by the two straight lines containing the right angle.

2. And in any parallelogrammic area let any one whatever of the parallelograms about its diameter with the two complements be called a **gnomon**.

The type of figure referred to as a *gnomon* in the second definition can be thought of as a square with a little corner missing (Figure 5.2). The term is also used in connection with number-theoretic problems (see [93, p. 45]). For details of the origin and use of the term "gnomon" see [51, Vol. 1, pp. 370–372].

PROPOSITION 5.

If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.

For let a straight line AB be cut into equal segments at C and into unequal segments at D; I say that the rectangle contained by AD, DB together with the square on CD is equal to the square on CB (Figure 5.3).

For let the square $CEFB$ be described on CB, [I. 46] and let BE be joined; through D let DG be drawn parallel to either CE or BF, through H again let KM be drawn parallel to either AB or EF, and again through A let AK be drawn parallel to either CL or BM. [I. 31]

Then, since the complement CH is equal to the complement HF, [I. 43] let DM be added to each; therefore the whole CM is equal to the whole DF.

But CM is equal to AL, since AC is also equal to CB; [I. 36] therefore AL is also equal to DF. Let CH be added to each; therefore the whole AH is equal to the gnomon NOP.

But AH is the rectangle AD, DB, for DH is equal to DB, therefore the gnomon NOP is also equal to the rectangle AD, DB.

Let LG, which is equal to the square on CD, be added to each; therefore the gnomon NOP and LG are equal to the rectangle contained by AD, DB and the square on CD.

But the gnomon NOP and LG are the whole square $CEFB$, which is described on CB; therefore the rectangle contained by AD, DB together with the square on CD is equal to the square on CB.

Therefore etc.

Q. E. D.

FIGURE 5.3. Proposition 5.

Translated into algebraic language, the proposition is easily understood. If we let $AC = CB = a$ and $CD = b$, then the proposition asserts that

$$(a+b)(a-b)+b^2 = a^2,$$

which is easily verified. Thus, as we mentioned in the Introduction, Proposition 5 is just another one of the rules of algebra, stated in geometric language. For a detailed discussion of the significance of this proposition the reader is encouraged to consult [173, pp. 120 ff.].

In Euclid's proof, the complement *CH* and the complement *HF* refer to the rectangles *CDHL* and *HMFG*, respectively. If one thinks of the square *CBFE* as containing the squares *LHGE* and *DBMH*, then the rectangles *CDHL* and *HMFG* are complements in the sense that they are the figures required to fill the rest of the square *CBFE*. (See the commentary in [51, Vol. 1, p. 341] following Proposition 43 of Book I.) Note also that Euclid identifies rectangles (including squares) by opposite corners; so, for instance, *DM* is the square *DBMH*. The gnomon ("carpenter's square") *NOP* is the figure *CBFGHL* consisting of the large square *CEFB* with the lower left corner missing.

For our purposes, the real significance of Proposition 5 is that it gives a geometric method for solving certain quadratic equations. There is no evidence, however, that Euclid felt the same way. Suppose we wish to solve the equation

$$ax - x^2 = x(a-x) = b^2$$

for x, where $a, b > 0$. Let us further assume that $b < a/2$. We draw the picture in Figure 5.4. Then we complete it in the spirit of Euclid's Proposition 5 (Figure 5.5).

Proposition 5 applied to this picture now tells us that the area of the rectangle with sides x and $a - x$ together with the area of the small square with side length c is equal to the area of the square with side length $a/2$. In algebraic notation,

$$x(a-x)+c^2 = (a/2)^2.$$

Applying the Pythagorean Theorem to the triangle in Figure 5.5 results in

$$b^2 + c^2 = (a/2)^2.$$

Combining the two equations yields

$$x(a-x) = b^2,$$

FIGURE 5.4. A quadratic equation.

FIGURE 5.5. Solving a quadratic equation.

as desired.

Exercise 5.8: Find a proof of Proposition 6 in Book II in the spirit of Euclid, which says: *If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.* Hint: See Figure 5.6.

Exercise 5.9: Proposition 11 in Book II states: *To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.* Which type of quadratic equation can be solved with the help of this proposition and how?

Exercise 5.10: Read about the golden ratio (or divine proportion, as it is sometimes called) and find its connection to Proposition 11.

Exercise 5.11: Figure 5.7 accompanies the proof of Proposition 11. Here, AB represents the given straight line. The cutting point H is constructed by Euclid as follows. First construct the square $ABDC$. Then bisect AC at E, and join E to B. Now extend CA until F, so as to make $EF = BE$. Construct the square $FGHA$ on AF, and extend GH to K.

Now complete the proof of Proposition 11 in the spirit of Euclid.

5.3 Cardano's Solution of the Cubic

Much of sixteenth-century Europe was in great turmoil. Wars and diseases such as the plague decimated the population and caused constant social and political upheavals. Nonetheless, it was a century of great artistic, literary, and scientific accomplishments. One of its renowned scholars was the Italian Girolamo Cardano. Known and sought after throughout Europe for his skills as a physician, he also distinguished himself as a natural philosopher, mathematician, and astrologer. Among his many books, one of the most wellknown today is his treatise on algebra, *Ars Magna* (The Great Art). However, his mathematical writings comprise only a small part of the ten volumes of his collected works, published in Lyons in 1663, under the title *Opera Omnia Hieronymi Cardani, Mediolanensis*. Cardano was a typical representative of the Renaissance. To quote from a biography:

FIGURE 5.6. Proposition 6.

FIGURE 5.7. Proposition 11.