Exercise 4.7: A *primitive* Pythagorean triple is one in which any two of the three numbers are relatively prime. Show that every multiple of a Pythagorean triple is again a Pythagorean triple, and that every Pythagorean triple is a multiple of a primitive one.

Exercise 4.8: Show that the sum of two odd squares is never a square, and use this fact to conclude that all Pythagorean triples have an even leg.

Exercise 4.9: Look up the Euclidean algorithm and use it to decide whether a Pythagorean triple is primitive or not.

Exercise 4.10: Show that FLT is true for all exponents n if it is true for $n = 4$ and all odd prime numbers n.

4.2   Euclid's Classification of Pythagorean Triples

A triple of positive integers (x, y, z) is called a *Pythagorean triple* if the integers satisfy the equation $x^2 + y^2 = z^2$. Such a triple is called *primitive* if x, y, z have no common factor. For instance, $(3, 4, 5)$ and $(5, 12, 13)$ are primitive triples, whereas $(6, 8, 10)$ is not primitive, but is a Pythagorean triple. The significance of primitive triples is that "multiples" of primitive ones account for all triples (see Exercise 4.7 in the previous section). The problem of finding Pythagorean triples occupied the minds of mathematicians as far back as the Babylonian civilization. Analysis of cuneiform clay tablets shows that the Babylonians were in possession of a systematic method for producing Pythagorean triples [127, pp. 36 ff.].

For instance, the tablet catalogued as Plimpton 322 in Columbia University's Plimpton Collection, dating from 1900–1600 B.C.E., contains a list of fifteen Pythagorean triples as large as $(12709, 13500, 18541)$. (Is this triple primitive?) For a detailed discussion of this tablet see, e.g., [64], [129]. There is reason to believe that the Babylonians might even have known the complete solution to the problem [131, pp. 175–79]. Other civilizations, such as those of China and India, also have studied the problem [93]. Clearly, Pythagorean triples are related to geometry via the Pythagorean Theorem, as a Pythagorean triple corresponds to a right triangle with integer sides.

The Pythagoreans, after whom the theorem is named, were an ancient Greek school that flourished around the sixth century B.C.E. Aristotle says that they "applied themselves to the study of mathematics, and were the first to advance that science; insomuch that, having been brought up in it, they thought that its principles must be the principles of all existing things" [85, p. 36]. Their motto is said to have been "all is number" [20, p. 54]. The particular interest of the Pythagoreans in relationships between whole numbers naturally led to the investigation of right triangles with integral sides. Proclus, a later commentator, who taught during the fifth century C.E. at the Neo-Platonic Academy in Athens, credits the Pythagoreans

PHOTO 4.7. Plimpton 322.

with the formula

$$(2n + 1, 2n^2 + 2n, 2n^2 + 2n + 1) ,$$

for any positive integer n, generating triples in which the longer leg and hypotenuse differ by one. He also describes another formula,

$$(2n, n^2 - 1, n^2 + 1)$$

for all $n \geq 2$, attributed to Plato [58, pp. 212–13], [85, p. 47], [134, p. 340] (Exercise 4.18).

   The first formula giving a complete classification of all triples appears in Euclid's *Elements*:

   To find two square numbers such that their sum is also square [51, vol. 3, p. 63].

This is the first text that we will examine.

   Very little is known about Euclid's life. The most that can be said with any certainty is that he lived about 300 B.C.E. and taught at the University in Alexandria. He probably attended Plato's Academy in Athens, which was at the time the center of Greek mathematics, and he was invited to Alexandria when the great library and museum were being set up at the direction of Ptolemy Soter. Under the rule of Ptolemy, Alexandria became a world center both in commerce and scholarship; Euclid was a founder of the great school of mathematics there. Proclus tells us that Euclid "put together the *Elements*, collecting many of Eudoxus's theorems, perfecting many of Theaetetus's, and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors" [85, p. 202]. There is an extensive entry on Euclid in [42], which is highly recommended. The *Elements* was intended as an introductory textbook on elementary mathematics for students at the University. It ended up as the

PHOTO 4.8. Pythagoras.

all-time number one mathematical bestseller, and was used as a textbook in a number of countries even into this century. We quote from the preface of one such textbook [170] used in England during the second half of the nineteenth century:

> In England, the text-book of Geometry consists of the *Elements* of Euclid; for nearly every official programme of instruction or examination explicitly includes some portion of this work. Numerous attempts have been made to find an appropriate substitute for the *Elements* of Euclid; but such attempts, fortunately, have hitherto been made in vain. The advantages attending a common standard of reference in such an important subject, can hardly be overestimated; and it is extremely improbable, if Euclid were once abandoned, that any agreement would exist as to the author who should replace him.

The *Elements* consists of thirteen books, which deal with a variety of different subjects. No manuscripts from Euclid's time are known, and the versions of the *Elements* in use today are translations of Arabic, Greek, and Latin versions of various time periods. The first printed edition of the *Elements* was published at Venice in 1482. It was the first printed mathematics book of any importance. For more details on the various translations see

PHOTO 4.9. First printed edition of Euclid, 1482.

[85, Ch. IX]. For a more detailed discussion of the structure of the *Elements* see the geometry chapter.

Euclid's solution to the problem of the classification of all Pythagorean triples appears as Lemma 1 in Book X, before Proposition 29.[2] Book X concerns itself primarily with the theory and application of irrationals, and is the first major treatise we have on this subject. Pythagorean triples are used in Proposition 29 to produce examples of irrationals with certain properties.

The reader should be cautioned that the following text is rather subtle and requires very careful reading. It is followed by an extensive mathemati-

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[2]Most of the statements in the *Elements* are called Propositions, followed by a proof. Then there are other statements, referred to as "Lemma." These are simply auxiliary results that are needed in the proof of a proposition, but that are stated and proved separately, so as not to interrupt or unduly lengthen the proof. They may be proved before or after the proposition in which they are used.

cal commentary. In order to understand it, one needs to know what *similar plane numbers* are, the central concept in Euclid's argument. First of all, a *plane* number is simply a formal product of two numbers, thought of as the sides of a rectangle, of which the plane number itself gives the area. Two plane numbers are *similar* if their respective sides are proportional with the same rational proportionality factor. Geometrically, the two plane numbers represent similar rectangles. For example, the two plane numbers $6 \cdot 10$ and $9 \cdot 15$ are similar.

Euclid, from

Elements

LEMMA 1
(before Proposition 29 in Book X)

To find two square numbers such that their sum is also a square.

Let two numbers AB, BC be set out, and let them be either both even or both odd.

Then since, whether an even number is subtracted from an even number, or an odd number from an odd number, the remainder is even [IX. 24, 26], therefore the remainder AC is even.

Let AC be bisected at D.

Let AB, BC also be either similar plane numbers, or square numbers, which are themselves also similar plane numbers.

Now the product of AB, BC together with the square on CD is equal to the square on BD [II. 6].

And the product of AB, BC is square, inasmuch as it was proved that, if two similar plane numbers by multiplying one another make some number the product is square [IX. 1].

Therefore two square numbers, the product of AB, BC, and the square on CD, have been found which, when added together, make the square on BD.

And it is manifest that two square numbers, the square on BD and the square on CD, have again been found such that their difference, the product of AB, BC, is a square, whenever AB, BC are similar plane numbers.

But when they are not similar plane numbers, two square numbers, the square on BD and the square on DC, have been found such that their difference, the product of AB, BC, is not square.

Q. E. D.

In order to translate Euclid's argument into modern algebraic notation, let $AB = u, BC = v, CD = d, BD = f$. From $[\text{II.6}]^3$ Euclid obtains the equation

$$uv + d^2 = f^2,$$

which alternatively follows via

$$uv = (f + d)(f - d) = f^2 - d^2,$$

since D bisects AC and thus $AD = CD = d$.

The remainder of Euclid's argument involves similar plane numbers. One can prove that two numbers are similar plane numbers if and only if their ratio is the square of a rational number (see the lemma at the end of the section). Euclid continues his analysis by noting that uv is a square if and only if u and v are similar plane numbers, which follows from the previous sentence. So far he has given a procedure that beginning with two similar plane numbers of the same parity constructs a Pythagorean triple. (The parity of a number is simply whether it is even or odd. Thus 4 is of even parity. That two numbers are of the same parity means that they are either both even or both odd.) (Observe that his plane numbers must be unequal; why?)

But in fact all Pythagorean triples arise in this way, as he somewhat cryptically explains in the last paragraph. Start with a difference $f^2 - d^2$, and factor it as uv, where $u = f + d$ and $v = f - d$. Since this u and v are clearly of the same parity, Euclid's construction can proceed using these numbers. Notice that this will reproduce the original d and f with which we started. The result is the equation $uv = f^2 - d^2$ as above, but as Euclid says in his last paragraph, this difference of squares, which is uv, will only itself be a square if u and v were similar plane numbers to begin with. Thus Euclid has set up a one-to-one correspondence between differences of squares and pairs of unequal numbers of the same parity, in such a way that differences of squares that are themselves square correspond to pairs that are similar plane numbers. Thus he has classified all Pythagorean triples in terms of similar plane numbers; so the problem of finding all Pythagorean triples reduces to that of finding all pairs of similar plane numbers of the same parity (Exercises 4.12–4.13).

Now we can see how to find the formulas of the Pythagoreans and Plato that we mentioned. Since

$$f = \frac{u + v}{2} \text{ and } d = \frac{u - v}{2},$$

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[3]This states: "If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line."

we have

$$\left(\frac{u+v}{2}\right)^2 - \left(\frac{u-v}{2}\right)^2 = uv.$$

The simplest way to choose u, v such that their product is a square is to let $u = k^2$ and $v = 1$. For f and d to be integers, we must have k odd, thus leading to the formula of the Pythagoreans. Letting $u = 2k^2$ and $v = 2$ gives the Platonic formula [51, Vol. 1, p. 385].

From Euclid's classification of Pythagorean triples we can derive the following classification of all primitive Pythagorean triples, which we will need for our next source. Suppose that the triple (d, e, f) is primitive. One of the numbers d, e has to be even (this follows from Exercise 4.8 in the Introduction); let us assume that it is e, by interchanging d and e if necessary. Now, as noted above, we can apply Euclid's method to this triple, obtaining

$$f^2 - d^2 = e^2 = uv,$$

with u, v similar plane numbers of the same parity. Thus u and v are both even, and it follows from the lemma below that they are of the form $u = mp^2, v = mq^2$. A common divisor of u and v also divides $u + v = 2f$ and $u - v = 2d$. But since d and f are relatively prime, that is, their greatest common divisor is 1, it follows that the greatest common divisor of u and v is 1 or 2. Together with the fact that u and v are even, this implies that $m = 2$, and that p and q are relatively prime. Thus, the triple is of the form

$$d = p^2 - q^2, \quad e = 2pq, \quad f = p^2 + q^2 \ ,$$

and moreover, p and q have different parity, since d is odd by primitivity of the triple.

To summarize, any primitive triple has this form (by interchanging d and e if necessary), where p is greater than q, and p and q are relatively prime and of different parity. Furthermore, any such choice of p and q results in a primitive triple (check this).

Lemma: *Two plane numbers are similar if and only if their ratio is the square of a rational number.*

Proof. Suppose that u and v are similar plane numbers. Then $u = xy$ and $v = zw$ for some numbers (that is, positive integers) x, y, z, w, which are the sides of u and v, respectively. For these sides to be proportional means that $x = ry$ and $z = rw$ for some rational number r. So

$$\frac{u}{v} = \frac{xy}{zw} = \frac{ry^2}{rw^2} = \left(\frac{y}{w}\right)^2,$$

and y/w is clearly rational. Conversely, suppose $u/v = (p/q)^2$, where we may assume that p and q are relatively prime. So

$$u = \frac{p^2}{q^2} v.$$

As u is an integer, we know that q^2 divides p^2v. Since p and q are relatively prime, q^2 must divide v; that is, $v = mq^2$ for some positive integer m. Similarly, p^2 must divide u, and it is easy to see that in fact $u = mp^2$. If we choose p and mp to be the sides of u, and q and mq to be the sides of v, we see that u and v are similar plane numbers.

Exercise 4.11: Prove that there are only a finite number of Pythagorean triples with a fixed leg.

Exercise 4.12: Show that there are exactly two pairs of similar plane numbers of same parity that will produce a given Pythagorean triple using Euclid's construction.

Exercise 4.13: Enumerate (that is, give a recipe for listing) all pairs of similar plane numbers with same parity. (Remember to show that your list is complete.) Use your list to enumerate all Pythagorean triples.

Exercise 4.14: Enumerate all primitive Pythagorean triples.

Exercise 4.15: Derive a general formula from Diophantus's treatment of Pythagorean triples [86], and compare it with Euclid's.

Exercise 4.16: Give a geometric demonstration of Euclid's statement that "the product of AB, BC together with the square on CD is equal to the square on BD"; that is,

$$AB \cdot BC + (CD)^2 = (BD)^2.$$

[Hint: Recall that the product of two numbers may be represented geometrically as the area of a rectangle with those numbers as the lengths of its sides.]

Exercise 4.17: Enumerate as many Pythagorean triples as you can that contain the number 1378.

Exercise 4.18: In the two formulas for Pythagorean triples, ascribed to the Pythagoreans and Plato, respectively, which triples, if any, appear on both lists? Are the triples generated by these lists primitive? Which are, and which are not? Are there an infinite number of primitive triples? Do the primitive triples in the above two lists exhaust all possible primitive triples?

4.3   Euler's Solution for Exponent Four

Without doubt Leonhard Euler is one of the world's mathematical giants, whose work profoundly transformed mathematics. He made extensive contributions to many mathematical subjects, including number theory, and