PHOTO 1.4. Euclid.

1.2 Euclid's Parallel Postulate

Euclid's *Elements* [51], written in thirteen "books" around 300 B.C.E., compiled much of the mathematics known to the classical Greek world at that time. It displayed new standards of rigor in mathematics, proving everything by proceeding from the known to the unknown, a method called synthesis. And it ensured that a geometrical way of viewing and proving things would dominate mathematics for two thousand years.

Book I contains familiar plane geometry, Book II some basic algebra viewed geometrically, and Books III and IV are about circles. Book V, on proportions, enables Euclid to work with magnitudes of arbitrary length, not just whole number ratios based on a fixed unit length. Book VI uses proportions to study areas of basic plane figures. Books VII, VIII, and IX are arithmetical, dealing with many aspects of whole numbers, such as prime numbers, factorization, and geometric progressions. Book X deals with irrational magnitudes. The final three books of the *Elements* study solid geometry. Book XI is about parallelepipeds, Book XII uses the method of exhaustion to study areas and volumes for circles, cones, and spheres, and Book

XIII constructs the five Platonic solids inside a sphere: the pyramid, octahedron, cube, icosahedron, and dodecahedron [42]. For more information about Euclid and his environment, see the number theory chapter.

Our focus will be specifically on the controversial parallel postulate in Book I and its ramifications, in particular the angle sum of triangles. Book I begins with twenty-three "definitions," five "postulates," and five "common notions." The first few definitions are fundamental ones like "A *point* is that which has no part," "A *line* is breadthless length," and "A *straight line* is a line that lies evenly with the points on itself." (Notice that a line is thus what we would call a curve, and that certain lines are then called straight.) While at first sight these definitions of the most basic objects in geometry appear to provide a good foundation, in fact they ultimately raise more questions than they answer. To use them we must first be able to work with terms like "part," "breadthless," "length," "evenly with," and we find that attempting to define these terms leads us even further backwards to yet more undefined notions. This process of defining the most basic terms leads to an infinite regression, and even begins to lead us in circles. So a twentieth-century underpinning for geometry (and other fields of mathematics as well; see the set theory chapter) follows a different approach [88], instead taking words like *point* and *line* as undefined terms, about which one simply supposes basic properties that dictate how they interact with each other. For Euclid, these assumed interactions are provided by his five postulates, already given in the Introduction. For instance, the first postulate, "To draw a straight line from any point to any point," dictates that two points always determine a line. Euclid's five common notions are axioms of a more general nature, not confined to geometry, but rather assumptions common to all the sciences as a method of arguing. For example, the first states "Things that are equal to the same thing are also equal to one another."

Euclid first develops as much as he can of plane geometry without using the parallel postulate (Postulate 5). The first 26 propositions of Book I deal primarily with familiar properties of lines, angles, and triangles. Even when he begins the theory of parallels, Propositions 27 and 28 first prove what results he could deduce about parallels without appealing to Postulate 5. While much of Euclid's *Elements* is a feat of organizing already existing knowledge into a systematic format of presentation and proof, the approach to the theory of parallels is apparently due to Euclid himself, particularly the genius of recognizing and adopting the parallel postulate as an unprovable assumption on which to base the theory [42]. To consider his theory carefully, and obtain his result on angle sums in triangles, we begin with his definitions of right angles and parallels, and restate the parallel postulate itself.

Euclid, from *Elements*

BOOK I.
DEFINITION 10.

When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

Note that this definition makes no reference to any units of angle measurement, as opposed to a definition such as "A right angle has measure 90 degrees." Of course, once the notion of degree measure of angles is introduced, Euclid's definition is equivalent to the "measure is 90 degrees" definition. (Why?) One might think of Euclid's use of "equal" above as geometric congruence. Drawing the figure described in the definition (Figure 1.4), we see that angles ACD and DCB are equal if they lie on top of each other when we fold along DC.

DEFINITION 23.

Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

POSTULATE 5.

That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

PROPOSITION 29.

A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles.

This is the first result in *Elements* whose proof requires Postulate 5. We leave the proof to the reader (Exercise 1.9). In Figure 1.5, alternate angles are illustrated by the pair AGH, GHD, exterior and interior opposite

FIGURE 1.4. Definition 10.

FIGURE 1.5. Proposition 29.

angles are the pair EGB, GHD, and interior angles on the same side are BGH, GHD. The converse of the first claim (about alternate angles) in Proposition 29 is also true, and Euclid already proved it as Proposition 27.[3] It is interesting that this converse does not depend on Postulate 5 (Exercise 1.10).

PROPOSITION 30.

Straight lines parallel to the same straight line are also parallel to one another.

PROPOSITION 31.

Through a given point to draw a straight line parallel to a given straight line.

Let A be the given point, and BC the given straight line; thus it is required to draw through the point A a straight line parallel to the straight line BC. [See Figure 1.6.]

Let a point D be taken at random on BC, and let AD be joined; on the straight line DA, and at the point A on it, let the angle DAE be constructed equal to the angle ADC [I. 23]; and let the straight line AF be produced in a straight line with EA.

Then, since the straight line AD falling on the two straight lines BC, EF has made the alternate angles EAD, ADC equal to one another, therefore EAF is parallel to BC [I. 27].

Therefore, through the given point A the straight line EAF has been drawn parallel to the given straight line BC.

In the proof, when Euclid makes use of a previously proven proposition, he refers to it in brackets.

Notice that Proposition 31 does not depend on the parallel postulate. In wondering why Euclid did not therefore state and prove it earlier, when he was proving everything he possibly could without the postulate, Proclus suggests the following [51, Vol. I, p. 316]. Proposition 31 implies that there is only one straight line through A parallel to BC, but does not actually prove this. However, this fact will follow from Proposition 30 (whose proof, which we have omitted, does require the parallel postulate). See Exercises 1.11 and 1.12.

Now we are ready for Euclid's theorem on the angle sum of triangles.

FIGURE 1.6. Proposition 31.

[3] The full statement of Proposition 27 is "If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another."

FIGURE 1.7. Proposition 32.

PROPOSITION 32.

In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.

Let ABC be a triangle, and let one side of it BC be produced to D [see Figure 1.7]

I say that the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC, and the three interior angles of the triangle ABC, BCA, CAB are equal to two right angles.

For let CE be drawn through the point C parallel to the straight line AB [I. 31].

Then, since AB is parallel to CE, and AC has fallen upon them, the alternate angles BAC, ACE are equal to one another [I. 29].

Again, since AB is parallel to CE, and the straight line BD has fallen upon them, the exterior angle ECD is equal to the interior and opposite angle ABC [I. 29].

But the angle ACE was also proved equal to the angle BAC;

therefore, the whole angle ACD is equal to the two interior and opposite angles BAC, ABC.

Let the angle ACB be added to each;

therefore, the angles ACD, ACB are equal to the three angles ABC, BCA, CAB.

But the angles ACD, ACB, are equal to two right angles [I.13];

therefore, the angles ABC, BCA, CAB are also equal to two right angles.

...

Euclid has proved, using the parallel postulate, that the angle sum in a triangle is always two right angles. In fact, as we discussed in the Introduction, this property of triangles is equivalent to the parallel postulate; i.e., one can also prove the converse implication, that if the angle sum is assumed to be two right angles, then the parallel postulate follows (without assuming it in advance). Thus proving the parallel postulate is equivalent to proving the angle sum theorem. Many later efforts at proving the parallel postulate homed in on precisely this approach, as we will see in our next source.

Exercise 1.8: Prove Proposition 15: "If two straight lines cut one another, they make the vertical angles equal to one another."

FIGURE 1.8. Proposition 15.

In Figure 1.8, the angles AED and CEB form one pair of vertical angles, and the angles AEC and BED form another. (The term "vertical" refers to the shared vertex E; as Heath notes in his commentary in [51, Vol. I, p. 278], "vertically opposite" is clearer.) Then compare your proof with Euclid's.

Exercise 1.9: Prove Proposition 29. Along the way you may find that you prove some other propositions of Euclid's (see, for example, Exercise 1.8). Then compare your proof with Euclid's.

Exercise 1.10: Prove Proposition 27: "If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another." You should not need to use the parallel postulate. Then compare your proof with Euclid's.

Exercise 1.11: Prove Proposition 30. Then compare your proof with Euclid's.

Exercise 1.12: Use Proposition 30 to show that the straight line one draws in Proposition 31 (through a given point, parallel to a given line) must be unique.

1.3 Legendre's Attempts to Prove the Parallel Postulate

Adrien-Marie Legendre was born to a well-to-do family, and received an excellent education at schools in Paris. His family's modest fortune allowed him to devote himself entirely to research, although he did teach mathematics for a time at the Ecole Militaire in Paris. In 1782 Legendre won a prize from the Berlin Academy for his essay on the subject "Determine the curve described by cannonballs and bombs, taking into consideration the resistance of the air; give rules for obtaining the ranges corresponding to different initial velocities and to different angles of projection." Then, as today, political and military considerations had great influence on the directions of mathematical research. This prize, along with work on celestial mechanics, gained Legendre election to the French Academy of Sciences in 1783, and his scientific output continued to grow. Legendre's favorite areas of research were celestial mechanics, number theory (see the number theory chapter), and the theory of elliptic functions, of which he should be considered the founder.

From 1787 Legendre was heavily involved in the Academy's work with the geodetic research of the Paris and Greenwich observatories, such as the linking of their meridians, and this stimulated him to work on spherical geometry, where he is known for his theorem on spherical triangles. Thus