mere points for other functions to be defined on, a metalevel analysis with applications in quantum physics.

At the close of the twentieth century, one of the hottest new fields in analysis is "wavelet theory," emerging from such applications as edge detection or texture analysis in computer vision, data compression in signal analysis or image processing, turbulence, layering of underground earth sediments, and computer-aided design. Wavelets are an extension of Fourier's idea of representing functions by superimposing waves given by sines or cosines. Since many oscillatory phenomena evolve in an unpredictable way over short intervals of time or space, the phenomenon is often better represented by superimposing waves of only short duration, christened wavelets. This tight interplay between current applications and a new field of mathematics is evolving so quickly that it is hard to see where it will lead even in the very near future [92].

We will conclude this chapter with an extraordinary modern twist to our long story. Recall that the infinitesimals of Leibniz, which had never been properly defined and were denigrated as fictional, had finally been banished from analysis by the successors of Cauchy in the nineteenth century, using a rigorous foundation for the real numbers. How surprising, then, that in 1960 deep methods of modern mathematical logic revived infinitesimals and gave them a new stature and role. In our final section we will read a few passages from the book *Non-Standard Analysis* [140] by Abraham Robinson (1918–1974), who discovered how to place infinitesimals on a firm foundation, and we will consider the possible consequences of his discovery for the future as well as for our evaluation of the past.

Exercise 3.1: Prove Hippocrates' theorem on the squaring of his lune.

Exercise 3.2: Research the history and eventual resolution of one of the three "classical problems" of antiquity.

Exercise 3.3: Find out what the "quadratrix of Hippias" is and how it was used in attempts to solve the problems of squaring the circle and trisecting the angle.

Exercise 3.4: Study Kepler's derivation [20, pp. 356 f.] of the area inside a circle. Critique his use of indivisibles. What are its strengths and weaknesses? Also study his matching of indivisibles to obtain the area inside an ellipse [20, pp. 356 f.]. Do you consider his argument valid? Why?

3.2 Archimedes' Quadrature of the Parabola

Archimedes (c. 287–212 B.C.E.) was the greatest mathematician of antiquity, and one of the top handful of all time. His achievements seem astounding even today. The son of an astronomer, he spent most of his life in Syracuse, on the island of Sicily, in present-day southern Italy, except for a likely period in Alexandria studying with successors of Euclid. In addition to spectacular mathematical achievements, his reputation during his lifetime derived from an impressive array of mechanical inventions, from the water snail (a screw for raising irrigation water) to compound pulleys, and fearful war instruments described in the Introduction. Referring to his principle of the lever, Archimedes boasted, "Give me a place to stand on, and I will move the earth." When King Hieron of Syracuse heard of this and asked Archimedes to demonstrate his principle, he demonstrated the efficacy of his pulley systems based on this law by easily pulling single-handedly a three-masted schooner laden with passengers and freight [93]. One of his most famous, but possibly apocryphal, exploits was to determine for the king whether a goldsmith had fraudulently alloyed a supposedly gold crown with cheaper metal. He is purported to have realized, while in a public bath, the principle that his floating body displaced exactly its weight in water, and, realizing that he could use this to solve the problem, rushed home naked through the streets shouting "Eureka, Eureka" (I have found it!).

The treatises of Archimedes contain a wide array of area, volume, and center of gravity determinations, including virtually all the best-known formulas taught in high school today. As mentioned in the Introduction, he was so pleased with his results about the sphere that he had one of

PHOTO 3.3. Archimedes.

them inscribed on his gravestone: The volume of a sphere is two-thirds that of the circumscribed cylinder, and astonishingly, the same ratio holds true for their surface areas. It was typical at the time to compare two different geometric objects in this fashion, rather than using formulas as we do today. Archimedes also laid the mathematical foundation for the fields of statics and hydrodynamics and their interplay with geometry, and frequently used intricate balancing arguments. A fascinating treatise of his on a different topic is *The Sandreckoner*, in which he numbered the grains of sand needed to fill the universe (i.e., a sphere with radius the estimated distance to the sun), by developing an effective system for dealing with large numbers. Even though he calculated in the end that only 10^{63} grains would be needed, his system could actually calculate with numbers as enormous as $\left((10^8)^{10^8}\right)^{10^8}$. Archimedes even modeled the universe with a mechanical planetarium incorporating the motions of the sun, the moon, and the "five stars which are called the wanderers" (i.e., the known planets) [42].

We will examine two remarkably different texts Archimedes wrote on finding the area of a "segment" of a parabola. A *segment* is the region bounded by a parabola and an arbitrary line cutting across the parabola. The portion of the cutting line between the two intersection points is called a chord, and forms what Archimedes calls the base of the segment. He states his beautiful result in a letter to Dositheus, a successor of Euclid's in Alexandria, prefacing his treatise *Quadrature of the Parabola* [3, pp. 233–34].

ARCHIMEDES to Dositheus greeting.

When I heard that Conon,[2] who was my friend in his lifetime, was dead, but that you were acquainted with Conon and withal versed in geometry, while I grieved for the loss not only of a friend but of an admirable mathematician, I set myself the task of communicating to you, as I had intended to send to Conon, a certain geometrical theorem which had not been investigated before but has now been investigated by me, and which I first discovered by means of mechanics and then exhibited by means of geometry. Now some of the earlier geometers tried to prove it possible to find a rectilineal area equal to a given circle and a given segment of a circle.... But I am not aware that any one of my predecessors has attempted to square the segment bounded by a straight line and a section of a right-angled cone [a parabola], of which problem I have now discovered the solution. For it is here shown that every segment bounded by a straight line and a section of a right-angled cone [a parabola] is four-thirds of the triangle which has the same base and equal height with the segment, and for the demonstration of this property the following lemma is assumed: that the excess by which the greater of (two) unequal areas exceeds the less can,

[2]Another successor of Euclid's.

by being added to itself, be made to exceed any given finite area. The earlier geometers have also used this lemma; for it is by the use of this same lemma that they have shown that circles are to one another in the duplicate ratio of their diameters, and that spheres are to one another in the triplicate ratio of their diameters, and further that every pyramid is one third part of the prism which has the same base with the pyramid and equal height; also, that every cone is one third part of the cylinder having the same base as the cone and equal height they proved by assuming a certain lemma similar to that aforesaid. And, in the result, each of the aforesaid theorems has been accepted no less than those proved without the lemma. As therefore my work now published has satisfied the same test as the propositions referred to, I have written out the proof and send it to you, first as investigated by means of mechanics, and afterwards too as demonstrated by geometry. Prefixed are, also, the elementary propositions in conics which are of service in the proof. Farewell.

Archimedes provides several points of view in demonstrating his result. In *Quadrature of the Parabola* he gives two proofs representing formal Greek methods, while yielding little insight into how the result might have been discovered. In the second, which he calls "geometrical," we will see the method of exhaustion in action. While Archimedes develops and uses many beautiful and fascinating features of parabolas in order to prove his result, most of which are unfamiliar to us today, we will focus primarily on how he combines these with the method of exhaustion, encouraging the reader to explore the geometric underpinnings.

Our other text, *The Method*, will reveal how Archimedes actually discovered his result by an imaginary physical balancing technique using indivisibles. While he considered this only heuristic, and not acceptable as proof, we shall see that it foreshadows later methods of the calculus by about two thousand years.

A parabola is an important curve in part because it can be described by a number of equivalent but very different properties, each simple and aesthetically pleasing. This reflects the fact that parabolas arise in many mathematical and physical situations. While the reader is probably familiar with a parabola in some form, it is surprising how different our typical view of it is today from that of two thousand years ago.

Parabolas are one of three types of curves (along with hyperbolas and ellipses) first studied as certain cross-sections created by a plane slicing through a cone, hence the name "conic sections" for these curves. While their discovery is attributed to the geometer Menaechmus around 350 B.C.E., we do not know how the relationship between their purely planar properties and their description as conic sections was discovered [43, pp. 56–57].

Greek mathematicians derived a planar "symptom" for each parabola [43, pp. 57 f.], a characteristic relation between the coordinates of any point on the curve, using measurements along a pair of coordinate directions (not necessarily mutually perpendicular!) to describe the positions of points. In modern algebraic symbolism, the symptom becomes an equation for the curve. It is fascinating to study how this was probably done for the various conic sections [43, pp. 57 f.] (Exercise 3.5). Of course, we know that in an appropriate modern (perpendicular) Cartesian coordinate system, the equation is simply $py = x^2$ (p a constant depending on how far from the vertex we slice the cone). One of the astonishing things Archimedes knew is that many features of a parabola, including its symptom, hold for oblique axes as well (Exercises 3.6, 3.7) [43, pp. 57 f.]. This offers a hint at the more modern subject of affine geometry [34].

We are ready to read selections from Archimedes' treatise *Quadrature of the Parabola* [3, pp. 233–37, 246–52]. The approach of the proof is to inscribe polygons inside the parabolic segment to approximate its area, and then use the method of exhaustion to confirm an exact, not merely approximate, value for its area. By a "tangent" to a parabola, Archimedes means a line touching it in exactly one point. He assumes that each point on a parabola has exactly one tangent line containing it (Exercise 3.8). He uses the word "diameter" to refer to any line parallel to the axis of symmetry of the parabola, and "ordinate" to refer to coordinate measurement along the oblique coordinate in the direction parallel to the tangent.

Archimedes, from
Quadrature of the Parabola

Definition. In segments bounded by a straight line and any curve I call the straight line the base, and the height the greatest perpendicular drawn from the curve to the base of the segment, and the vertex the point from which the greatest perpendicular is drawn.

Proposition 20.

If Qq be the base, and P the vertex, of a parabolic segment, then the triangle PQq is greater than half the segment PQq. [See Figure 3.2]

For the chord Qq is parallel to the tangent[3] at P, and the triangle PQq is half the parallelogram formed by Qq, the tangent at P, and the diameters through Q, q.

FIGURE 3.2. Proposition 20.

[3] Read Propositions 1 and 18 of Archimedes' treatise [3], and follow Exercise 3.9, to see why this beautiful fact holds.

Therefore the triangle PQq is greater than half the segment.

Cor. It follows that *it is possible to inscribe in the segment a polygon such that the segments left over are together less than any assigned area.*

This corollary refers to one of the cornerstones of the method of exhaustion; we will postpone discussing it until Archimedes elaborates in Proposition 24.

Proposition 22.

If there be a series of areas A, B, C, D, ... each of which is four times the next in order, and if the largest, A, be equal to the triangle PQq inscribed in a parabolic segment PQq and having the same base with it and equal height, then

$$(A + B + C + D + \cdots) < (\textit{area of segment } PQq)\,.$$

For, since $\Delta PQq = 8\Delta PRQ = 8\Delta Pqr$ (see Figure 3.3), where R, r are the vertices of the segments cut off by PQ, Pq, then as in the last proposition,[4]

$$\Delta PQq = 4\,(\Delta PQR + \Delta Pqr)\,.$$

Therefore, since $\Delta PQq = A$,

$$\Delta PQR + \Delta Pqr = B.$$

In like manner we prove that the triangles similarly inscribed in the remaining segments are together equal to the area C, and so on.

Therefore $A + B + C + D + \cdots$ is equal to the area of a certain inscribed polygon, and is therefore less than the area of the segment.

Proposition 23.

Given a series of areas A, B, C, D, ..., Z, of which A is the greatest, and each is equal to four times the next in order, then

$$A + B + C + \cdots + Z + \tfrac{1}{3}Z = \tfrac{4}{3}A.$$

We ask the reader to prove this algebraic result (Exercise 3.11).

The finale now combines Propositions 20, 22, and 23 to prove the theorem.

Proposition 24.

Every segment bounded by a parabola and a chord Qq is equal to four-thirds of the triangle which has the same base as the segment and equal height.

FIGURE 3.3. Proposition 22.

[4]Study Propositions 3, 19, and 21.

FIGURE 3.4. Proposition 24.

Suppose

$$K = \tfrac{4}{3}\Delta PQq,$$

where P is the vertex of the segment; and we have then to prove that the area of the segment is equal to K. [See Figure 3.4]

For, if the segment be not equal to K, it must either be greater or less.

I. Suppose the area of the segment greater than K.

If then we inscribe in the segments cut off by PQ, Pq triangles which have the same base and equal height, i.e., triangles with the same vertices R, r as those of the segments, and if in the remaining segments we inscribe triangles in the same manner, and so on, we shall finally have segments remaining whose sum is less than the area by which the segment PQq exceeds K.

Therefore the polygon so formed must be greater than the area K; which is impossible, since [Prop. 23]

$$A + B + C + \cdots + Z < \tfrac{4}{3}A,$$

where

$$A = \Delta PQq.$$

Thus the area of the segment cannot be greater than K.

II. Suppose, if possible, that the area of the segment is less than K.

If then $\Delta PQq = A$, $B = \frac{1}{4}A$, $C = \frac{1}{4}B$, and so on, until we arrive at an area X such that X is less than the difference between K and the segment, we have

$$\begin{aligned} A + B + C + \cdots + X + \tfrac{1}{3}X &= \tfrac{4}{3}A \qquad \text{[Prop. 23]} \\ &= K. \end{aligned}$$

Now, since K exceeds $A + B + C + \cdots + X$ by an area less than X, and the area of the segment by an area greater than X, it follows that

$$A + B + C + \cdots + X > \text{ (the segment)};$$

which is impossible, by Prop. 22 above.

Hence the segment is not less than K.

Thus, since the segment is neither greater nor less than K,

$$\text{(area of segment } PQq) = K = \tfrac{4}{3}\Delta PQq.$$

Early in the proof, Archimedes asserts that if the (area of the) segment does not equal K, it must be either greater or less, and he proceeds to rule out both these possibilities. This was a very common method of proof in his time, known now as double *reductio ad absurdum*. (What does this Latin phrase mean, and how does it describe the method?)

In parts I and II, polygons of computable area are inscribed that are sufficiently close in area to the segment to provide the needed contradictions to the possibility that the area of the segment could be greater or less than K, even though sufficiently close is necessarily of an arbitrary, unknown nature. This is the essence of the method of exhaustion; the area is being exhausted by the polygons. This is really an inappropriate term, though, since any polygon necessarily leaves some area unfilled; the method was designed precisely to get around this problem without confronting the completing of the filling process. Comprehending the connection between the infinite filling process and the area of the entire segment became the goal of a two thousand year struggle, only resolved in the nineteenth century.

How does Archimedes know that his process of including ever more triangles will ultimately provide a polygon differing in area from the segment by as little as needed? This is the Corollary he noted after Proposition 20. How does he know that Proposition 20 means that the area left over can be made as small as needed, in his words "less than any assigned area"? We find the answer by rereading Archimedes' introductory letter!

He tells Dositheus that "the following lemma is assumed" and also "a certain lemma similar to the aforesaid"; and that these lemmas were used by "earlier geometers" to prove their results on areas of circles and volumes of spheres and cones. The second lemma he refers to here had been stated by Euclid in *Elements* [51, Vol. 3, p. 14], as Proposition 1 in Book X:

> Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out.

Clearly, this is exactly what Archimedes uses to obtain his Corollary to Proposition 20 and thus his claim in Proposition 24. Intuitively, it says that by successively halving a magnitude, we can eventually, but always after some finite number of halvings, make it smaller than some previously assigned magnitude. The previously assigned magnitude, although perhaps small and unknown, is fixed, and this is what makes it possible to eventually undershoot it, since it does not recede as the halving process progresses.

This assumed lemma on the part of Archimedes is called the "dichotomy principle." While it may seem quite reasonable, this is deceptive, since its great power actually amounts to ruling out the existence of infinitesimally small magnitudes, i.e., those that are smaller than any fraction of a unit and yet still not zero. The dichotomy principle was an assumption about the fundamental nature of magnitudes, which Archimedes could not prove; it was only set on a firm foundation much later, when a truly satisfactory definition for real numbers was discovered (Exercise 3.12).

To summarize, Archimedes combined the exhaustion method with a deep geometric understanding of parabolas and a clever summation of a series of terms with identical successive ratios (today called a "geometric" series; see the chapter Appendix), to demonstrate that the exact area of a parabolic segment is four-thirds of a certain triangle (Exercises 3.13, 3.14). His presentation is an example of "synthesis," in which simple pieces are systematically put together to yield something more complex. It leaves us puzzled, though, about how Archimedes discovered his result. Our second text will actually reveal the answer to this riddle.

Exercise 3.5: Derive the symptom (equation) of the parabola using modern algebraic notation, and compare with the spirit of how the ancients may have done it [43, pp. 57 f.].

Exercise 3.6: Learn about the Greek method of "application of areas," and explain how Apollonius used it in enlarging the notion of conic sections [20, 127, 173][43, pp. 51 f., 61 f.]. Explain how this accounts for the terms parabola, hyperbola, ellipse.

Exercise 3.7: Learn about how Apollonius expanded the idea of conic sections to cutting a right-angled cone at any angle and to oblique cones, and still (surprisingly) obtained the same curves as before [43, pp. 59 f.]. Describe the details in your own words using modern notation.

Exercise 3.8: Show that each point on a parabola has exactly one tangent line. Use just the symptom. Any use of calculus (e.g., a derivative) would of course be cheating by a couple of millennia. Hint: Archimedes' Proposition 2 from *Quadrature of the Parabola* [3] should suggest which line to focus your attention on.

Exercise 3.9: Prove Propositions 1, 2, and 3 of Archimedes' treatise *Quadrature of the Parabola* [3]. We will outline one path to doing so, from a modern point of view, although you may wish to find your own.

- From Exercise 3.8 we have a complete description of the tangent line to a parabola at any point.
- From the equation $py = x^2$ for the symptom of a parabola, verify Proposition 3 directly algebraically. Go one step further to obtain the equation for the symptom of the parabola using the new oblique coordinates. Notice how the constant p changes in the new coordinates, and how this change depends on the angle of the relevant tangent line for the new coordinates. You might wish to do this part by formally introducing the new coordinates, seeing how to convert between old and new (this particular type of change is called an "affine" change of coordinates), and then transforming the equation for the symptom into the new coordinates.
- Now either prove the general oblique form of Propositions 1 and 2 by using Proposition 3 and some algebra, or carefully create an argument that because the symptom is true for both oblique and right-angled coordinates from Proposition 3, the claims of Propositions 1 and 2 will follow for oblique axes, since we already know they are true for the standard axes.

Exercise 3.10: Consider the situation of Proposition 2 from *Quadrature of the Parabola* [3], but using only standard right-angled coordinate axes. The length of TV is called the "subtangent" because it is the length of the portion of the axis corresponding to that portion of the tangent line from the point of tangency to the axis. We have seen that the length of the subtangent is twice the coordinate PV. Another important line is the "normal," the line at a right angle to the tangent line at Q. Then the "subnormal" is defined by analogy to the subtangent. Find the subnormal. What does the subnormal surprisingly *not* depend on?

Exercise 3.11:

- Derive your own algebraic proof of Proposition 23. Generalize this to any series with arbitrary ratio between the terms, rather than always four to one.
- Archimedes seems to have proven Proposition 23 only for a sum of twenty-six areas $A, \ldots, Z$. Discuss. How could we phrase this more generally today? Why didn't he do so?
- Can you obtain from Proposition 23 the value of a certain infinite sum? Archimedes' method of proof is designed to avoid actually doing this, since infinite sums were considered unrigorous.

Exercise 3.12: The dichotomy principle has an equivalent form as the first lemma in Archimedes' letter, asserting nonexistence of infinitely large magnitudes, which today is called the postulate of Eudoxus. Study it and its use [84, 85], and show that the dichotomy principle and the postulate of Eudoxus are equivalent (Hint: Look at Euclid, Book X, Proposition 1 for help.)

Exercise 3.13: Compare and contrast Archimedes' proof of his result on the area of a circle [3, pp. 91 f.] with the exhaustion proof he has given for the area of a segment of a parabola. Note where he uses yet another "assumption" in that proof, namely Assumption 6 in *On the Sphere and Cylinder, I* [3, p. 4]. Discuss why he must make this assumption, and what would be needed to elevate this assumption to something one could actually prove.

Exercise 3.14: Archimedes proves that the area of a segment of a parabola is four-thirds that of a certain triangle. But this is of limited usefulness unless one can determine the dimensions of the triangle for a given segment, in order to compute its area. Show how this can always be done.

3.3 Archimedes' Method

Our second source comes from *The Method of Treating Mechanical Problems*, an extraordinary manuscript with an astonishing history. While it