Exercise 3.10: Consider the situation of Proposition 2 from *Quadrature of the Parabola* [3], but using only standard right-angled coordinate axes. The length of TV is called the "subtangent" because it is the length of the portion of the axis corresponding to that portion of the tangent line from the point of tangency to the axis. We have seen that the length of the subtangent is twice the coordinate PV. Another important line is the "normal," the line at a right angle to the tangent line at Q. Then the "subnormal" is defined by analogy to the subtangent. Find the subnormal. What does the subnormal surprisingly *not* depend on?

Exercise 3.11:

- Derive your own algebraic proof of Proposition 23. Generalize this to any series with arbitrary ratio between the terms, rather than always four to one.
- Archimedes seems to have proven Proposition 23 only for a sum of twenty-six areas $A, \ldots, Z$. Discuss. How could we phrase this more generally today? Why didn't he do so?
- Can you obtain from Proposition 23 the value of a certain infinite sum? Archimedes' method of proof is designed to avoid actually doing this, since infinite sums were considered unrigorous.

Exercise 3.12: The dichotomy principle has an equivalent form as the first lemma in Archimedes' letter, asserting nonexistence of infinitely large magnitudes, which today is called the postulate of Eudoxus. Study it and its use [84, 85], and show that the dichotomy principle and the postulate of Eudoxus are equivalent (Hint: Look at Euclid, Book X, Proposition 1 for help.)

Exercise 3.13: Compare and contrast Archimedes' proof of his result on the area of a circle [3, pp. 91 f.] with the exhaustion proof he has given for the area of a segment of a parabola. Note where he uses yet another "assumption" in that proof, namely Assumption 6 in *On the Sphere and Cylinder, I* [3, p. 4]. Discuss why he must make this assumption, and what would be needed to elevate this assumption to something one could actually prove.

Exercise 3.14: Archimedes proves that the area of a segment of a parabola is four-thirds that of a certain triangle. But this is of limited usefulness unless one can determine the dimensions of the triangle for a given segment, in order to compute its area. Show how this can always be done.

3.3 Archimedes' Method

Our second source comes from *The Method of Treating Mechanical Problems*, an extraordinary manuscript with an astonishing history. While it was alluded to in ancient commentaries, it was thought nonexistent or irretrievably lost until its remarkable rediscovery by the Danish scholar J. L. Heiberg at the turn of the twentieth century [43, pp. 44 f.], [3, Supplement].

Heiberg's attention was drawn to an 1899 report about a palimpsest with originally mathematical contents in the library of the monastery of the Holy Sepulchre, in Jerusalem. A palimpsest is a parchment, tablet, or other surface, that has been written on more than once, with the previous text(s) imperfectly erased, and therefore still visible. Of course, in times past, before paper was plentiful, this was a common practice. A few lines of erased text quoted in the report convinced Heiberg that the underlying text was by Archimedes, and he was able to examine and photograph the parchment in Constantinople. It contained an Archimedean text written in a tenth century hand, and an imperfect attempt had been made to wash it off and replace it with a religious text in the twelfth to fourteenth century. Heiberg succeeded in deciphering most of the underlying manuscript, which contains versions of previously known works by Archimedes, and the almost complete text of the long lost *Method*. It is interesting that this only extant copy was first preserved by religious efforts, then obliterated for another religious purpose, and now through an incredible stroke of luck is available to us today.[5]

The significance of the rediscovered manuscript is explained by two distinguished scholars.

E.J. Dijksterhuis writes:

> Greek mathematics is characterized—and in this respect, too, it founded a tradition which was to last down to our own time—by a care of the form of the mathematical argument which, superficially viewed, seems almost exaggerated. It demands the inexorably proceeding, irrefutably persuading sequence of logical conclusions constituting the synthetic method of demonstration, but to this it sacrifices the reader's wish to gain also an insight into the method by which the result was first discovered. It is this wish, however, which Archimedes meets in his *Method:* he will reveal how he himself, long before he knew how to prove his theorems, became convinced of their truth [43, p. 315].

And T.L. Heath says:

> [H]ere we have a sort of lifting of the veil, a glimpse of the interior of Archimedes' workshop as it were. He tells us how he discovered certain theorems in quadrature and cubature, and he is at the same time careful to insist on the difference between (1) the means which

[5] As this book goes to print in October, 1998, the palimpsest has reappeared after almost a century in the hands of private collectors, and is being auctioned at Christie's in New York for about one million dollars.

may be sufficient to suggest the truth of theorems, although not furnishing scientific proofs of them, and (2) the rigorous demonstrations of them by irrefragable geometrical methods which must follow before they can be finally accepted as established; to use Archimedes' own terms, the former enable theorems to be *investigated* but not be *proved* [3, Supplement, pp. 6–7].

Archimedes' explanation of how he discovered the area of a parabolic segment involves two ideas very different from Eudoxean exhaustion: a "mechanical balancing" method, and "summation of indivisibles." Let us begin with Archimedes' prefatory letter:

Archimedes to Eratosthenes[6] greeting.

I sent you on a former occasion some of the theorems discovered by me, merely writing out the enunciations and inviting you to discover the proofs, which at the moment I did not give. The enunciations of the theorems which I sent were as follows....

Seeing moreover in you, as I say, an earnest student, a man of considerable eminence in philosophy, and an admirer [of mathematical inquiry], I thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge. This is a reason why, in the case of the theorems the proof of which Eudoxus was the first to discover, namely that the cone is a third part of the cylinder, and the pyramid of the prism, having the same base and equal height, we should give no small share of the credit to Democritus, who was the first to make the assertion with regard to the said figure though he did not prove it. I am myself in the position of having first made the discovery of the theorem now to be published [by the method indicated], and I deem it necessary to expound the method partly because I have already spoken of it and I do not want to be thought to have uttered vain words, but equally because I am persuaded that it will be of no little service

[6]Eratosthenes was a great scholar in many fields, including mathematics and astronomy. He made the most accurate calculation in antiquity of the circumference of the earth, based on measurements of the sun's shadow at different latitudes. After many years in Athens, Eratosthenes was appointed chief librarian at the Museum/University in Alexandria, and it was to him that Archimedes sent *The Method*.

to mathematics; for I apprehend that some, either of my contemporaries or of my successors, will, by means of the method when once established, be able to discover other theorems in addition, which have not yet occurred to me.

First then I will set out the very first theorem which became known to me by means of mechanics, namely that

Any segment of a section of a right-angled cone (i.e., a parabola) is four-thirds of the triangle which has the same base and equal height, and after this I will give each of the other theorems investigated by the same method. Then, at the end of the book, I will give the geometrical [proofs of the propositions].

Archimedes proceeds to Proposition 1 of *The Method*, his parabolic area result.

The Method makes heavy use of balancing principles from mechanics (the movement of bodies under the action of forces). Here Archimedes leaves pure mathematics and appeals to natural science considered mathematically, claiming to draw mathematical conclusions from the physical science of weights, centers of gravity, and equilibria. He was the first to establish a systematic connection between mathematics and mechanics, developing much of that portion of mechanics today called statics. From intuitively acceptable postulates, he developed a variety of results on static equilibria, most importantly the

Principle of the Lever

> Two magnitudes balance at distances reciprocally proportional to the magnitudes [3, *On the Equilibrium of Planes*, Book I, Propositions 6,7].

To apply this principle we must understand how to interpret the appropriate point of positioning for an arbitrary geometric magnitude, called its "center of gravity" by Archimedes; intuitively, it is the point upon which the magnitude itself will balance in equilibrium. The foundational issues involved in properly interpreting this idea are intricate and fascinating (Exercise 3.15). It is not hard to believe from symmetry that the center of gravity of a line segment is the midpoint of the segment, or that the center of gravity of a rectangle is the point where the diagonals intersect. But what about a triangle? This is the type of object Archimedes studies, and he proves (Book I, Proposition 13) that its center of gravity is the point where the median lines intersect (recall that a median connects a vertex to the midpoint of the opposite side). Archimedes mentions in Book I, Proposition 15, that this point is two-thirds of the way along each median (Exercise 3.16). Now we are ready for *The Method*.

Archimedes, from
The Method of Treating of Mechanical Problems

Proposition 1.

Let ABC be a segment of a parabola bounded by the straight line AC and the parabola ABC, and let D be the middle point of AC. [See Fig. 3.5] Draw the straight line DBE parallel to the axis of the parabola and join AB, BC.

FIGURE 3.5. Proposition 1.

Then shall the segment ABC be $\frac{4}{3}$ of the triangle ABC.

From A draw AKF parallel to DE, and let the tangent to the parabola at C meet DBE in E and AKF in F. Produce CB to meet AF in K, and again produce CK to H, making KH equal to CK.

Consider CH as the bar of a balance, K being its middle point.

Let MO be any straight line parallel to ED, and let it meet CF, CK, AC in M, N, O and the curve in P.

Now, since CE is a tangent to the parabola and CD the semi-ordinate,

$$EB = BD;^{7}$$

"for this is proved in the Elements [of Conics]*."

Since FA, MO are parallel to ED, it follows that

$$FK = KA, \; MN = NO.$$

Now, by the property of the parabola, "proved in a lemma,"

$$\begin{aligned} MO : OP &= CA : AO && \text{[cf. } \textit{Quadrature of Parabola}\text{, Prop. 5]}^{8} \\ &= CK : KN && \text{[Eucl. VI. 2]} \\ &= HK : KN. \end{aligned}$$

Take a straight line TG equal to OP, and place it with its center of gravity at H, so that $TH = HG$; then, since N is the center of gravity of the straight line MO,

and

$$MO : TG = HK : KN,$$

it follows that TG at H and MO at N will be in equilibrium about K.

[*On the Equilibrium of Planes*, I. 6, 7]

Similarly, for all other straight lines parallel to DE and meeting the arc of the parabola, (1) the portion intercepted between FC, AC with its middle point in KC and (2) a length equal to the intercept between the curve and AC placed with its center of gravity at H will be in equilibrium about K.

Therefore K is the center of gravity of the whole system consisting (1) of all the straight lines as MO intercepted between FC, AC and placed as they actually are in the figure and (2) of all the straight lines placed at H equal to the straight lines as PO intercepted between the curve and AC.

And, since the triangle CFA is made up of all the parallel lines like MO, and the segment CBA is made up of all the straight lines like PO within the curve,

---

[7] Read Propositions 1 and 2 in *Quadrature of the Parabola* to see why this is true.

*I.e., the works on conics by Aristaeus and Euclid....

[8] This surprising result is explored in Exercise 3.17.

it follows that the triangle, placed where it is in the figure, is in equilibrium about K with the segment CBA placed with its center of gravity at H.

Divide KC at W so that $CK = 3KW$;
then W is the center of gravity of the triangle ACF; "for this is proved in the books on equilibrium."

[Cf. *On the Equilibrium of Planes* I. 15]

$$
\begin{array}{lcl}
\text{Therefore } \Delta ACF : (\text{segment } ABC) & = & HK : KW \\
 & = & 3:1. \\
\text{Therefore segment } ABC & = & \frac{1}{3}\Delta ACF. \\
\text{But } \Delta ACF & = & 4\Delta ABC. \\
\text{Therefore segment } ABC & = & \frac{4}{3}\Delta ABC.
\end{array}
$$

Now the fact here stated is not actually demonstrated by the argument used; but that argument has given a sort of indication that the conclusion is true. Seeing then that the theorem is not demonstrated, but at the same time suspecting that the conclusion is true, we shall have recourse to the geometrical demonstration which I myself discovered and have already published.

Archimedes' "mechanical" argument is beautiful in its simplicity, and displays the mark of a genius, expert at the principle of the lever and centers of gravity. He was clearly comfortable with the idea that a region is made up of all its parallel lines, and his *Method* depends precisely on the delicate interplay between the summation of indivisibles and the application of the Principle of the Lever to both the indivisibles and the regions.

Archimedes viewed his method only as a heuristic discovery tool, reverting to geometric Eudoxean exhaustion when he published a rigorous proof. What remains tantalizingly unclear even today is whether it is the mechanical (balancing) or indivisible aspect of the method that taints it as a method of rigorous proof by classical Greek standards. Modern scholars provide compelling arguments and evidence in several directions on these questions [43, pp. 319 ff.], [99]; this is highly recommended reading (Exercise 3.18).

With hindsight, the challenge of making *The Method* rigorous lies not so much in mathematicizing the apparent appeal to physics in the mechanical aspect of the treatise, for this is easily done. Rather, the use of indivisibles is the sticky point. Greek mathematics had struggled with this in the broader context of the infinite for several centuries, and it would be many more centuries before these questions would be tackled on a new level (Exercise 3.19).

Exercise 3.15: Investigate the controversy over the assumptions Archimedes makes in developing and using his "Principle of the Lever" [43, pp. 286 ff.].

Exercise 3.16: Prove that the medians of a triangle intersect two-thirds of the way along each median.

Exercise 3.17: Learn about Proposition 5 of *The Method* [3], and prove it directly from Propositions 1, 2, 3 of *Quadrature of the Parabola* [3]. Does modern coordinate terminology help?

Exercise 3.18: Investigate the controversy today over Archimedes' use of mechanical and indivisible methods.

Exercise 3.19: The Fundamental Theorem of Calculus, whose seventeenth-century proof by Leibniz we will study later, enables one to calculate the area "under" a portion of a parabola. Considering the parabola $y = kx^2$ and the region bounded by this curve, the x-axis, and the verticals $x = a$ and $x = b$, the Fundamental Theorem will yield the result $k(b^3 - a^3)/3$ for its area. Obtain this value directly from Archimedes' theorem *Quadrature of the Parabola*, without using Leibniz's Fundamental Theorem of Calculus.

3.4 Cavalieri Calculates Areas of Higher Parabolas

Bonaventura Cavalieri was one of several early-seventeenth-century mathematicians who developed their own methods of using indivisibles, combined with emerging algebraic knowledge, to calculate many new areas and volumes.

As a boy Cavalieri entered the Gesuati religious order (not to be confused with the Jesuits) [42], [157, p. 106], and through a monk who had studied with Galileo he was introduced to geometry, studied at the University of Pisa, and became a lifelong disciple of Galileo's. For a number of years he was prior of the monastery in Parma, and by 1627 he announced to Galileo that he had finished a book on indivisibles, *Geometria indivisibilibus continuorum nova quadam ratione promota* (Geometry by indivisibles of the continua advanced by a new method). Galileo successfully recommended him for a professorship at the University of Bologna, saying "few, if any, since Archimedes, have delved as far and as deep into the science of geometry" [42]. "He has discovered a new method for the study of mathematical truths" [157, p. 106].

Although all the writings of Archimedes known to mathematicians of that era were based on the method of exhaustion (recall that *The Method* lay undiscovered until the twentieth century), they believed that the ancients must have had another method of discovery. Cavalieri's contemporary Evangelista Torricelli wrote:

> I should not dare affirm that this geometry of indivisibles is actually a new discovery. I should rather believe that the ancient geometricians availed themselves of this method in order to discover the more difficult theorems, although in their demonstration they may have preferred another way, either to conceal the secret of their art or to afford no occasion for criticism by invidious detractors. What-