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ON $p^a$-PURE SEQUENCES OF ABELIAN GROUPS*

John M. Irwin, Carol L. Walker, and Elbert A. Walker

1. Introduction

1. PRELIMINARIES

The concept of purity and its generalizations have been very useful in Abelian group theory. In recent years these concepts have come to play an increasingly important role in the homological aspects of the theory. This paper is an investigation of a generalization of purity that arose homologically.

In [5], L. Fuchs pointed out that the set $\text{Pext}(A,B)$ of elements of $\text{Ext}(A,B)$ represented by pure short exact sequences is a subgroup, and indeed is the subgroup of elements of infinite height, of $\text{Ext}(A,B)$. This latter fact seems remarkable, and provided the initial motivation for the investigations carried out in this paper. Consequently, one of the central considerations here is a discussion of the short exact sequences which represent the elements of $p$-height $a$ in $\text{Ext}(A,B)$ (that is, the elements in $p^a\text{Ext}(A,B)$), where $p$ is any prime and $a$ is any ordinal number. Some properties of these sequences are described in Theorems 14 to 16, partially answering a question brought up by Fuchs in [6].

It is well known that from any exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and any group $X$, there result exact sequences

$$0 \longrightarrow \text{Hom}(X,A) \longrightarrow \text{Hom}(X,B) \longrightarrow \text{Hom}(X,C) \longrightarrow \text{Ext}(X,A)$$

$$\longrightarrow \text{Ext}(X,B) \longrightarrow \text{Ext}(X,C) \longrightarrow 0,$$ and

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\[ 0 \rightarrow \text{Hom}(C,X) \rightarrow \text{Hom}(B,X) \rightarrow \text{Hom}(A,X) \rightarrow \text{Ext}(C,X) \]
\[ \rightarrow \text{Ext}(B,X) \rightarrow \text{Ext}(A,X) \rightarrow 0. \]

D. K. Harrison noted in \[7\] that if
\[ 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \]
represents an element of $\text{Pext}(C,A)$ (that is, is pure exact), then for any group $X$, the sequences
\[ 0 \rightarrow \text{Hom}(X,A) \rightarrow \text{Hom}(X,B) \rightarrow \text{Hom}(X,C) \rightarrow \text{Pext}(X,A) \]
\[ \rightarrow \text{Pext}(X,B) \rightarrow \text{Pext}(X,C) \rightarrow 0, \text{ and} \]
\[ 0 \rightarrow \text{Hom}(C,X) \rightarrow \text{Hom}(B,X) \rightarrow \text{Hom}(A,X) \rightarrow \text{Pext}(C,X) \]
\[ \rightarrow \text{Pext}(B,X) \rightarrow \text{Pext}(A,X) \rightarrow 0, \]
are exact, where the maps in these sequences are the restrictions to the appropriate subgroups of the maps in the above sequences. It is natural then to investigate the exactness of
\[ 0 \rightarrow \text{Hom}(X,A) \rightarrow \text{Hom}(X,B) \rightarrow \text{Hom}(X,C) \rightarrow p\alpha\text{Ext}(X,A) \]
\[ \rightarrow \text{p}\alpha\text{Ext}(X,B) \rightarrow \text{p}\alpha\text{Ext}(X,C) \rightarrow 0, \text{ and} \]
\[ 0 \rightarrow \text{Hom}(C,X) \rightarrow \text{Hom}(B,X) \rightarrow \text{Hom}(A,X) \rightarrow p\alpha\text{Ext}(C,X) \]
\[ \rightarrow \text{p}\alpha\text{Ext}(B,X) \rightarrow \text{p}\alpha\text{Ext}(A,X) \rightarrow 0, \]
when
\[ 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \]
represents an element of $p\alpha\text{Ext}(C,A)$. These sequences turn out to be exact when $\alpha < \omega + \omega$, as shown by Theorem 17.

The exactness of these and other natural sequences that arise, the discussion of the elements of $p\alpha\text{Ext}(A,B)$ mentioned in the previous paragraph, and the injectives and projectives for $p\alpha\text{Ext}(A,B)$ are the main concern in this paper.

2. DEFINITIONS

The following definitions and conventions will be used throughout this discussion. All groups considered will be Abelian. Two short exact sequences of groups

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\[ 0 \to A \xrightarrow{f} X \xrightarrow{g} B \to 0 \]

and
\[ 0 \to A \xrightarrow{f'} Y \xrightarrow{g'} B \to 0 \]

are said to be equivalent if there exists a homomorphism \( h : X \to Y \) with the diagram
\[
\begin{array}{ccc}
0 & \to & A \\
\downarrow & & \downarrow h \\
0 & \to & B \\
\end{array}
\]

with \( \text{id} \) commuting. Such an \( h \) is necessarily an isomorphism. We will consider \( \text{Ext}(B, A) \) as the set of equivalence classes of short exact sequences of the form
\[ 0 \to A \xrightarrow{f} X \xrightarrow{g} B \to 0 \]

with addition as defined by R. Baer [1]. See also [4], pages 289-293. When convenient, given an extension
\[ 0 \to A \xrightarrow{f} X \xrightarrow{g} B \to 0 \]

of \( A \) by \( B \) we will assume \( A \) is a subgroup of \( X \) and \( f \) is the inclusion map. Let \( G \) be a group and \( p \) a prime. For any ordinal \( \alpha \), \( p^\alpha G \) is defined inductively by letting \( p^0 G = G \), \( p^\alpha G = p(p^{\beta}G) \) if \( \beta + 1 = \alpha \), and \( p^\alpha G = \bigcap_{\beta < \alpha} p^\beta G \) if \( \alpha \) is a limit ordinal. Thus one obtains a chain \( \{ p^\alpha G \}_{\alpha} \) of subgroups of \( G \). There exists an ordinal \( \delta \) such that \( p^\alpha G = p^\delta G \) for \( \alpha \geq \delta \).

The subgroup \( p^\delta G \) is the \( p \)-divisible subgroup of \( G \), and is denoted by \( p^\infty G \). If \( \alpha \) is an ordinal or the symbol \( \infty \), a group \( P \) is \( p^\alpha \)-projective if \( p^\alpha \text{Ext}(P, A) = 0 \) for all groups \( A \), and a group \( I \) is \( p^\alpha \)-injective if \( p^\alpha \text{Ext}(B, I) = 0 \) for all groups \( B \). If \( A \) is a subgroup of \( X \), \( A \) is \( p^\alpha \)-pure in \( X \) if the exact sequence
\[ 0 \to A \to X \to X/A \to 0 \]

represents an element of \( p^\alpha \text{Ext}(X/A, A) \). The exact sequence is called a \( p^\alpha \)-pure sequence. A \( p^\alpha \)-projective resolution of a group \( B \) is an exact sequence.
\[ \ldots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow B \rightarrow 0 \]

where \( P_i \) is \( p^\alpha \)-projective, and \( \text{Im} f_{i+1} \) is \( p^\alpha \)-pure in \( P_i \). A \( p^\alpha \)-injective resolution of a group \( A \) is an exact sequence

\[ 0 \rightarrow A \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \ldots \]

where \( I_i \) is \( p^\alpha \)-injective and \( \text{Im} \ e_i \) is \( p^\alpha \)-pure in \( I_i \). It is known \([11]\) that every group \( A \) has \( p^\alpha \)-projective and \( p^\alpha \)-injective resolutions.

II. The Group \( p^n\text{Ext}(B,A) \) for Finite Ordinals \( n \)

1. A CHARACTERIZATION OF THE ELEMENTS OF \( p^n\text{Ext}(B,A) \)

   This investigation of the subgroups \( p^\alpha \text{Ext}(B,A) \) begins with the subgroups \( p^n\text{Ext}(B,A) \), \( n \) finite, although a few facts concerning \( p^\alpha \text{Ext}(B,A) \) will be included here for later use. R. J. Nunke has provided \([10]\), Theorem 5.1, pg. 280) a description of the short exact sequences that represent the elements of \( p^n\text{Ext}(B,A) \). His theorem, rephrased to fit the present context yields

   \[ \text{Theorem 1. The exact sequence} \]

   \[ 0 \rightarrow A \rightarrow X \rightarrow B \rightarrow 0 \]

   represents an element of \( p^n\text{Ext}(B,A) \) if and only if \( A \cap p^mX = p^mA \) for \( m \leq n \).

   \[ \text{Corollary 1. The exact sequence} \]

   \[ 0 \rightarrow A \rightarrow X \rightarrow B \rightarrow 0 \]

   represents an element of \( p^\alpha\text{Ext}(B,A) \) if and only if \( A \cap p^nX = p^nA \) for all positive integers \( n \).

   \[ \text{Proof. The corollary follows immediately from the} \]

   \[ \text{fact that for any group } G, p^\alpha G = \bigcap_1^n p^G. \]
2. PROJECTIVES AND INJECTIVES FOR $p^n\text{Ext}(B,A)$

The $p^n$-projectives and the $p^n$-injectives are generally known. The $p^n$-injectives are described in terms of cotorsion groups in [11]. Cotorsion groups were introduced by D. K. Harrison, and the reader is referred to [7] for a discussion of them. A description in more elementary terms of the $p^n$-projective and the $p^n$-injective groups is given in

Theorem 2. A group $P$ is $p^n$-projective if and only if $P = F \oplus R$, where $F$ is a free group and $R$ is a group satisfying $p^nR = 0$. A group $I$ is $p^n$-injective if and only if $I = D \oplus R$, where $D$ is a divisible group and $R$ is a group satisfying $p^nR = 0$.

In homological considerations, it is usually very important to know whether objects (in this case, groups) have projective and injective resolutions. If they do have, it may be equally important to know how to construct these resolutions. Nunke has shown [11] that for any ordinal $\alpha$, $p^\alpha$-projective and $p^\alpha$-injective resolutions exist for every group. Theorem 3 gives an explicit construction of such resolutions for a finite. For a group $G$ and any integer $n$, $G[n]$ will denote the subgroup consisting of elements $x$ of $G$ such that $n x = 0$.

Theorem 3. Let $B$ be any group, and let $\xymatrix{ F \ar[r]^f & B & 0}$ be exact with $F$ a free group. Then the sequence

$\xymatrix{ 0 & A & F \oplus B[p^n] & B & 0 }$

is a $p^n$-projective resolution of $B$, where $g(x,y) = f(x) + y$, and $A = \ker g$.

Proof. The sequence is exact by its construction, and $F \oplus B[p^n]$ and $A$ are $p^n$-projective by Theorem 2. By Theorem 1, we need only show that $A \cap p^m(F \oplus B[p^n]) = p^mA$ for $m \leq n$. Let $(x,y)$ be in $A \cap p^m(F \oplus B[p^n])$. Then $(x,y) = p^m(u,v)$, with $(u,v)$ in $F \oplus B[p^n]$. Now $p^m(u,v)$ is in $A = \ker g$. 
which gives \( g(p^n(u,v)) = 0 = p^n(f(u) + v) \), and hence 
\( (f(u) + v) = w \) is in \( B[p^n] \) which is contained in \( B[p^n] \). 
Thus \( (u,v - w) \) is in \( A \) and \( p^n(u,v - w) = p^n(u,v) = (x,y) \) is in \( p^nA \). The other inclusion is obvious.

**Theorem 4.** Let \( A \) be any group and let \( 0 \to A \xrightarrow{f} D \to A \xrightarrow{g} 0 \) be exact, with \( D \) divisible. Then the sequence
\[
0 \to A \to D \otimes A/p^nA \to (D \otimes A/p^nA)/g(A) \to 0
\]
is a \( p^n \)-injective resolution of \( A \), where \( g(a) = (f(a), a + p^nA) \).

**Proof.** Similar to the proof of Theorem 3, we need only show that \( g(A) \cap p^n(D \otimes A/p^nA) = p^n g(A) \) for \( m \leq n \). Let 
\( (x,y) \) be in \( g(A) \cap p^n(D \otimes A/p^nA) \). Then \( (x,y) = p^n(d,a + p^nA) = g(w), w \) in \( A \). Hence \( p^n(d,a + p^nA) = (f(w),w + p^nA) \),
\( p^n(a + p^nA) = w + p^nA \), and \( (p^ma - w) = p^n v \) for some \( v \) in \( A \).
Thus \( w = p^n(a - p^{n-m}v) \), and we have that \( (x,y) = p^n(f(a - p^{n-m}v), a - p^{n-m}v + A) = p^n g(a - p^{n-m}v) \) is in \( p^n g(A) \). The other inclusion is obvious.

3. **LONG EXACT SEQUENCES WITH \( p^n \text{Ext}(B,A) \)**

The following theorem can be proved directly with little difficulty. However, its proof is omitted here, since in Section IV this theorem will be shown to hold for all ordinals \( \alpha < \omega + \omega \).

**Theorem 5.** If
\[
0 \to A \to B \to C \to 0
\]
represents an element of \( p^\alpha \text{Ext}(C,A) \) and \( \alpha \leq \omega \), then for any group \( X \), the sequences
\[
0 \to \text{Hom}(X,A) \to \text{Hom}(X,B) \to \text{Hom}(X,C) \to p^\alpha \text{Ext}(X,A)
\]
\[
\to p^\alpha \text{Ext}(X,B) \to p^\alpha \text{Ext}(X,C) \to 0\], and
\[
0 \to \text{Hom}(C,X) \to \text{Hom}(B,X) \to \text{Hom}(A,X) \to p^\alpha \text{Ext}(C,X)
\]
are exact.

\[
\to p^\alpha \text{Ext}(B,X) \to p^\alpha \text{Ext}(A,X) \to 0
\]
4. A Generalization

The theorems in this section can be generalized as follows. Let $J$ be an non-empty set $\{p_i^{n_i}\}$ of powers of distinct primes. Let $J\text{-Ext}(B,A) = \bigcap_i p_i^{n_i} \text{Ext}(B,A)$. Then Theorems 1 to 5 have direct analogues for $J\text{-Ext}(B,A)$. For example, the analogue to Theorem 2 is

**Theorem 2'.** A group $P$ is $J$-projective if and only if $\quad P = F \oplus R$, where $F$ is a free group and $R = \sum R_i$ with $p_i^{n_i} R_i = 0$. A group $I$ is $J$-injective if and only if $I = D \oplus R$, where $D$ is a divisible group and $R = \prod R_i$ with $p_i^{n_i} R_i = 0$.

The statements of the other analogous theorems and their proofs follow easily, and are omitted. Additional theorems concerning $p^n\text{Ext}(B,A)$ will appear in Section 4 as special cases of theorems for $p^\alpha\text{Ext}(B,A)$.

III. The Group $p^\alpha\text{Ext}(B,A)$

1. A Characterization of the Elements of $p^\alpha\text{Ext}(B,A)$ and $D\text{Ext}(B,A)$

We begin this section by determining the exact sequences

$$0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0$$

which represent the elements of $p^\alpha\text{Ext}(B,A)$. The $p$-component of a group $G$ will be denoted by $G_p$, the torsion subgroup of $G$ by $G_t$, and the divisible subgroup of $G$ by $G_d$. For an exact sequence

$$0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0,$$

let $X(p)$ be the subgroup of $X$ such that $X(p)/A = (X/A)_p$. First we establish the following lemma which is needed in its full generality in a later section.
Lemma 1. For any ordinal \( \alpha \),
\[
p^\alpha \text{Ext}(B, A)/p^\infty \text{Ext}(B, A) \cong p^\alpha \text{Ext}(B_p, A).
\]
The isomorphism is induced by mapping the element of \( \text{Ext}(B, A) \) represented by
\[
0 \rightarrow A \rightarrow X \rightarrow B \rightarrow 0
\]
onto the element of \( p^\alpha \text{Ext}(B_p, A) \) represented by
\[
0 \rightarrow A \rightarrow X(p) \rightarrow B_p \rightarrow 0.
\]
For any ordinal \( \alpha \geq \omega \),
\[
p^\alpha \text{Ext}(B, A)/p^\omega \text{Ext}(B, A) \cong p^\alpha \text{Ext}(B_p, A_p).
\]
The isomorphism is induced by mapping the element of \( p^\alpha \text{Ext}(B, A) \) represented by
\[
0 \rightarrow A \rightarrow X \rightarrow B \rightarrow 0
\]
onto the element of \( p^\alpha \text{Ext}(B_p, A_p) \) represented by
\[
0 \rightarrow A_p \rightarrow X_p \rightarrow B_p \rightarrow 0.
\]

Proof. The exact sequence
\[
0 \rightarrow B_p \rightarrow B \rightarrow B/B_p \rightarrow 0
\]
yields the exact sequence
\[
\text{Ext}(B/B_p, A) \rightarrow \text{Ext}(B, A) \rightarrow \text{Ext}(B_p, A). \quad \text{f}
\]
By [10], Theorem 4.2, the image of \( \text{Ext}(B/B_p, A) \) in \( \text{Ext}(B, A) \) is \( p^\infty \text{Ext}(B, A) \). Thus \( \text{Ext}(B, A)/p^\infty \text{Ext}(B, A) \cong \text{Ext}(B_p, A) \).
Since \( p^\alpha \text{Ext}(B, A) \leq p^\alpha \text{Ext}(B, A) \) for all \( \alpha \),
\[
p^\alpha(\text{Ext}(B, A)/p^\alpha \text{Ext}(B, A)) = p^\alpha \text{Ext}(B, A)/p^\infty \text{Ext}(B, A) \cong p^\alpha \text{Ext}(B_p, A).
\]
The homomorphism \( f \) is the map described in the statement of the first part of the lemma.

For any group \( G \), it is easy to see that \( G_p \) and \( G_t \) are \( p^\alpha \)-pure in \( G \). From the \( p^\alpha \)-pure sequence
\[
0 \rightarrow A_p \rightarrow A \rightarrow A/A_p \rightarrow 0
\]
one has the exact sequence
\[
\text{Hom}(B_p, A/A_p) \rightarrow p^\alpha \text{Ext}(B_p, A_p) \rightarrow p^\alpha \text{Ext}(B_p, A)
\]
\[
\rightarrow p^\alpha \text{Ext}(B_p, A/A_p) \rightarrow 0.
\]
The \( p^\alpha \)-pure sequence
0 → \frac{A_t}{A_p} → \frac{A}{A_p} → \frac{A}{A_t} → 0

yields the exact sequence
\[ p^{\alpha}\text{Ext}(B_p, \frac{A_t}{A_p}) → p^{\alpha}\text{Ext}(B_p, A/A_p) → p^{\alpha}\text{Ext}(B_p, A/A_t) → 0. \]

Clearly \( p^{\alpha}\text{Ext}(B_p, \frac{A_t}{A_p}) = 0 \). Using Theorem 62.2, Lemma 63.1, and 0) page 246 in [5], we conclude that \( p^{\alpha}\text{Ext}(B_p, A/A_t) = \text{Pext}(B_p, A/A_t) = 0 \). Thus \( p^{\alpha}\text{Ext}(B_p, A/A_p) = 0 \).

Also \( \text{Hom}(B_p, A/A_p) = 0 \), and it follows that \( p^{\alpha}\text{Ext}(B_p, A) \cong p^{\alpha}\text{Ext}(B_p, A_p) \). Since \( p^{\alpha}G \subseteq p^{\alpha}G \) for any group \( G \), it follows that
\[ p^{\alpha}\text{Ext}(B, A)/p^{\alpha}\text{Ext}(B, A) = p^{\alpha}(\text{Ext}(B, A)/p^{\alpha}\text{Ext}(B, A)) \]
\[ \cong p^{\alpha}\text{Ext}(B_p, A) \cong p^{\alpha}\text{Ext}(B_p, A_p). \]

Any ordinal \( \alpha \geq \omega \) may be written in the form \( \omega + \beta \), and \( p^\beta(p^{\alpha}G) = p^{\alpha+\beta}G \). (See [8], Lemma 8.) It follows that
\[ p^{\alpha}\text{Ext}(B, A)/p^{\alpha}\text{Ext}(B, A) \cong p^{\alpha}\text{Ext}(B_p, A_p) \]
for all \( \alpha \geq \omega \).

To prove the last assertion of the lemma, it must be shown that the element \( x \) in \( p^{\alpha}\text{Ext}(B, A) \) represented by the exact sequence
\[ 0 → A → X → B → 0 \]
goes onto, by means of \( g^{-1}f \), the element \( w \) in \( p^{\alpha}\text{Ext}(B_p, A_p) \) represented by the exact sequence
\[ 0 → A_p → X_p → B_p → 0. \]

(That this last sequence is exact follows from the fact that \( A \) is \( p^{\alpha} \)-pure in \( X \) and Corollary 1.) The element \( f(x) \) in \( \text{Ext}(B_p, A) \) is represented by the exact sequence
\[ 0 → A → X(p) → B_p → 0. \]

From the \( p^{\alpha} \)-purity of
\[ 0 → A → X → B → 0, \]
it follows that \( X(p) = A + X_p \). The element \( g(w) \) in \( p^{\alpha}\text{Ext}(B_p, A) \) is represented by the exact sequence
\[ 0 → A → \frac{(A \oplus X_p)}{K} → B_p → 0, \]
where \( K = \{(a, -a): a \in A_p\} \) and the maps are the obvious ones. The map \( k: (A \oplus X_p)/K → (A + X_p): \]

[77]
\[ a + x_p \text{ yields an equivalence} \]
\[
\begin{array}{c}
0 \longrightarrow A \longrightarrow (A \oplus X_p)/\mathcal{K} \longrightarrow B_p \longrightarrow 0 \\
\downarrow \text{id} \quad \downarrow k \quad \downarrow \text{id} \\
0 \longrightarrow A \longrightarrow A + X_p \longrightarrow B_p \longrightarrow 0.
\end{array}
\]

Since \( \mathcal{K} \) is an isomorphism, the lemma follows.

An immediate consequence of Lemma 1 is

**Theorem 6.** The exact sequence
\[
0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0
\]
represents an element in \( p^\infty \text{Ext}(B, A) \) if and only if the sequence
\[
0 \longrightarrow A_p \longrightarrow X_p \longrightarrow B_p \longrightarrow 0
\]
is splitting exact.

Let \( \text{Dext}(B, A) \) denote the divisible subgroup of \( \text{Ext}(B, A) \).
It is clear that \( \text{Dext}(B, A) = \bigcap_{p} p^\infty \text{Ext}(B, A) \).

Thus the exact sequence
\[
0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0
\]
represents an element of \( \text{Dext}(B, A) \) if and only if
\[
0 \longrightarrow A_p \longrightarrow X_p \longrightarrow B_p \longrightarrow 0
\]
is splitting exact for all primes \( p \).
Since \( G_t = \sum_{p} G_p \), this implies

**Theorem 7.** The exact sequence
\[
0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0
\]
represents an element of \( \text{Dext}(B, A) \) if and only if
\[
0 \longrightarrow A_t \longrightarrow X_t \longrightarrow B_t \longrightarrow 0
\]
is splitting exact.

Although it may be proved directly, we remark that
Theorem 7 yields immediately that \( \text{Pext}(B, A) = \text{Dext}(B, A) \)
whenever \( A \) is torsion free, since if a sequence
\[
0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0
\]
is pure exact, then the sequence of torsion subgroups.
is exact.

2. **PROJECTIVES AND INJECTIVES FOR** $p^\infty \text{Ext}(B, A)$ **AND** $D \text{ext}(B, A)$

The $p^\infty$-projectives and $p^\infty$-injectives will now be determined, beginning with the projectives.

**Theorem 8.** A group $P$ is $p^\infty$-projective if and only if $P = F \oplus T$ where $F$ is a free group and $T$ is a $p$-group.

**Proof.** If $P = F \oplus T$ where $F$ is free and $T$ is a $p$-group, then $p^\infty \text{Ext}(P, X) = p^\infty \text{Ext}(F \oplus T, X) \cong p^\infty \text{Ext}(T, X)$, and $p^\infty \text{Ext}(T, X) = 0$ for all $X$ by [10], Corollary 4.2.

Suppose $P$ is $p^\infty$-projective. The exact sequence

$$0 \longrightarrow P \longrightarrow P \longrightarrow P/P_p \longrightarrow 0$$

yields, by [10], Theorem 4.2, the exact sequence

$$\text{Hom}(P_p, X) \longrightarrow \text{Ext}(P/P_p, X) \longrightarrow p^\infty \text{Ext}(P, X) = 0.$$ 

Thus when $X$ is torsion free, $\text{Ext}(P/P_p, X) = 0$. There exists an exact sequence

$$0 \longrightarrow K \longrightarrow G \longrightarrow P/P_p \longrightarrow 0$$

with $G$ free, and $\text{Ext}(P/P_p, K) = 0$ implies that this sequence splits. Hence $P/P_p$ is free, and the assertion follows.

**Theorem 9.** A group $I$ is $p^\infty$-injective if and only if $I = D \oplus C$, where $D$ is divisible and $C$ is cotorsion, that is $qC = C$ for all primes $q \neq p$. In fact, $C = A \otimes B$, where $A_\mathbb{Z} = A_p$, $A \cong \text{Ext}(\mathbb{Z}(p^\infty), A_p)$, and $B$ is a summand of a product of copies of the $p$-adic integers.

**Proof.** (A number of the facts needed in this proof are provided by R. J. Nunke in [10] and by D. K. Harrison in [7].) Suppose $I = D \oplus C$, where $D$ is divisible and $C$ is cotorsion with $qC = C$, $q \neq p$. Then $p^\infty \text{Ext}(X, D \oplus C) \cong p^\infty \text{Ext}(X, C)$. The exact sequence

$$0 \longrightarrow X_t \longrightarrow X \longrightarrow X/X_t \longrightarrow 0$$

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yields the exact sequence.

\[ \text{Ext}(X/X_t, C) = 0 \rightarrow \text{Ext}(X, C) \rightarrow \text{Ext}(X_t, C) \rightarrow 0. \]

Hence \( p^\infty \text{Ext}(X, C) \cong p^\infty \text{Ext}(X_t, C) \). The group \( C \) is \( q \)-divisible, \( q \not= p \), and by [10], Theorem 4.5, \( \text{Ext}(X_t, C) \) is \( q \)-divisible. Hence \( p^\infty \text{Ext}(X_t, C) \) is divisible. But since \( X_t \) is torsion \( \text{Ext}(X_t, C) \) is reduced. Hence \( p^\infty \text{Ext}(X, C) \cong p^\infty \text{Ext}(X_t, C) = 0 \) for all groups \( X \), and \( I \) is \( p^\infty \)-injective.

Suppose \( I \) is \( p^\infty \)-injective. If \( Y \) is torsion free, then \( \text{Ext}(Y, I) \) is divisible. Thus \( \text{Ext}(Y, I) = p^\infty \text{Ext}(Y, I) = 0 \), and it follows that \( I = D \oplus C \) with \( D \) divisible and \( C \) cotorsion. Now \( C \cong \text{Ext}(Q/Z, C) = \text{Ext}(Z(p^\infty), C) \cong \prod_{q} \text{Ext}(Z(q^\infty), C) = C_p \oplus R \), where \( C_p = \text{Ext}(Z(p^\infty), C) \) and \( R = \prod_{q \not= p} \text{Ext}(Z(q^\infty), C) \). By [10], \( qC_p = C_p \) if \( q \not= p \), and \( pR = R \). The group \( C_p \) is cotorsion since it is a summand. Now \( \text{Ext}(X, R) = p^\infty \text{Ext}(X, R) = 0 \) for all groups \( X \). Thus \( R = 0 \) since \( R \) is reduced, and \( C \) is cotorsion, with \( qC = C \) for primes \( q \not= p \).

There is a decomposition \( C = A \oplus B \), where \( A \) is adjusted and \( B \) is torsion free. Since for \( q \not= p \), \( qC = C \), and \( C \) is reduced, it follows that \( qA = A \) and \( A_t = A_p \). Since \( A \) is adjusted, \( A \cong \text{Ext}(Q/Z, A) \cong \text{Ext}(Z(p^\infty), A) \). The group \( B \) is a summand of a product of \( q \)-adic integers for various primes \( q \). But \( qB = B \) for \( q \not= p \) implies that \( B \) is a summand of a product of copies of the \( p \)-adic integers.

The projectives and injectives for \( \text{Dext}(B, A) \) are somewhat easier to determine than those for \( p^\infty \text{Ext}(B, A) \). These groups will be called \( D \)-projective and \( D \)-injective, respectively, and are described in

**Theorem 10.** A group \( P \) is \( D \)-projective if and only if \( P = F \oplus T \), where \( F \) is free and \( T \) is torsion. A group \( I \) is \( D \)-injective if and only if \( I = D \oplus C \), where \( D \) is divisible and \( C \) is cotorsion.

**Proof.** Suppose \( P = F \oplus T \), where \( F \) is free and \( T \) is torsion. Then \( \text{Dext}(P, X) \cong \text{Dext}(T, X) = (\text{Ext}(T, X))_d \cong \)
\((\Pi \text{Ext}(T_p, X))_d = \Pi_{p} \text{Ext}(T_p, X)\) and
\(\text{Dext}(T_p, X) \subseteq p^{\infty} \text{Ext}(T_p, X) = 0\) for all \(p\). Hence \(\text{Dext}(P, X) = 0\).

Suppose \(P\) is \(D\)-projective. The exact sequence
\[
0 \longrightarrow P_t \longrightarrow P \longrightarrow P/P_t \longrightarrow 0
\]
and the fact that \(\text{Ext}(P/P_t, X)\) is divisible for all \(X\) yield the exact sequence
\[
\text{Hom}(P_t, X) \longrightarrow \text{Ext}(P/P_t, X) \longrightarrow \text{Dext}(P, X) = 0.
\]
Thus when \(X\) is torsion free, \(\text{Ext}(P/P_t, X) = 0\), and it follows that \(P/P_t\) is free. Hence \(P \cong P/P_t \oplus P_t\), and so \(P\) is as asserted.

Suppose \(I = D \oplus C\), where \(D\) is divisible and \(C\) is cotorsion. As in the proof of Theorem 9, \(C \cong \Pi_{p} CP\), where \(qCP = CP\) if \(q \neq p\), and \(\text{Dext}(X,I) \cong \text{Dext}(X,C) \cong \Pi_{p} (\cap q^{\infty} \text{Ext}(X,CP))\).

But \(\cap q^{\infty} \text{Ext}(X,CP) \subseteq p^{\infty} \text{Ext}(X,CP) = 0\) by Theorem 9, hence \(I\) is \(D\)-injective.

Suppose \(I\) is \(D\)-injective. Then for \(X\) torsion free, \(\text{Dext}(X,I) = \text{Ext}(X,I) = 0\). Thus \(I\) is \(D \oplus C\) where \(D\) is divisible and \(C\) is cotorsion.

We proceed now to the construction of \(p^{\infty}\)-projective and \(p^{\infty}\)-injective resolutions. Let \(F \xrightarrow{f} A \longrightarrow 0\) be exact with \(F\) free. Define \(g: F \oplus A_p \longrightarrow A; \ g(x, y) = f(x) + y\). Let \(Q_p\) be the subgroup of elements of \(Q\) whose denominators are powers of \(p\). The exact sequence
\[
0 \longrightarrow Z \longrightarrow Q_p \longrightarrow Z(p^{\infty}) \longrightarrow 0
\]
yields the exact sequence
\[
\text{Hom}(Q_p, B) \longrightarrow \text{Hom}(Z, B) \sim B \longrightarrow \text{Ext}(Z(p^{\infty}), B) \longrightarrow \text{Ext}(Q_p, B) \longrightarrow 0.
\]
Note that \(\text{Ker } h \subseteq p^{\infty} B\) since \(pQ_p = Q_p\), and \(\text{Ext}(Z(p^{\infty}), B)/f(B) \cong \text{Ext}(Q_p, B)\) is divisible. Let \(B \subseteq D\), \(D\) divisible, and define \(k: B \longrightarrow D \oplus \text{Ext}(Z(p^{\infty}), B); \ k(b) = b + h(b)\).

\textbf{Theorem 11.} \textit{The sequence}
\[
(\mathcal{Y}) \; 0 \longrightarrow X \longrightarrow F \oplus A_p \xrightarrow{E} A \longrightarrow 0,
\]
where $K = \ker g$, is a $p^\infty$-projective resolution of $A$. The sequence

\begin{align*}
0 & \longrightarrow B \xrightarrow{k} D \oplus \text{Ext}(Z(p^\infty), B) \\
& \quad \longrightarrow (D \oplus \text{Ext}(Z(p^\infty), B))/k(b) \longrightarrow 0
\end{align*}

is a $p^\infty$-injective resolution of $B$.

**Proof.** The sequence (1) is exact by construction, and $F \oplus A_p$ is $p^\infty$-projective by Theorem 8. Since $(\ker g) \cap A_p = 0$, the map $K \longrightarrow F: (x, y) \longmapsto x$ is a monomorphism of $K$ into $F$. Hence $K$ is free and so is $p^\infty$-projective. The sequence is $p^\infty$-pure by Theorem 6.

The sequence (2) is exact by construction. Since $\text{qExt}(Z(p^\infty), B) = \text{Ext}(Z(p^\infty), B)$ for $q \neq p$, and $\text{Ext}(Z(p^\infty), B)$ is cotorsion [7], $D \oplus \text{Ext}(Z(p^\infty), B)$ is $p^\infty$-injective, by Theorem 9. Let $(d, x) + k(b)$ be in $(D \oplus \text{Ext}(Z(p^\infty), B))/k(b)$. The divisibility of $\text{Ext}(Z(p^\infty), B)/h(B)$ and $D$ yield $x = nx + h(b)$ and $(d, x) + k(b) = (d - b, nx) + k(b) = n(d, b_n) + k(b)$, where $n$ is any non-zero integer, $b$ and $b_n$ are in $B$, and $d_n$ is in $D$. Thus $(D \oplus \text{Ext}(Z(p^\infty), B))/k(b)$ is divisible, and hence $p^\infty$-injective.

To show (2) is $p^\infty$-pure, we use Theorem 6. Let $Z(p)$ denote the cyclic group of order $p$. By a standard homological formula (See [4], p. 116),

\[ \text{Hom}(Z(p), \text{Ext}(Q_p, B)) \oplus \text{Ext}(Z(p), \text{Hom}(Q_p, B)) = \text{Hom}(\text{Tor}(Z(p), Q_p), B) \oplus \text{Ext}(Z(p) \otimes Q_p, B). \]

But $\text{Tor}(Z(p), Q_p) = Z(p) \otimes Q_p = 0$, so that $\text{Hom}(Z(p), \text{Ext}(Q_p, B)) = 0$. Hence $\text{Ext}(Q_p, B)_p = 0$. From the definition of $h$, it follows that $h(B_0) = \text{Ext}(Z(p^\infty), B)_p$, and that $h(B)$ is $p^\infty$-pure in $\text{Ext}(Z(p^\infty), B)$. Let $(d, x)$ be in $D \oplus \text{Ext}(Z(p^\infty), B)$ and $b$ be in $B$. If $p^n(d, x) = (b, h(b))$, then $h(b) = p^n x = p^n h(b_1)$ for some $b_1$ in $B$. Thus $b - p^n b_1$ is in Ker $C_{p^\infty}$, so $b - p^n b_1 = p^n b_2$ for some $b_2$ in $B$. Hence $(b, h(b)) = p^n(b_1 + b_2, h(b_1 + b_2))$ is in $p^n k(B)$, and (2) is $p^\infty$-pure.

This implies that the sequence

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Theorem 12. The sequence

\[ 0 \longrightarrow K \longrightarrow F \otimes A_t \xrightarrow{F} A \longrightarrow 0, \]

where \( K = \text{Ker } g \), is a \( D \)-projective resolution of \( A \). The sequence

\[ 0 \longrightarrow B \xrightarrow{k} D \otimes \text{Ext}(Q/Z, B) \]
\[ \longrightarrow (D \otimes \text{Ext}(Q/Z, B)) \otimes k(B) \longrightarrow 0 \]

is a \( D \)-injective resolution of \( B \).

The proof is omitted since it is similar to, but much easier than, the proof of Theorem 11. It is worth noting that \( K \) is free and \((D \otimes \text{Ext}(Q/Z, B)) \otimes k(B)\) is divisible.

3. LONG EXACT SEQUENCES WITH \( p^\infty \text{Ext}(B, A) \) AND \( D \text{ext}(B, A) \)

Theorem 13. If

\[ 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \]

represents an element of \( p^\infty \text{Ext}(C, A) \), then the sequences

(1) \[ 0 \longrightarrow \text{Hom}(X, A) \longrightarrow \text{Hom}(X, B) \longrightarrow \text{Hom}(X, C) \longrightarrow p^\infty \text{Ext}(X, A) \]
\[ \longrightarrow p^\infty \text{Ext}(X, B) \longrightarrow p^\infty \text{Ext}(X, C) \longrightarrow 0, \] and
(2) $0 \longrightarrow \text{Hom}(C, X) \longrightarrow \text{Hom}(B, X) \longrightarrow \text{Hom}(A, X) \longrightarrow \phi^\infty \text{Ext}(C, X) \longrightarrow \phi^\infty \text{Ext}(B, X) \longrightarrow \phi^\infty \text{Ext}(A, X) \longrightarrow 0$

are exact.

If

$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$

represents an element of $\text{Dext}(C, A)$, then the sequences

(3) $0 \longrightarrow \text{Hom}(X, A) \longrightarrow \text{Hom}(X, B) \longrightarrow \text{Hom}(X, C) \longrightarrow \text{Dext}(X, A) \longrightarrow \text{Dext}(X, B) \longrightarrow \text{Dext}(X, C) \longrightarrow 0$, and

(4) $0 \longrightarrow \text{Hom}(C, X) \longrightarrow \text{Hom}(B, X) \longrightarrow \text{Hom}(A, X) \longrightarrow \text{Dext}(C, X) \longrightarrow \text{Dext}(B, X) \longrightarrow \text{Dext}(A, X) \longrightarrow 0$

are exact.

Proof. Let

$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$

represent an element of $\text{Dext}(C, A)$. The exact sequence

$0 \longrightarrow X \longrightarrow X/X_t \longrightarrow 0$

yields the following commutative diagram with exact rows and columns.

\[
\begin{array}{ccccccccc}
\text{Hom}(X, A) & \longrightarrow & \text{Ext}(X, A) & \longrightarrow & \text{Ext}(X, B) & \longrightarrow & \text{Ext}(X, C) & \longrightarrow & 0 \\
\text{Hom}(C, X) & \longrightarrow & \text{Ext}(X, A) & \longrightarrow & \text{Ext}(X, B) & \longrightarrow & \text{Ext}(X, C) & \longrightarrow & 0 \\
\text{Hom}(X_t, C) & \longrightarrow & \text{Ext}(X/X_t, A) & \longrightarrow & \text{Ext}(X/X_t, B) & \longrightarrow & \text{Ext}(X/X_t, C) & \longrightarrow & 0 \\
\end{array}
\]

Observing that $\text{Dext}(X, A) = \text{Im } f_1$, $\text{Dext}(X, B) = \text{Im } f_2$, and $\text{Dext}(X, C) = \text{Im } f_3$, then routine diagram chasing gives (3) exact.

Since

$0 \longrightarrow A_t \longrightarrow B_t \longrightarrow C_t \longrightarrow 0$

is split exact, the sequence

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0 → A/A_t → B/B_t → C/C_t → 0

is exact. This yields the following commutative diagram with exact rows and columns.

\[
\begin{array}{ccc}
\text{Hom}(B_t, X) & \to & \text{Hom}(A_t, X) \to 0 \\
\text{Hom}(A/A_t, X) & \to & \text{Ext}(C/C_t, X) \to \text{Ext}(B/B_t, X) \to \text{Ext}(A/A_t, X) \to 0 \\
\text{Hom}(A, X) & \to & \text{Ext}(C, X) \to \text{Ext}(B, X) \to \text{Ext}(A, X) \to 0 \\
\text{Hom}(A_t, X) & \to & \text{Ext}(C_t, X) \to \text{Ext}(B_t, X) \to \text{Ext}(A_t, X) \to 0 \\
0 & \to & 0 \to 0
\end{array}
\]

The exactness of (4) follows from this diagram in a manner similar to the proof of the exactness of (3).

Using the exact sequences

\[
0 \to X_p \to X \to X/X_p \to 0 \quad \text{and} \quad 0 \to A_p \to B_p \to C_p \to 0,
\]

the exactness of (1) and (2) are obtained in a manner analogous to the proof of exactness of (3) and (4).

IV. The Group p^\alpha Ext(B, A)

1. PROPERTIES OF THE ELEMENTS OF p^\alpha Ext(B, A)

This section describes several properties of the short exact sequences that represent elements of p^\alpha Ext(B, A). The following definitions and properties are needed in the proof of Theorem 14 below.

A subgroup S of a group X is essential in X if for any subgroup T of X, S \cap T = 0 implies T = 0. A homomorphism f:X \to Y is essential if Ker f is essential in X. The set of all essential homomorphisms from X into Y is a subgroup of Hom(X, Y), and is denoted Shom(X, Y). If X is a p-group, then pHom(X, Y) ⊂ Shom(X, Y). If X is a p-group and if either X or Y is p-divisible then pHom(X, Y) = Shom(X, Y). If S and T are subgroups of X such that S \cap T = 0, S + T is essential in X, and S is neat in X, then S is maximal disjoint from T.
If sequences
\[ 0 \longrightarrow A \longrightarrow C \longrightarrow C/A \longrightarrow 0, \] \[ 0 \longrightarrow A \longrightarrow D \longrightarrow D/A \longrightarrow 0 \]
are \(p^\beta\)-pure sequences with \(C\) and \(D\) \(p^\beta\)-injective, then the images of the connecting homomorphisms
\[ \text{Shom}(B,C/A) \longrightarrow \text{Ext}(B,A) \] \[ \text{Shom}(B,D/A) \longrightarrow \text{Ext}(B,A) \]
are the same subgroup of \(\text{Ext}(B,A)\). The proofs of these assertions are straightforward. Proofs of these or similar assertions appear in [8].

**Lemma 2.** Suppose \(A \subset S \subset C\). If \(A\) is \(p^\alpha\)-pure in \(C\), then \(A\) is \(p^\alpha\)-pure in \(S\).

**Proof.** The natural injection map \(S/A \longrightarrow C/A\) yields a map \(\text{Ext}(C/A,A) \longrightarrow \text{Ext}(S/A,A)\), and hence a map \(p^\alpha\text{Ext}(C/A,A) \longrightarrow p^\alpha\text{Ext}(S/A,A)\). The sequence
\[ 0 \longrightarrow A \longrightarrow C \longrightarrow C/A \longrightarrow 0 \]
represents an element in \(p^\alpha\text{Ext}(C/A,A)\) which maps onto the element of \(p^\alpha\text{Ext}(S/A,A)\) represented by
\[ 0 \longrightarrow A \longrightarrow S \longrightarrow S/A \longrightarrow 0. \]
Hence \(A\) is \(p^\alpha\)-pure in \(S\).

**Theorem 14.** Let
\[ (1) \quad 0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0 \]
be a short exact sequence. If \((1)\) represents an element of \(p^\alpha\text{Ext}(B,A)\) then for each \(\beta < \alpha\) there exists a subgroup \(K_\beta\) of \(X(p)\) such that \(A\) is maximal disjoint from \(K_\beta\) in \(X(p)\) and \((A \otimes K_\beta)\) is \(p^\beta\)-pure in \(X(p)/K_\beta\). If \(\alpha \leq \omega + \omega\) or if \(B\) is divisible then the existence of subgroups \(K_\beta\) with the properties above implies that \((1)\) represents an element of \(p^\alpha\text{Ext}(B,A)\).

**Proof.** It follows from Lemma 1 that \((1)\) represents an element of \(p^\alpha\text{Ext}(B,A)\) if and only if
\[ O \longrightarrow A \longrightarrow X(p) \longrightarrow B_p \longrightarrow 0 \]

represents an element of \( p^{\alpha}{\text{Ext}}(B_p, A) \). For convenience, we will assume \( B \) is a \( p \)-group. Suppose (1) represents an element of \( p^{\alpha}{\text{Ext}}(B, A) \). The existence of the desired \( K_\beta \)'s will be shown by induction on \( \alpha \), and for \( \alpha = 0 \) the assertion is vacuously true. Assume \( \alpha \geq 1 \) and that the assertion is true for all ordinals \( \beta < \alpha \). If \( \alpha \) is a limit ordinal, the desired \( K_\beta \)'s clearly exist. Suppose \( \alpha = \delta + 1 \). Let \( 0 \longrightarrow A \longrightarrow C \) be exact with \( A \) \( p^\delta \)-pure in \( C \) and \( C \) \( p^\delta \)-injective. (See [11] for the existence of such a group \( C \).) Then

\[
\text{Hom}(B, C/A) \longrightarrow p^\delta{\text{Ext}}(B, A) \longrightarrow 0
\]

is exact. (See Lemma 4, which is proved independently of this theorem.) Thus

\[
p\text{Hom}(B, C/A) \longrightarrow p(p^\delta{\text{Ext}}(B, A)) = p^\delta{\text{Ext}}(B, A) \longrightarrow 0
\]

is exact, and there exists an \( f \) in \( p\text{Hom}(B, C/A) \subset \text{Shom}(B, C/A) \) which maps onto the element of \( p^{\alpha}{\text{Ext}}(B, A) \) represented by (1). The group \( X \) may be assumed to be \( \{(c, b) : c \in C, b \in B, f(b) = c + A\} \). (\( A \) is mapped into \( X \) by \( a \mapsto (a, 0) \), and we identify \( A \) with its image in \( X \). \( X \) is mapped onto \( B \) by \( (c, b) \mapsto b \).) Let \( K_\delta = \{(0, b) : (0, b) \in X\} \). The mapping \( K_\delta \longrightarrow \text{Ker } f : (0, b) \longrightarrow b \) is an isomorphism. \( A \cap K_\delta = 0 \), and \( \text{Ker } f \) is essential in \( B \). The commutative diagram

\[
\begin{array}{ccc}
\text{Ker } f & \longrightarrow & \mathbb{E} \\
\downarrow & & \downarrow \\
(K_\delta \oplus A)/A & \longrightarrow & X/A
\end{array}
\]

with the vertical arrows natural isomorphisms yields \( (K_\delta \oplus A)/A \) essential in \( X/A \). It follows immediately that \( K_\delta \oplus A \) is essential in \( X \). Since \( A \) is neat in \( X \), it follows that \( A \) is maximal disjoint from \( K_\delta \).

Let \( g : X/K_\delta \longrightarrow C : g((c, b) + K_\delta) = c \), and let \( \text{Im } g = S \). The commutative diagram

\[
\begin{array}{ccc}
(A \oplus K_\delta)/K_\delta & \longrightarrow & A \\
\downarrow & & \downarrow \\
X/K_\delta & \longrightarrow & S
\end{array}
\]
with the vertical arrows injection maps, and the top arrow the natural map, together with the fact that $A$ is $p^\delta$-pure in $S$ (Theorem 18), imply that $(A \otimes K_\delta)/K_\delta$ is $p^\delta$-pure in $X/K_\delta$. The induction hypothesis gives the desired $K_\beta$ for $\beta < \delta$.

Suppose (1) is exact, and for each $\beta < \alpha$ there exists $K_\beta \subset X$ such that $A$ is maximal disjoint from $K_\beta$ and $(A + K_\beta)/K_\beta$ is $p^\delta$-pure in $X/K_\beta$. We wish to show under additional hypothesis that (1) represents an element of $p^\alpha \text{Ext}(B, A)$. We proceed by induction on $\alpha$. If $\alpha = 0$, the assertion is vacuously true. Assume $\alpha \geq 1$, and that it is true for $\beta < \alpha$. If $\alpha$ is a limit ordinal the assertion follows easily. Suppose $\alpha = \delta + 1$. Since $\delta < \alpha$, we have a $K_\delta$ satisfying the conditions above. Let

$$0 \longrightarrow X/K_\delta \longrightarrow C/K_\delta$$

be exact with $X/K_\delta$ $p^\delta$-pure in $C/K_\delta$ and $C/K_\delta$ $p^\delta$-injective. By hypothesis,

$$0 \longrightarrow (A + K_\delta)/K_\delta \longrightarrow X/K_\delta \longrightarrow X/(A + K_\delta) \longrightarrow 0$$

represents an element of $p^\delta \text{Ext}(X/(A + K_\delta), (A + K_\delta)/K_\delta)$. It follows from [12] that $(A + K_\delta)/K_\delta$ is $p^\delta$-pure in $C/K_\delta$. Define $f: B \longrightarrow C/(A + K_\delta)$ as the composition of the natural maps

$$B \longrightarrow X/A \longrightarrow X/(A + K_\delta) \longrightarrow C/(A + K_\delta).$$

Then $\text{Ker } f$ is easily seen to be essential in $B$. The $p^\delta$-pure exact sequence

$$0 \longrightarrow A \longrightarrow C/K_\delta \longrightarrow C/(A + K_\delta) \longrightarrow 0$$

yields the exact sequence

$$\text{Hom}(B, C/(A + K_\delta)) \longrightarrow p^\delta \text{Ext}(B, A) \longrightarrow 0$$

and hence the sequence

$$\text{pHom}(B, C/(A + K_\delta)) \longrightarrow p(p^\delta \text{Ext}(B, A)) = p^\alpha \text{Ext}(B, A) \longrightarrow 0$$

is exact. The element $f$ in $\text{Shom}(B, C/(A + K_\delta))$ maps onto the element in $p^\delta \text{Ext}(B, A)$ represented by

$$0 \longrightarrow A \longrightarrow Y \longrightarrow B \longrightarrow 0$$

where $Y = \{(c + K_\delta, b) : (c + K_\delta) \in C/K_\delta, b \in B, f(b) = c + K_\delta\}$.
and the maps are the obvious ones. Identifying $B$ with $X/A$, define $g : X \to Y : g(x) = (x + K_A, x + A)$. Then $g$ yields an equivalence between (1) and (2).

If $B$ is $p$-divisible then $\text{Shom}(B, C/(A + K_A)) = \text{pHom}(B, C/(A + K_A))$ so that (2) represents an element of $p\alpha\text{Ext}(B, A)$ and hence so does (1).

Assume $\alpha \leq \omega + \omega$. Then there exists a $p^\omega$-pure exact sequence

$$0 \to A \to E \to E/A \to 0$$

with $E$ $p^\omega$-injective and $E/A$ divisible (see Theorem 34, which is proved independently of this theorem). Now

$$\text{Hom}(B, E/A) \to p\alpha\text{Ext}(B, A) \to 0$$

is exact and hence

$$\text{Shom}(B, E/A) = \text{pHom}(B, E/A) \to p(p\alpha\text{Ext}(B, A)) = p\alpha\text{Ext}(B, A) \to 0$$

is exact. Now $\text{Image}(\text{Shom}(B, C/(A + K_A)) \to \text{Ext}(B, A)) = \text{Image}(\text{Shom}(B, E/A) \to \text{Ext}(B, A)) = p\alpha\text{Ext}(B, A)$, so that (1) represents an element of $p\alpha\text{Ext}(B, A)$. This completes the induction.

It is interesting to note that for $\alpha \leq \omega$, an exact sequence

$$0 \to A \to X \to B \to 0$$

represents an element of $p\alpha\text{Ext}(B, A)$ if and only if

$$0 \to A_p \to X_p \to B_p \to 0$$

represents an element of $p\alpha\text{Ext}(B_p, A_p)$. This follows from the second part of Lemma 1.

The $\alpha$-th Ulm subgroup of $\text{Ext}(B, A)$ is $\bigcap p^{\alpha\text{Ext}(B, A)}$.

Theorem 14 characterizes these Ulm subgroups when $B$ is divisible, partially answering a question mentioned by L. Fuchs ([6], page 124).

A slight modification of the proof of Theorem 14 yields an alternate statement which is needed later.

Theorem 14'. Let $\alpha = \beta + n$ with $\beta$ a limit ordinal and $n$ finite, and let
be a short exact sequence. If (1) represents an element of $p^\alpha \text{Ext}(B,A)$ then for each $\delta < \beta$ there exists a subgroup $K_\delta$ of $X(\rho)$ such that $A$ is maximal disjoint from $K_\delta$ in $X(\rho)$.

$(A \oplus K_\delta)/K_\delta$ is $\rho^\delta$-pure in $X(\rho)/K_\delta$, and if $n > 0$, there exists $K_\beta$ in $X(\rho)$ such that $A$ is maximal disjoint from $K_\beta$ in $X(\rho)$.

$(A \oplus K_\beta)/K_\beta$ is $\rho^\beta$-pure in $X(\rho)/K_\beta$, and $X(\rho^n) = A[\rho^n] + K_\beta[\rho^n]$. If either $B_\rho$ is divisible or $\alpha \not\geq \omega + \omega$, then the existence of subgroups $K_\delta$ and $K_\beta$ satisfying the properties above implies that (1) represents an element of $p^\alpha \text{Ext}(B,A)$.

Theorems 14 and 14' describe inductively a property of short exact sequences that represent elements in $p^\alpha \text{Ext}(B,A)$. The proofs of the following two theorems utilize this property, and the properties of $p^\alpha$-pure sequences which are described in the following theorems will perhaps give more insight into their nature.

Lemma 3. If the sequence

$$\begin{array}{c}
0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0
\end{array}$$

is exact, then $X(\rho) \cap p^\alpha X = p^\alpha (X(\rho))$ for all ordinals $\alpha$.

Proof. The proof is by induction on $\alpha$. For $\alpha = 0$ the assertion is trivial. Assume $\alpha > 0$ and that $X(\rho) \cap p^\beta X = p^\beta (X(\rho))$ for all $\beta < \alpha$. If $\alpha$ is a limit ordinal, $X(\rho) \cap p^\alpha X = \bigcap_{\beta < \alpha} (X(\rho) \cap p^\beta X) = \bigcap_{\beta < \alpha} (p^\beta (X(\rho))) = p^\alpha (X(\rho))$.

Suppose $\alpha = \beta + 1$. Let $x$ be in $X(\rho) \cap p^\alpha X = X(\rho) \cap p^\beta X$.

Then $x = py$, with $y$ in $p^\beta X$. But $x$ is in $X(\rho)$ and $(X/X(\rho))_{p^\beta} = 0$ implies that $y$ is in $X(\rho)$. By the induction hypothesis, $y$ is in $p^\beta (X(\rho))$, and hence $x = py$ is in $p^\alpha (X(\rho))$. Thus $X(\rho) \cap p^\alpha X \subset p^\alpha (X(\rho))$. The other inclusion is clear.

Theorem 15. If the sequence

$$(1) \begin{array}{c}
0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0
\end{array}$$

represents an element of $p^\alpha \text{Ext}(B,A)$, then $A \cap p^\beta X = p^\beta A$ for all $\beta \not\geq \alpha$. 

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Proof. Suppose that (1) represents an element of \( p^\alpha \Ext(B, A) \). We show that \( A \cap p^\beta x = p^\beta A \) for \( \beta \leq \alpha \) by induction on \( \alpha \). If \( \alpha = 0 \), the equality is trivial, and if \( \alpha = 1 \), it follows from Theorem 1. Assume \( \alpha > 1 \) and that the equality holds for all \( \beta < \alpha \). If \( \alpha \) is a limit ordinal, \( A \cap p^\alpha X = \bigcap_{\beta < \alpha} (A \cap p^\beta X) = A \cap \bigcap_{\beta < \alpha} (p^\beta A) = p^\alpha A \). Suppose \( \alpha = \beta + 1 \). Let \( a \) be in \( A \cap p^\alpha X \). \( x \) in \( p^\beta X \). By Theorem 14, there is a subgroup \( K_\beta \) of \( X(p) \) such that \( A \) is maximal disjoint from \( K_\beta \) and \( (A \cap K_\beta) / K_\beta \) is \( p^\beta \)-pure in \( X / K_\beta \). Since \( A \) is maximal disjoint from \( K_\beta \) in \( X(p) \) and \( px \) is in \( A \), then \( x \) is in \( X(p) \) and there are elements \( w \) in \( A \) and \( y \) in \( K_\beta \) such that \( w + x = y \). Now \( x \) in \( X(p) \cap p^\beta X \) implies that \( x + K_\beta \) is in \( p^\beta (X(p) / K_\beta) \) and \(-w + K_\beta = x + K_\beta \) is in \( ((A \cap K_\beta) / K_\beta) \cap p^\beta (X(p) / K_\beta) = p^\beta ((A \cap K_\beta) / K_\beta) = (p^\alpha A \cap K_\beta) / K_\beta \) by the induction hypothesis. Thus \( w \) is in \( p^\beta A \). But \( pw + px = py \) in \( A \cap K = 0 \) implies \( a = px = -pw \) is in \( p(p^\alpha A) = p^\alpha A \). This completes the proof.

Theorem 16. Let \( B \) be a \( p \)-group. If

\[
(1) \quad 0 \longrightarrow A \longrightarrow Y \longrightarrow B \longrightarrow 0
\]

represents an element of \( p^\alpha \Ext(B, A) \), where \( \alpha = \beta + \gamma \) with \( \beta \) a limit ordinal and \( \gamma < \omega \), then

(a) \( p^\delta (X / A) = (p^\delta X + A) / A \) for \( \delta < \beta \), and

(b) \( (p^\beta (X / A)) [p^\gamma] = ((p^\beta X + A) / A)[p^\gamma] \).

Proof. To prove (a), we induct on \( \delta \). We may assume \( \beta > 0 \). If \( \delta = 0 \), (a) holds. Assume \( \epsilon < \beta \) and (a) holds for all \( \delta < \epsilon \). Suppose \( \epsilon \) is a limit ordinal. Let \( x + A \) be in \( p^\epsilon (X / A) \), and let \( o(x + A) = p^m \). For a divisible group \( D \) containing \( A \) we have the exact sequence

\[
\Hom(B, D / A) \longrightarrow \Ext(B, A) \longrightarrow 0.
\]

Since \( \epsilon + m < \beta \), there is an element \( f \) in \( \Hom(B, D / A) \) whose image in \( p^\epsilon \Ext(B, A) \) is represented by

\[
0 \longrightarrow A \longrightarrow Y \longrightarrow B \longrightarrow 0,
\]

where \( Y = \{(d, b): f(b) = d + A\} \), and \( p^m f \) maps onto the element represented by
It may be assumed that $X = \{(d,b) : p^n f(b) = d + A\}$, and we identify $A$ with the subgroup $\{(a,0) : a \in A\}$ of $X$. The maps in these sequences are the obvious ones. Now $x = (d,b)$ is in $X$, and $p^m x$ is in $A$, so that $p^m b = 0$. Thus $(p^m f)(b) = 0$, and so $(d,0)$ is in $A$, $(0,b)$ is in $X$, and $x + A = (0,b) + A$. We show that $(0,b)$ is in $p^e X$. For $\delta < \epsilon$, $(0,b) + A = x + A$ is in $p^\delta (X/A) = (p^\delta X + A)/A$ by the induction hypothesis. Thus $(0,b) = (a,b) + (-a,0)$, where $(a,b)$ is in $p^\delta X$. The homomorphism $X \rightarrow Y$: $(u,v) \rightarrow (u,p^m v)$ takes $(a,b)$ in $p^\delta X$ onto $(a,p^m b) = (a,0)$ in $(p^\delta Y) \cap A = p^\delta A$. Thus $(0,b) = (a,b) + (-a,0)$ is in $p^\delta X + p^\delta A = p^\delta X$ for all $\delta < \epsilon$. Hence $(0,b)$ is in $p^e X$, and $x + A = (0,b) + A$ is in $(p^e X + A)/A$. The other inclusion is clear.

If $\epsilon = \delta + 1$, $p^e X/A = p(p^\delta X/A) = p((p^\delta X + A)/A) = (p(p^\delta X) + A)/A = (p^e X + A)/A$. Thus $(a)$ is proved.

Since $(b)$ is trivial for $\beta = 0$, we take $\beta > 0$. Let $(x + A)$ be in $(p^\beta (X/A))[p^n]$. Since $A$ is $p^n$-pure in $X$, we may assume $p^n x = 0$. By Theorem 14', there exists a subgroup $K$ of $X$ such that $A$ is maximal disjoint from $K$,

$$0 \rightarrow (A \oplus K)/K \rightarrow X/K \rightarrow X/(A \oplus K) \rightarrow 0$$

is $p^\beta$-pure, and $X[p^n] = A[p^n] \oplus K[p^n]$. Now $x = a + k$ with $a$ in $A[p^n]$, $k$ in $K[p^n]$. By $(a)$, $p^\delta (X/A) = (p^\delta X + A)/A$ for $\delta < \beta$, so $k + A = x^\delta + A$ with $x^\delta$ in $p^\delta X$, and $k = x^\delta - a^\delta$, $a^\delta$ in $A$. But $a^\delta + K = x^\delta + K$ is in $((A \oplus K)/K) \cap p^\delta (X/K) = p^\delta ((A \oplus K)/K) = (p^\delta A \oplus K)/K$ implies that $a^\delta$ is in $p^\delta A$. Therefore $k = x^\delta - a^\delta$ is in $p^\delta X$ for all $\delta < \beta$, and so $k$ is in $p^\beta X$. Hence $x + A = k + A$ is in $(p^\beta X + A)/A$. The other inclusion is clear.

**Corollary 2.** Let $B$ be a $p$-group. If

$$0 \rightarrow A \rightarrow X \rightarrow B \rightarrow 0$$

represents an element of $p^\alpha \text{Ext}(B,A)$, where $\alpha = \beta + n$ with $\beta$ a limit ordinal and $n$ finite, then

$$(1) \quad O \rightarrow p^\delta A \rightarrow p^\delta X \rightarrow p^\delta B \rightarrow O$$

is $p^\alpha$-pure exact for all ordinals $\delta < \beta$. 

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Proof. By Theorems 15 and 16, \( p^\delta(X/A) = (p^\delta X + A)/A \cong (p^\delta X)/(p^\delta A) \cap A = (p^\delta X)/(p^\delta A) \). It follows that (1) is exact. Again using Theorem 15, \((p^\delta A) \cap (p^m(p^\delta X)) = ((p^\delta A) \cap A) \cap (p^{\delta + m}X) = (p^\delta A) \cap (A \cap (p^{\delta + m}X)) = (p^\delta A) \cap (p^{\delta + m}A) = p^{\delta + m}A = p^m(p^\delta A)\), for \( m \leq \omega \). Thus (1) is \( p^\omega \)-pure exact.

Sequences which satisfy the properties of Theorems 15 and 16 are of special interest and will be discussed again later. The question arises whether or not the properties described in Theorems 15 and 16 actually characterize the sequences which represent elements of \( p^\alpha \text{Ext}(B,A) \), when \( B \) is a \( p \)-group. It was pointed out by Nunke that they do not, as is shown in the following example.

Let \( \overline{B} = (\prod_{n=1}^\infty C(p^n))_{\omega} \), where \( C(p^n) \) is a cyclic group of order \( p^n \). The group \( \overline{B} \) is not a direct sum of cyclic groups. Let

\[
\begin{array}{c}
0 \rightarrow K \rightarrow C \rightarrow \overline{B} \rightarrow 0
\end{array}
\]

be \( p^\omega \)-pure with \( C \) and \( K \) each a direct sum of cyclic groups (See Theorem 31 for the construction of such a sequence). The sequence (1) represents a non-zero element of \( p^\alpha \text{Ext}(B,K) \). Since \( \overline{B} \) is a \( p \)-group, \( p^\alpha \text{Ext}(\overline{B},K) = 0 \), so for some ordinal \( \beta \), \( p^\beta \text{Ext}(\overline{B},K) = 0 \). In particular, (1) does not represent an element of \( p^\alpha \text{Ext}(\overline{B},K) \). However for all ordinals \( \alpha, K \cap p^\alpha C = p^\alpha K \) and \( p^\alpha(C/K) = (p^\alpha C + K)/K \). For \( \alpha < \omega \) this is true since (1) represents an element of \( p^\alpha \text{Ext}(\overline{B},K) \) and for \( \alpha \geq \omega \), this is true since all of the terms above are zero.

2. LONG EXACT SEQUENCES WITH \( p^\alpha \text{Ext}(B,A) \)

The next theorem is concerned with long exact sequences involving \( p^\alpha \text{Ext}(B,A) \), and the following lemma will be needed.

Lemma 4. Let

\[
\begin{array}{c}
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0
\end{array}
\]

represent an element of \( p^\alpha \text{Ext}(C,A) \). Then for any group \( X \), the image of the connecting homomorphism \( \text{Hom}(X,C) \rightarrow \text{Ext}(X,A) \) is a subgroup of \( p^\alpha \text{Ext}(X,A) \), and the image of the connecting homomorphism \( \text{Hom}(A,X) \rightarrow \text{Ext}(C,X) \) is a subgroup of \( p^\alpha \text{Ext}(C,X) \).
Proof. Let $f$ be in $\text{Hom}(X,C)$. Then $f$ induces a map $\text{Ext}(C,A) \rightarrow \text{Ext}(X,A)$. The element represented by

$$0 \rightarrow A \rightarrow B \rightarrow \mathcal{E} \rightarrow C \rightarrow 0$$

maps onto the element represented by

$$0 \rightarrow A \rightarrow Y \rightarrow X \rightarrow 0$$

where $Y = \{(b,x) : \mathcal{E}(b) = f(x)\}$, and the maps are the natural ones. This is precisely the image of $f$ under the map $\text{Hom}(X,C) \rightarrow \text{Ext}(X,A)$, and thus $f$ is mapped into $p^a\text{Ext}(X,A)$. A similar proof yields the second assertion.

Theorem 17. If

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is $p^a$-pure exact, then for any group $X$ the sequences

(1) $0 \rightarrow \text{Hom}(X,A) \rightarrow \text{Hom}(X,B) \rightarrow \text{Hom}(X,C) \rightarrow p^a\text{Ext}(X,A)$

$$\rightarrow p^a\text{Ext}(X,B) \rightarrow p^a\text{Ext}(X,C),$$

(2) $0 \rightarrow \text{Hom}(C,X) \rightarrow \text{Hom}(B,X) \rightarrow \text{Hom}(A,X) \rightarrow p^a\text{Ext}(C,X)$

$$\rightarrow p^a\text{Ext}(B,X) \rightarrow p^a\text{Ext}(A,X)$$

are exact. If $a < \omega + \omega$, the sequences

(3) $p^a\text{Ext}(X,B) \rightarrow p^a\text{Ext}(X,C) \rightarrow 0$, and

(4) $p^a\text{Ext}(B,X) \rightarrow p^a\text{Ext}(A,X) \rightarrow 0$

are exact.\(^{2}\)

Proof. By Lemma 4 and the fact that homomorphisms do not decrease height, the sequences above make sense, and then clearly

$$0 \rightarrow \text{Hom}(X,A) \rightarrow \text{Hom}(X,B) \rightarrow \text{Hom}(X,C) \rightarrow p^a\text{Ext}(X,A)$$

$$\rightarrow p^a\text{Ext}(X,B)$$

and

$$0 \rightarrow \text{Hom}(C,X) \rightarrow \text{Hom}(B,X) \rightarrow \text{Hom}(A,X) \rightarrow p^a\text{Ext}(C,X)$$

$$\rightarrow p^a\text{Ext}(B,X)$$

\(^{2}\) Nurse (unpublished) has shown that (3) and (4) are exact for countable ordinals $a$. 

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are exact. The exactness of the sequences
\[ p^\alpha \Ext(X,A) \to p^\alpha \Ext(X,B) \to p^\alpha \Ext(X,C) \]
and
\[ p^\alpha \Ext(C,X) \to p^\alpha \Ext(B,X) \to p^\alpha \Ext(A,X) \]
was proved by Nunke [12].

If \( \alpha < \omega + \omega \), then subgroups of \( p^\alpha \)-projectives are \( p^\alpha \)-projective [11]. Let
\[
\begin{array}{c}
0 \to K \to P \to X \to 0
\end{array}
\]
be a \( p^\alpha \)-projective resolution of \( X \). This yields a commutative diagram
\[
\begin{array}{ccc}
\Hom(K,A) & \to & \Hom(K,B) \\
\downarrow & & \downarrow \\
p^\alpha \Ext(X,A) & \to & p^\alpha \Ext(X,B) \\
\downarrow & & \downarrow \\
0 & \to & p^\alpha \Ext(X,C) \\
\end{array}
\]
with exact rows and columns. Now diagram chasing proves exactness of (3) for \( \alpha < \omega + \omega \).

Let
\[
\begin{array}{c}
0 \to X \to I \to I/X \to 0
\end{array}
\]
be \( p^\alpha \)-pure exact with \( I \) \( p^\alpha \)-injective, \( \alpha < \omega + \omega \). Then for any group \( Y \), we have
\[
0 = p^\alpha \Ext(Y,I) \to p^\alpha \Ext(Y,I/X) \to 0
\]
exact by (3), so that \( p^\alpha \Ext(Y,I/X) = 0 \). Now we have the commutative diagram
\[
\begin{array}{ccc}
\Hom(C,I/X) & \to & \Hom(B,I/X) \\
\downarrow & & \downarrow \\
p^\alpha \Ext(C,X) & \to & p^\alpha \Ext(B,X) \\
\downarrow & & \downarrow \\
0 & \to & p^\alpha \Ext(A,X) \\
\end{array}
\]
with exact rows and columns. Diagram chasing yields exactness of (4) for \( \alpha < \omega + \omega \).

3. PROPERTIES OF \( p^\alpha \)-PURITY
The following theorem lists several additional properties of \( p^\alpha \)-purity. These properties are closely related
to the exactness of the long sequences in the previous theorem.

**Theorem 1.8.** Let $A$, $B$, and $C$ be groups with $A \subset C \subset B$. The following hold.

(a) If $A$ is $p^a$-pure in $B$, then $A$ is $p^a$-pure in $C$.

(b) If $A$ is $p^a$-pure in $B$ and $S$ is a subgroup of $A$, then $A/S$ is $p^a$-pure in $B/S$.

(c) If $A$ is $p^a$-pure in $B$ and $C/A$ is $p^a$-pure in $B/A$, then $C$ is $p^a$-pure in $B$.

(d) If $A$ is $p^a$-pure in $C$, and $C$ is $p^a$-pure in $B$, then $A$ is $p^a$-pure in $B$.

(e) If $A$ is $p^a$-pure in $B$ and $H$ is $p^a$-pure in $K$, then $A \Theta H$ is $p^a$-pure in $B \Theta K$.

(f) Let $H$ and $K$ be subgroups of a group $G$. If $H \cap K$ is $p^a$-pure in $K$, then $H$ is $p^a$-pure in $H + K$.

**Proof.** See Lemma 2 for the proof of (a). If the sequence

$$0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0$$

represents an element of $p^a\text{Ext}(B/A, A)$, then under the map $p^a\text{Ext}(B/A, A) \rightarrow p^a\text{Ext}(B/A, A/S)$ induced by the map $A \rightarrow A/S$, the image of the element is represented by

$$0 \rightarrow A/S \rightarrow B/S \rightarrow B/A \rightarrow 0.$$ 

Thus $A/S$ is $p^a$-pure in $B/S$, and (b) holds. Property (d) is given in [12]. Also in [12], it is stated that $p^a\text{Ext}$ is half exact on $p^a$-pure exact sequences. By Theorem 1.1 in [3], this implies that (c) holds. To prove (e), notice that $A \Theta H$ is $p^a$-pure in $B \Theta H$, and $B \Theta H$ is $p^a$-pure in $B \Theta K$.

Thus (e) follows from (d).

Let $H$ and $K$ be subgroups of a group $G$. Under the map $p^a\text{Ext}((H + K)/H, H \cap K) \rightarrow p^a\text{Ext}((H + K)/H, H)$, the element represented by the $p^a$-pure sequence

$$0 \rightarrow H \cap K \rightarrow K \rightarrow (K + H)/H \rightarrow 0$$

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maps onto the element represented by the sequence
\[ 0 \longrightarrow H \longrightarrow (H \oplus K)/M \longrightarrow (K + H)/H \longrightarrow 0, \]
where \( M = \{(x,-x) : x \in H \cap K\} \), and the maps are the obvious ones. The map \((H \oplus K)/M \rightarrow H + K\) that takes \((h,k) + M\) onto \( h + k \) is an isomorphism, and yields an equivalent of the sequences
\[ 0 \longrightarrow H \longrightarrow (H \oplus K)/M \longrightarrow (K + H)/H \longrightarrow 0 \]
and
\[ 0 \longrightarrow H \longrightarrow H + K \longrightarrow (K + H)/H \longrightarrow 0. \]
Thus the latter sequence is \( p^2 \)-pure exact, and (f) holds.

4. THE GROUP \( E_\alpha(B,A) \)

We now return to the discussion of sequences satisfying the properties described in Theorems 15 and 16. Let \( S_\alpha(B,A) \), where \( \alpha = \beta + n \) with \( \beta \) a limit ordinal and \( n \) finite, be the class of exact sequences
\[ 0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0 \]
which satisfy the properties
(i) \( A \cap p^\delta X = p^\delta A \) for all \( \delta \leq \alpha \)
(ii) \( p^\delta(X/A) = (p^\delta X + A)/A \) for \( \delta < \beta \)
(iii) \( p^\beta(X/A)[p^n] = ((p^\beta X + A)/A)[p^n] \)

Let \( E_\alpha(B,A) \) be the set of equivalence classes of sequences in \( S_\alpha(B,A) \). It will be shown that \( E_\alpha(B,A) \) is a subgroup of \( \text{Ext}(B,A) \), and long exact sequences will be obtained similar to those which hold for \( p^2\text{Ext}(B,A) \). From Theorems 15 and 16, \( p^2\text{Ext}(B,A) \subseteq E_\alpha(B,A) \) for any group \( A \) and any \( p \)-group \( B \).

**Theorem 19.** Let \( A, C \) and \( B \) be groups with \( A \subseteq C \subseteq B \). The following hold.

(a) \( 0 \longrightarrow C \longrightarrow G \oplus H \longrightarrow H \longrightarrow 0 \) belongs to \( S_\alpha(H,G) \) for all groups \( G \) and \( H \).

(b) If \( 0 \longrightarrow H \longrightarrow X \longrightarrow K \longrightarrow 0 \) is equivalent to a sequence belonging to \( S_\alpha(K,H) \), then it is in \( S_\alpha(K,H) \).
(c) If \(0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0\) belongs to \(S_{p^A}(B/A, A)\), then \(0 \rightarrow A \rightarrow C \rightarrow C/A \rightarrow 0\) is in \(S_{p^A}(C/A, A)\).

(d) If \(0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow B/C \rightarrow 0\) and \(0 \rightarrow C/A \rightarrow B/A \rightarrow B/C \rightarrow 0\) are in \(S_{p^A}(B/A, A)\) and \(S_{p^A}(B/C, C/A)\) respectively, then \(0 \rightarrow C \rightarrow B \rightarrow B/C \rightarrow 0\) belongs to \(S_{p^A}(B/C, C)\).

(e) If \(0 \rightarrow C \rightarrow B \rightarrow B/C \rightarrow 0\) belongs to \(S_{p^A}(B/C, C)\), then \(0 \rightarrow C/A \rightarrow B/A \rightarrow B/C \rightarrow 0\) belongs to \(S_{p^A}(B/C, C/A)\).

(f) If \(0 \rightarrow A \rightarrow C \rightarrow C/A \rightarrow 0\) and \(0 \rightarrow C \rightarrow B \rightarrow B/C \rightarrow 0\) are in \(S_{p^A}(C/A, A)\) and \(S_{p^A}(B/C, C)\) respectively, then \(0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0\) belongs to \(S_{p^A}(B/A, A)\).

Proof. That (a) and (b) hold is clear. To prove (c) suppose

\[0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0\]

belongs to \(S_{p^A}(B/A, A)\), and \(C\) is a subgroup of \(B\) with \(A \subseteq C\). If \(\alpha \leq \beta\), then

\[p^\alpha A \subseteq A \cap p^\alpha C \subseteq A \cap p^\beta C \subseteq p^\alpha A \cap C = p^\alpha A,\]

and thus \(A \cap p^\alpha C = p^\alpha A\). For property (ii), let \(\delta < \beta\). We induct on \(\delta\). If \(\delta = 0\), the assertion is trivial. Assume that for all \(\epsilon < \delta\), \(p^\epsilon(C/A) = (p^\epsilon C + A)/A\). If \(\delta = \epsilon + 1\) for some ordinal \(\epsilon\), then

\[p^\delta(C/A) = p(p^\epsilon(C/A)) = p((p^\epsilon C + A)/A) = (p(p^\epsilon C + A))/A = (p^\delta C + A)/A.\]

Suppose \(\delta\) is a limit ordinal. Let \(c + A\) be in \(p^\delta(C/A)\). Now

\[p^\delta(C/A) = \bigcap_{\epsilon < \delta} p^\epsilon(C/A) = \bigcap_{\epsilon < \delta} (p^\epsilon C + A)/A.\]

Since

\[p^\delta(C/A) \subseteq p^\delta(B/A) = (p^\delta B + A)/A,\]

it may be assumed that \(c\) is in \(p^\delta B\). Given \(\epsilon < \delta\), then \(c + A = c_\epsilon + A\) with \(c_\epsilon\) in \(p^\epsilon C\), and \(c = c_\epsilon + a_\epsilon\) for some \(a_\epsilon\) in \(A\). Now \(a_\epsilon = c - c_\epsilon\) is in \(p^\epsilon B \cap A = p^\epsilon A\), so that \(c = c_\epsilon + a_\epsilon\) is in \(p^\epsilon C + p^\epsilon A = p^\epsilon C\). Thus \(c\) is in \(\bigcap_{\epsilon < \delta} p^\epsilon C = p^\delta C\), and \(c + A\) is in \((p^\delta C + A)/A\). The other inclusion is obvious. To get property (iii), let \(c + A\) be in \((p^\beta(C/A))[p^n]\). As above,
we may assume that \( c \) is in \( p^{\delta}B \) and that \( c = c_5 + a_5 \), with \( c_5 \) in \( p^{\delta}C \) and \( a_5 \) in \( A \) for each \( \delta < \beta \). Thus \( a_5 \) is in \( p^{\delta}B \cap A = p^{\delta}A \), and \( c \) is in \( p^{\delta}C \) so that \( c \) is in \( p^{\delta}C \cap p^{\delta}B \). The other inclusion is obvious.

To prove (d), assume

\[ A \rightarrow B \rightarrow (B/A) \rightarrow O \]

and

\[ C \rightarrow (B/A) \rightarrow (B/C) \rightarrow O \]

are in \( S_\beta \alpha(B/A, A) \) and \( S_\beta \alpha(B/C, C/A) \) respectively. Let \( \delta < \beta \), and let \( c \) be in \( C \cap p^{\delta}B \). Then \( c + A \) is in \( (C/A) \cap p^{\delta}(B/A) = p^{\delta}(C/A) \). By part (c), the sequence

\[ (C/A) \rightarrow (B/A) \rightarrow (B/C) \rightarrow O \]

belongs to \( S_\beta \alpha(C/A, A) \), so that \( p^{\delta}(C/A) = (p^{\delta}C + A)/A \), and \( c + a \) is in \( p^{\delta}C \) for some \( a \) in \( A \). Then \( a = (c + a) - c \) is in \( p^{\delta}B \cap A = p^{\delta}A \subset p^{\delta}C \), so that \( c = (c + a) - a \) is in \( p^{\delta}C \).

Since \( p^{\delta}C \subset C \cap p^{\delta}B \), we conclude that \( C \cap p^{\delta}B = p^{\delta}C \) for \( \delta < \beta \). Now \( C \cap p^{\delta}B = C \cap (C \cap p^{\delta}B) = C \cap (C \cap p^{\delta}B) = C \cap p^{\delta}C = p^{\delta}C \).

Given \( 0 \leq m \leq n \), let \( c \) be in \( p^{m}(p^{\beta}B) \cap C \). Then \( c = p^{m}b \) for some \( b \) in \( p^{\beta}B \), and \( c + A \) is in \( (C/A) \cap p^{\beta+m}(B/A) = p^{\beta+m}(C/A) = p^{m}(p^{\beta}(C/A)) \), so \( c + A = p^{m}(c' + A) \) with \( c' + A \) in \( p^{\beta}(C/A) \), and \( c = p^{m}b = p^{m}c' + a \) for some \( a \) in \( A \). Now \( (b - c') + A \) is in \( (p^{\beta}(B/A))[p^{m}] \subseteq (p^{\beta}(B/A))[p^{n}] = ((p^{\beta}B + A)/A)[p^{n}] \), thus \( b - c' = b' + a' \) with \( b' \) in \( p^{\beta}B \) and \( a' \) in \( A \). Then \( b - b' = c' + a' \) is in \( C \cap p^{\beta}B = p^{\beta}C \).

Also \( a - p^{m}a' = p^{m}b' \) is in \( A \cap p^{\beta+m}B = p^{\beta+m}A \), so \( a - p^{m}a' = p^{m}a'' \) with \( a'' \) in \( p^{\beta}A \). This yields \( c = p^{m}c' + a = p^{m}(b - b' - a'') + a = p^{m}(b - b') + p^{m}a'' \) is in \( p^{m}(p^{\beta}C) \). The other inclusion is obvious, so \( C \cap p^{\delta}B = p^{\delta}C \) for all \( \delta \geq \alpha \).

To prove the other properties hold for

\[ C \rightarrow B \rightarrow (B/C) \rightarrow O, \]

let \( \delta < \beta \), and let \( b + C \) be in \( p^{\delta}(B/C) \). Then \( (b + A) + C/A \) is in \( p^{\delta}((B/A)/(C/A)) = (p^{\delta}(B/A) + C/A)/(C/A) = ((p^{\delta}B + A)/A + C/A)/(C/A) = ((p^{\delta}B + C)/A)/(C/A), \)

so that \( b = b' + c \) with \( b' \) in \( p^{\delta}B \) and \( c \) in \( C \). It follows that \( p^{\delta}(B/C) = (p^{\delta}(B + C))/C \). If \( b + C \) is in \( (p^{\delta}(B/C))[p^{n}] \),
a similar proof shows that $b$ is in $p^\delta B + C$, so that

$$(p^\beta(B/C))[p^n] = ((p^\beta B + C)/C)[p^n].$$

To prove (e), suppose

$$0 \longrightarrow C \longrightarrow B \longrightarrow B/C \longrightarrow 0$$

belongs to $S_\psi\alpha(B/C, C)$ and $A$ is a subgroup of $C$. Let $\delta \leq \beta$, and assume $(C/A) \cap p^\varepsilon(B/A) = p^\varepsilon(C/A)$ for $\varepsilon < \delta$. (Equality holds trivially for $\varepsilon = 0$.) If $\delta$ is a limit ordinal, by induction, $(C/A) \cap p^\delta(B/A) = (C/A) \cap (\bigcap_{\varepsilon < \delta} p^\varepsilon(B/A)) = (C/A) \cap (\bigcap_{\varepsilon < \delta} p^\varepsilon(C/A)) = p^\delta(C/A)$. Suppose $\delta = \varepsilon + 1$, and let $c + A$ be in $(C/A) \cap p^\delta(B/A) = (C/A) \cap (p^\varepsilon(B/A))$. Then $c + A = p(x + A)$ with $x + A$ in $p^\varepsilon(B/A)$, and $x + C$ is in $p^\varepsilon(B/C) = (p^\varepsilon B + C)/C$. Thus $x = y + w$ with $y$ in $p^\varepsilon B$, $w$ in $C$ and $w + A$ is in $(p^\varepsilon(B/A)) \cap (C/A) = p^\varepsilon(C/A)$, using the induction hypothesis. Also, $py$ is in $C \cap p^\delta B = p^\delta C$, so that $py + A$ is in $p^\delta(C/A)$. Hence $c + A = (py + A) + (pw + A)$ is in $p^\varepsilon(C/A)$. The other inclusion is obvious, so $(C/A) \cap p^\delta(B/A) = p^\delta(C/A)$ for $\delta \leq \beta$.

Now let $0 \leq m \leq n$, and let $c + A$ be in $(C/A) \cap p^{\beta + m}(B/A)$. Then $c + A = p^m x + A$ with $x + A$ in $p^\beta(B/A)$. Since $p^m x$ is in $C$ and $m \leq n$, $x + C$ is in $(p^\beta(B/C))[p^n] = ((p^\beta B + C)/C)[p^n]$, and $x = y + w$ with $y$ in $p^\beta B$, $w$ in $C$. Now $y + A = (x - y) + A$ is in $(C/A) \cap p^\beta(B/A) = p^\beta(C/A)$, and $p^m y$ is in $C \cap p^{\beta + m} B = p^{\beta + m} C$. This implies $p^m y = p^m v$ for some $v$ in $p^\beta C$. Hence $c + A = p^m(y + w) + A = (p^m v + A) + (p^m w + A)$ is in $p^{\beta + m}(C/A)$. The other inclusion is clear, so we have $(C/A) \cap p^\beta(B/A) = p^\delta(C/A)$ for all ordinals $\delta \leq \alpha$.

Now let $\delta < \beta$. If $(b + A) + C/A$ is in $p^\delta((B/A)/(C/A))$, then $b + C$ is in $p^\delta(B/C) = (p^\varepsilon B + C)/C$, so $b = b' + c$ with $b'$ in $p^\varepsilon B$, $c$ in $C$. Thus $(b + A) + C/A = (b' + A) + C/A$ is in $((p^\varepsilon B + A)/(A) + (C/A))/(C/A) \subset (p^\varepsilon(B/A) + (C/A))/(C/A)$. The reverse inclusion is clear. A similar proof shows that $(p^\beta((B/A)/(C/A))[p^n] = ((p^\beta(B/A) + (C/A))/(C/A))[p^n]$.

For the proof of (f), suppose

$$0 \longrightarrow A \longrightarrow C \longrightarrow C/A \longrightarrow 0$$

and

$$0 \longrightarrow C \longrightarrow B \longrightarrow B/C \longrightarrow 0$$

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are in \( S_p\alpha(C/A, A) \) and \( S_p\alpha(B/C, C) \) respectively. Then if
\( \delta \leq \alpha \), \( A \cap p^\delta B = A \cap C \cap \delta \cap p^\delta B = A \cap p^\delta C = p^\delta A \). Let \( \delta < \beta \), and let \( y + A \) be in \( p^\delta(B/A) \). Then \( y + C \) is in \( p^\delta(B/C) = (p^\delta B + C)/C \), and \( y = y' + x \) with \( y' \) in \( p^\delta B \) and \( x \) in \( C \). Also \( x + A = (y - y') + A \) is in \( (p^\delta(B/A)) \cap (C/A) = p^\delta(C/A) = (p^\delta C + A)/A \) (using (e)). Thus \( x + A = x' + A \) with \( x' \) in \( p^\delta C \), and \( y + C = (y' + C) + (x + C) \) is in \( (p^\delta B + A)/A \). The other inclusion is clear, and we have \( p^\delta(B/A) = (p^\delta B + A)/A \).

A similar proof shows that \( (p^\delta(B/A))[p^n] = ((p^\delta B + A)/A)[p^n] \).

Using the properties just established in Theorem 19, we obtain with little difficulty

**Theorem 20.** For each ordinal \( \alpha \), \( E_{p\alpha}(X, Y) \) is a subgroup of \( \text{Ext}(X, Y) \). If

\[
\begin{array}{c}
O \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow O
\end{array}
\]

represents an element of \( E_{p\alpha}(C, A) \), then the sequences

\[
\begin{array}{c}
O \longrightarrow \text{Hom}(G, A) \longrightarrow \text{Hom}(G, B) \longrightarrow \text{Hom}(G, C) \longrightarrow E_{p\alpha}(G, A)
\end{array}
\]

\[
\longrightarrow E_{p\alpha}(B, G) \longrightarrow E_{p\alpha}(G, C)
\]

and

\[
\begin{array}{c}
O \longrightarrow \text{Hom}(C, G) \longrightarrow \text{Hom}(B, G) \longrightarrow \text{Hom}(A, G) \longrightarrow E_{p\alpha}(C, G)
\end{array}
\]

\[
\longrightarrow E_{p\alpha}(B, G) \longrightarrow E_{p\alpha}(A, G)
\]

are exact for all groups \( G \).

**Proof.** It follows from Theorem 19 that the family of monomorphisms \( Y \longrightarrow W \) such that

\[
\begin{array}{c}
O \longrightarrow Y \longrightarrow W \longrightarrow X \longrightarrow O
\end{array}
\]

belongs to \( S_{p\alpha}(X, Y) \) form an h.f. class \([8]\), so that
\( E_{p\alpha}(X, Y) \) is a subgroup of \( \text{Ext}(X, Y) \), and by Theorem 3.1 of \([8]\), the sequences (1) and (8) are exact.

5. **OTHER LONG EXACT SEQUENCES WITH \( p^\alpha\text{Ext}(B, A) \) AND \( \text{Ext}(B, A) \)**

We now look at other long exact sequences involving \( \text{Ext} \) and \( p^\alpha\text{Ext} \) and obtain several interesting corollaries.
Theorem 21. If $A$ is a subgroup of $\phi^G$, then the image of the map $f: \text{Ext}(B,A) \to \text{Ext}(B,G)$ induced by the inclusion $A \subseteq G$ is a subgroup of $\phi^\text{Ext}(B,G)$, and hence the sequence

(1) \hspace{1cm} 0 \to \text{Hom}(B,A) \to \text{Hom}(B,G) \to \text{Hom}(B,G/A) \to \text{Ext}(B,A) \to \phi^\text{Ext}(B,G) \to \phi^\text{Ext}(B,G/A) \to 0

is exact for all $B$. If $\alpha < \omega + \omega$, then the sequence

(2) \hspace{1cm} \phi^\text{Ext}(B,G) \to \phi^\text{Ext}(B,G/A) \to 0

is exact for all $B$.

Proof. The proof is by induction on $\alpha$. Let $\alpha = 1$ and suppose

(3) \hspace{1cm} 0 \to G \to X \to B \to 0

represents an element in the image of $f$. If

0 \to A \to Y \to B \to 0

represents an element of $\text{Ext}(B,A)$ which maps onto (3), we may assume $X = (G \otimes Y)/K$, where $K = \{(a,-a): a \in A\}$. Let $x$ be in $G \cap pX$. (We identify $G$ with its image in $X$.) Then $x = (u,0) + K = p((v,y) + K)$ for some $u, v$ in $G$, $y$ in $Y$. Then $u = pv + a$ for some $a$ in $A$. But $a = pw$ for some $w$ in $G$ since $A \subseteq pG$, so $u = pv + pw$, and $(u,0) + K = p((v+w,0) + K)$ is in $pG$. The other inclusion is obvious, so $G \cap pX = pG$. Now by Theorem 1, (3) represents an element of $\phi\text{Ext}(B,G)$, and the assertion holds for $\alpha = 1$. Assume the assertion holds for $\alpha < \beta$. If $\beta = \delta + 1$, the maps

$A \to \phi^G \to G$ and the induction hypothesis yield maps

$\text{Ext}(B,A) \to \text{Ext}(B,\phi^G) = \text{Ext}(B, \phi^G) \to \text{Ext}(B, \phi^G) \to \phi\text{Ext}(B,G)) = \phi\text{Ext}(B,G)$, and the composition of these maps, $\text{Ext}(B,A) \to \phi\text{Ext}(B,G) \subseteq \text{Ext}(B,G)$, is the map $f$.

If $\beta$ is a limit ordinal, the induction hypothesis yields the map $\text{Ext}(B,A) \to \phi\text{Ext}(B,A)$ for each $\delta < \beta$, and thus the image of $f$ is a subgroup of $\bigcap_{\delta < \beta} \phi\text{Ext}(B,G)) = \phi\text{Ext}(B,G)$. Hence the assertion holds for $\alpha = \beta$, and the exactness of (1) follows.

Suppose $\alpha < \omega + \omega$. Let

0 \to G/A \to X \to B \to 0

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represent an element of $p^\alpha \text{Ext}(B,G/A)$. There is an element in $\text{Ext}(B,G)$ mapping onto it. Let

$$0 \to G \to Y \to E \to 0$$

represent such an element. Then the sequences

$$0 \to G/A \to Y/A \to B \to 0$$

and

$$0 \to G/A \to X \to B \to 0$$

are equivalent, so

$$0 \to G/A \to Y/A \to B \to 0$$

is $p^\alpha$-pure exact. Let $w$ be in $G \cap p^n Y$ where $n$ is finite and $n \leq \alpha$. Then $w + A$ is in $(G/A) \cap p^n(Y/A) = p^n(G/A)$, so $w = p^nv + a$ for some $v$ in $G$, $a$ in $A \subset p^\alpha G \subset p^nG$, and $w$ in $p^nG$. It follows that $G \cap p^n Y = p^nG$. If $\alpha < \omega$, this implies that

$$0 \to G \to Y \to B \to 0$$

represents an element of $p^\alpha \text{Ext}(B,G)$, and we are done.

Suppose $\alpha = \omega + m$, with $m < \omega$. Then by the proof above,

$$p^\alpha \text{Ext}(B,G) \to p^\alpha \text{Ext}(B,G/A) \to 0$$

is exact, and it follows easily that

$$p^m(p^\alpha \text{Ext}(B,G)) \to p^m(p^\alpha \text{Ext}(B,G/A)) \to 0$$

is exact, and this is the desired sequence.

**Theorem 22.** Let $A$ be a subgroup of a group $G$. Then $A$ is a subgroup of $G^\perp = \bigcap_p p^\omega G$, if and only if the sequence

$$0 \to \text{Hom}(B,A) \to \text{Hom}(B,G) \to \text{Hom}(B,G/A) \to \text{Ext}(B,A) \to \text{Pext}(B,G) \to \text{Pext}(B,G/A) \to 0$$

is exact for all groups $B$.

**Proof.** The proof of one implication is similar to the proof of Theorem 31, so is omitted. If the image of $\text{Ext}(B,A) \to \text{Ext}(B,G)$ is a subgroup of $\text{Pext}(B,G)$ for all $B$, let $A \subset D$ with $D$ divisible. Then under the map $\text{Ext}(D/A,A) \to \text{Pext}(D/A,G)$ the element represented by

$$0 \to A \to D \to D/A \to 0$$
maps onto the element of $\text{Pext}(D/A, G)$ represented by the sequence

$$
0 \longrightarrow G \xrightarrow{f} (G \otimes D)/M \xrightarrow{g} D/A \longrightarrow 0
$$

where $M = \{(a, -a) : a \in A\}$, $f(x) = (x, 0) + M$ for $x$ in $G$, and $g((x, d) + M) = d + A$ for $x$ in $G$, $d$ in $D$. Let $a$ be in $A$ and let $n$ be any integer. Then $a = nd$ for some $d$ in $D$, and $f(a) = (a, 0) + M = (0, a) + M = n((0, d) + M)$ is in $f(G) \cap n((G \otimes D)/M) = n(f(G))$. Thus $f(a) = n((x, 0) + M)$ for some $x$ in $G$ and it follows that $a = nx$ is in $nG$. Thus $a$ is in $\cap nG = G^1$, and $A \subseteq G^1$.

**Corollary 3.** If $B$ is a direct sum of cyclic groups, then

$$
\text{Ext}(B, A) \cong \text{Ext}(B, A/A^1)
$$

for all groups $A$.

**Proof.** By Theorem 22, the sequence

$$
\text{Hom}(B, A/A^1) \longrightarrow \text{Ext}(B, A^1) \xrightarrow{f} \text{Pext}(B, A) = 0
$$

is exact. Thus in the exact sequence

$$
\text{Ext}(B, A^1) \xrightarrow{f} \text{Ext}(B, A) \xrightarrow{g} \text{Ext}(B, A/A^1) \longrightarrow 0,
$$

$\text{Im} f = 0$ implies $g$ is an isomorphism.

A slight modification of the above proof yields

**Corollary 4.** If $B$ is $\alpha$-projective, then

$$
\text{Ext}(B, A) \cong \text{Ext}(B, A/\alpha A)
$$

for all groups $A$.

**Corollary 5.** For any group $A$ and any divisible group $D$,

$$
\text{Pext}(D, A) \cong \text{Pext}(D, A/A^1) \otimes \text{Ext}(D, A^1).
$$

**Proof.** By Theorem 22, the sequence

$$
\text{Hom}(D, A/A^1) = 0 \longrightarrow \text{Ext}(D, A^1) \longrightarrow \text{Pext}(D, A) \longrightarrow \text{Pext}(D, A/A^1) \longrightarrow 0
$$

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is exact. For any prime \( p \),

\[
(\operatorname{Ext}(\mathbb{Z}(p^\infty), A/A^1))_p \cong (A/A^1)_p
\]

(see [7]) has no elements of infinite height, that is \( \operatorname{Pext}(\mathbb{Z}(p^\infty), A/A^1) \) is torsion free. The divisible group \( D \) is a direct sum of copies of the rationals and copies of \( \mathbb{Z}(p^\infty) \) for various primes \( p \). Since \( \operatorname{Ext}(\mathbb{Q}, X) \) is torsion free for all groups \( X \) [7], it follows that \( \operatorname{Pext}(D, A/A^1) \) is torsion free. Then since \( \operatorname{Ext}(D, A^1) \) is cotorsion plus divisible, the sequence above splits.

In a similar manner one can prove

**Corollary 6.** Let \( D \) be a divisible group with \( D \) a \( p \)-group. Then for any group \( A \), \( \operatorname{Ext}(D, p^\alpha A) \) is isomorphic to a summand of \( p^\alpha \operatorname{Ext}(D, A) \). If \( \alpha < \omega + \omega \), then

\[
\operatorname{Ext}(D, A) \cong \operatorname{Ext}(D, A/p^\alpha A) \oplus \operatorname{Ext}(D, p^\alpha A).
\]

This implies immediately

**Corollary 7.** If \( A^1 \cong B^1 \) and \( A/A^1 \cong B/B^1 \), then

\[
\operatorname{Pext}(D, A) \cong \operatorname{Pext}(D, B)
\]

for all divisible groups \( D \).

If \( \alpha < \omega + \omega \), \( p^\alpha A \cong p^\alpha B \), and \( A/p^\alpha A \cong B/p^\alpha B \), then

\[
\operatorname{Ext}(E, A) \cong \operatorname{Ext}(E, B)
\]

for all divisible groups \( E \) such that \( E_t = E_p \).

In light of Theorem 22, it is only natural to ask the question whose answer is given in

**Theorem 23.** Let

\[
0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0
\]

be exact. Then

\[
0 \longrightarrow \operatorname{Hom}(B/A, G) \longrightarrow \operatorname{Hom}(B, G) \longrightarrow \operatorname{Hom}(A, G) \longrightarrow \operatorname{Ext}(B/A, G)
\]

\[
\longrightarrow \operatorname{Pext}(B, G) \longrightarrow \operatorname{Pext}(A, G) \longrightarrow 0
\]

is exact for all groups \( G \) if and only if \( A \) contains \( B_t \).

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Proof. Suppose \( B_t \subseteq A \), and
\[
0 \to G \to X \to B/A \to 0
\]
represents an element of \( \text{Ext}(B/A,G) \). The image of this element in \( \text{Ext}(B,G) \) is represented by
\[
0 \to G \to Y \to B \to 0,
\]
where \( Y = \{(x,b) : x \in X, b \in B, f(x) = b + A\} \), and the maps are the obvious ones. We identify \( G \) with its image \( \{(w,0) : w \in G\} \) in \( Y \). Let \((w,0)\) be in \( G \cap nY \), \((w,0) = n(x,b)\), such that \( f(x) = b + A \). Then \( nb = 0 \) implies that \( b \) is in \( A \) since \( B_t \subseteq A \), and \( f(x) = 0 \) implies that \( x \) is in \( G \). Therefore \((w,0) = n(x,0)\) is in \( nG \), and we have
\[
0 \to G \to Y \to B \to 0
\]
pure exact. It remains to show that \( \text{Pext}(B,G) \to \text{Pext}(A,G) \) is an epimorphism. Let
\[
0 \to G \to C \to D \to 0
\]
be a pure injective resolution of \( G \) with \( D \) divisible (the existence of such a resolution is proved independently in Theorem 29). This yields a commutative diagram
\[
\begin{array}{ccc}
\text{Hom}(B,D) & \to & \text{Hom}(A,D) \\
\downarrow & & \downarrow \\
\text{Pext}(B,G) & \to & \text{Pext}(A,G) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]
with exact row and columns. Diagram chasing yields the desired result.

Now assume the image of the map \( \text{Ext}(B/A,G) \to \text{Ext}(B,G) \) is a subgroup of \( \text{Pext}(B,G) \) for all groups \( G \), and let \( G \) be torsion free. Let
\[
(1) \quad 0 \to G \to X \to (B_t + A)/A \to 0
\]
represent an element of \( \text{Ext}((B_t + A)/A,G) \). The inclusion map \((B_t + A)/A \to B/A\) yields the exact sequence
\[
\text{Ext}(B/A,G) \to \text{Ext}((B_t + A)/A,G) \to 0
\]
Let
\[
(2) \quad 0 \to G \to X' \xrightarrow{f} B/A \to 0
\]
represent an element mapping onto the element represented by (1). We assume \( X \subseteq X' \). From the map \( \text{Ext}(B/A, G) \) \( \text{Pext}(B, G) \), we get the pure exact sequence

\[ 0 \rightarrow G \rightarrow Y \rightarrow B \rightarrow 0 \]

where \( Y = \{(x, b): x \in X', b \in B, f(x) = b + A\} \), and the maps are the obvious ones. Now let \( w \) be in \( G \cap nX \), \( w = nx \) for some \( x \) in \( X \). Since \( x \) is in \( X \), \( f(x) \) is in \( (B_t + A)/A \), say \( f(x) = t + A \), with \( t \) in \( B_t \). Let \( m = o(t) \), so that \( m(w, 0) = mn(y, t) \in G \cap mnY = mnG \), and \( m(w, 0) = mn(v, 0) \) with \( v \) in \( G \).

But \( m(w - nv) = 0 \) and \( G \) torsion free implies \( w = nv \). Thus \( G \) is pure in \( X \). This result implies that \( \text{Ext}((B_t + A)/A, G) = \text{Pext}((B_t + A)/A, G) \) for all torsion free groups \( G \), so that \( \text{Ext}((B_t + A)/A, G) \) is divisible whenever \( G \) is torsion free. But \( (B_t + A)/A \) torsion implies \( \text{Ext}((B_t + A)/A, G) \) is reduced, so \( \text{Ext}((B_t + A)/A, G) = 0 \) whenever \( G \) is torsion free. Now take a free resolution

\[ 0 \rightarrow K \rightarrow F \rightarrow (B_t + A)/A \rightarrow 0. \]

Since \( K \) is torsion free, this sequence splits, implying that \( (B_t + A)/A = 0 \) and hence that \( B_t \subseteq A \).

**Corollary 8.** Let \( A \) be a subgroup of a group \( B \). If \( C \) is algebraically compact, then

\[ \text{Ext}(B, C) \cong \text{Ext}(A, C) \]

whenever \( B_t \subseteq A \).

**Proof.** This follows at once from the exactness of

\[ \text{Ext}(B/A, C) \xrightarrow{f} \text{Pext}(B, C) \rightarrow 0 \]

and

\[ \text{Ext}(B/A, C) \xrightarrow{f} \text{Ext}(B, C) \rightarrow \text{Ext}(A, C) \rightarrow 0. \]

A proof similar to that of Theorem 23 yields

**Theorem 24.** Let

\[ 0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0 \]

be exact. Then the sequence
\[ 0 \rightarrow \text{Hom}(B/A, G) \rightarrow \text{Hom}(B, G) \rightarrow \text{Hom}(A, G) \rightarrow \text{Ext}(B/A, G) \]
\[ \rightarrow p^n \text{Ext}(B, G) \rightarrow p^n \text{Ext}(A, G) \rightarrow 0 \]

is exact for all groups $G$ if and only if $B[p^n] \subseteq A$.

6. **PROJECTIVES AND INJECTIVES FOR $p^a \text{Ext}(B, A)$**

We conclude with a brief discussion of the $p^a$-injectives and $p^a$-projectives. For completeness' sake the following two theorems are included, both due to Nunke.

**Theorem 25.** A $p^a$-injective group has the form $D \oplus C$ with $D$ divisible, $C$ cotorsion, and $p^aC = 0$. A $p^a$-projective group has the form $F \oplus T$ with $F$ free and $T$ a $p$-group.

The first statement is given by Nunke in [11]. The second statement follows from Theorem 8, since a $p^a$-projective group is also $p^\infty$-projective.

It has been mentioned earlier that for each group $A$ there are $p^a$-pure extensions
\[ 0 \rightarrow K \rightarrow P \rightarrow A \rightarrow 0 \]
and
\[ 0 \rightarrow A \rightarrow I \rightarrow E \rightarrow 0 \]
with $P$ $p^a$-projective and $I$ $p^a$-injective. (See [11].) By iteration one can obtain $p^a$-projective and $p^a$-injective resolutions for each group $A$.

**Theorem 26.** The following statements are equivalent.

1. If $C$ is cotorsion and $p^aC = 0$, then $C$ is $p^a$-injective.

2. Each group $A$ is a $p^a$-pure subgroup of a $p^a$-injective group such that no $p^a$-injective is properly contained between them.

3. An inverse limit of reduced $p^a$-injectives is $p^a$-injective.

4. Every subgroup of a $p^a$-projective is $p^a$-projective.
These properties are true for all \( \alpha < \omega + \omega \), but not for \( \omega + \omega \) itself.

The next theorems give perhaps more insight into \( p^{\alpha} \)-projectives.

**Lemma 5.** If \( p^\alpha G = 0 \) and \( S \) is a subgroup of \( G[p] \), then \( p^{\alpha+1}(G/S) = 0 \).

**Proof.** Since \( G/G[p] \cong pG \) and \( p^\alpha(pG) \subseteq p^\alpha G = 0 \), then \( p^\alpha(G/G[p]) = 0 \). Now \( G/(G[p]) \cong (G/S)/(G[p]/S) \), so \( 0 = p^\alpha(G/(G[p])) \cong p^\alpha((G/S)/(G[p]/S)) \supset (p^\alpha(G/S) + (G[p]/S))/(G[p]/S) \) implies \( p^\alpha(G/S) \subseteq G[p]/S \), so that \( p^{\alpha+1}(G/S) = p(p^\alpha(G/S)) \subseteq p(G[p]/S) = 0 \).

**Theorem 27.** If

\[
0 \longrightarrow S \longrightarrow P \longrightarrow P/S \longrightarrow 0
\]

is exact with \( pS = 0 \) and \( P/S \) \( p^\alpha \)-projective, then \( P \) is \( p^{\alpha+1} \)-projective. If \( \alpha < \omega + \omega \) and \( P \) is \( p^{\alpha+1} \)-projective, then there is a subgroup \( S \subseteq P[p] \) with \( P/S \) \( p^\alpha \)-projective.

**Proof.** Suppose

\[
0 \longrightarrow S \longrightarrow P \longrightarrow P/S \longrightarrow 0
\]

is exact with \( pS = 0 \) and \( P/S \) \( p^\alpha \)-projective. If \( \alpha \) is finite, it follows immediately from Theorem 2 that \( P \) is \( p^{\alpha+1} \)-projective, so we may assume \( \alpha \geq \omega \). For any group \( X \) there is an exact sequence

\[
0 \longrightarrow \text{Hom}(S,X) \overset{f}{\longrightarrow} \text{Ext}(P/S,X) \overset{g}{\longrightarrow} \text{Ext}(P,X) \longrightarrow \text{Ext}(S,X) \longrightarrow 0.
\]

By Lemma 5, \( p^\alpha \text{Ext}(P/S,X) = 0 \), and \( p(f(\text{Hom}(S,X))) = 0 \) imply that \( p^{\alpha+1}(\text{Im} g) \cong p^{\alpha+1}(\text{Ext}(P/S,X)/f(\text{Hom}(S,X))) = 0 \). Now \( p(\text{Ext}(P,X)/\text{Im} g) \cong p\text{Ext}(S,X) = 0 \), so that \( p\text{Ext}(P,X) \subseteq \text{Im} g \) and \( p^{\alpha+1}(p\text{Ext}(P,X)) \subseteq p^{\alpha+1}(\text{Im} g) = 0 \). Since \( \alpha \geq \omega \), \( \alpha = \omega + \alpha \), and \( p^{\alpha+1}\text{Ext}(P,X) = p^{\alpha+1}\text{Ext}(P,X) = p^{\alpha+1}\text{Ext}(P,X) = 0 \), implying that \( P \) is \( p^{\alpha+1} \)-projective.

Let \( \alpha < \omega + \omega \), and let \( P \) be \( p^{\alpha+1} \)-projective. If \( \alpha < \omega \) then \( p^{\alpha+1}P = 0 \), and then \( p^\alpha(P/(P[p])) = 0 \) implies
$P/(P[p])$ is $p^g$-projective. Assume $a \leq \omega$. Let

$$
\begin{array}{ccc}
0 & \rightarrow & P \xrightarrow{\epsilon} D \rightarrow D/P \rightarrow 0
\end{array}
$$

be a divisible resolution of $P$, and let

$$
\begin{array}{ccc}
0 & \rightarrow & K \rightarrow G \rightarrow D \rightarrow 0
\end{array}
$$

be a $p^g$-projective resolution of $D$. Multiplying the element represented by (1) by $p$, we obtain the element of $p^{a+1}\text{Ext}(D,K)$ represented by

$$
\begin{array}{ccc}
0 & \rightarrow & K \rightarrow Y \xrightarrow{\epsilon} D \rightarrow 0,
\end{array}
$$

where $Y = (K \oplus G)/M$ with $M = \{(pk,-k) : k \in K\}$, and the maps are the obvious ones. Define $f: Y \rightarrow G$: $f((k,v) + M) = k + pv$. Then $f$ maps $Y$ onto $K + pG$ and, since $G/K$ is divisible, $K + pG = G$, and $f$ is an epimorphism. Thus we have the exact sequence

$$
\begin{array}{ccc}
0 & \rightarrow & S \rightarrow Y \xrightarrow{f} G \rightarrow 0,
\end{array}
$$

where $S = \text{Ker } f$. If $(k,v) + M$ is in $S$, then $k = -pv$ and $(pk,pv) = (pk,-k)$ is in $M$. Thus $pS = 0$. Under the map $p^{a+1}\text{Ext}(D,K) \rightarrow p^{a+1}\text{Ext}(P,K)$, the element represented by

$$
\begin{array}{ccc}
0 & \rightarrow & K \rightarrow Y \xrightarrow{\epsilon} D \rightarrow 0
\end{array}
$$

maps onto the element of $p^{a+1}\text{Ext}(P,K)$ represented by

$$
\begin{array}{ccc}
0 & \rightarrow & K \rightarrow Y' \rightarrow P \rightarrow 0,
\end{array}
$$

where $Y' = g^{-1}(P)$. This sequence splits since $P$ is $p^{a+1}$-projective, and we may identify $P$ with a summand of $Y'$. Thus we have the exact sequence

$$
\begin{array}{ccc}
0 & \rightarrow & (S \cap P) \rightarrow P \rightarrow P/(S \cap P) \rightarrow 0
\end{array}
$$

with $p(S \cap P) = 0$. Moreover, since $P/(S \cap P) \cong (P + S)/S \subseteq Y/S$, which is $p^a$-projective, $P/(S \cap P)$ is $p^a$-projective by Theorem 26, (4).

It is well known that a group $P$ is pure-projective if and only if $P$ is a direct sum of cyclic groups. Also, a group $P$ is $p^g$-projective if and only if $P$ is a direct sum of cyclic groups with $P_t = P_p$. With a slight modification in the proof of Theorem 27, one obtains
Theorem 27'. Let $\eta$ be a finite ordinal. A group $P$ is $p^{\omega+\eta}$-projective if and only if there is a subgroup $S \subseteq P[p^\eta]$ with $P/S$ a direct sum of cyclic groups, and with $(P/S)_{\mu} = (P/S)_{\mu}$. 

Lemma 6. Let $$0 \longrightarrow A \longrightarrow X \overset{f}{\longrightarrow} G \longrightarrow 0$$ represent an element of $p^{\alpha} \text{Ext}(G,A)$, and let $$0 \longrightarrow B \longrightarrow Y \overset{g}{\longrightarrow} G \longrightarrow 0$$ be any exact sequence. Then the sequence $$0 \longrightarrow H \longrightarrow X \otimes Y \overset{h}{\longrightarrow} G \longrightarrow 0$$ represents an element of $p^{\alpha} \text{Ext}(G,H)$, where $h(x,y) = f(x) + g(y)$ and $H = \ker h$.

Proof. There is a homomorphism $k: A \longrightarrow H$ with $k(a) = (a,0)$. This induces a map $$p^{\alpha} \text{Ext}(G,A) \longrightarrow p^{\alpha} \text{Ext}(G,H)$$ and the image of (1) under this map may be represented by the exact sequence 

\begin{equation}
0 \longrightarrow H \overset{s}{\longrightarrow} (H \otimes X)/N \overset{t}{\longrightarrow} G \longrightarrow 0
\end{equation}

where $N = \{(a,0,-a) : a \in A^1, s(x,y) = (x,y,0) + N \text{ for } (x,y) \text{ in } H, \text{ and } t((x,y,x') + N) = f(x')\}$. Define $u: (H \otimes X)/N \longrightarrow X \otimes Y$ by $u((x,y,x') + N) = (x + x', y)$. Then $u$ is a homomorphism, and for $(x,y)$ in $H$, $u(x,y) = u((x,y,0) + N) = (x,y)$. If $(x,y,x') + N$ is in $(H \otimes X)/N$, then $hu((x,y,x') + N) = h(x + x', y) = f(x + x') + g(y) = f(x') = t((x,y,x') + N)$. Now the commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & H \\
& \downarrow{id} & \searrow{u} & \downarrow{id} \\
0 & \longrightarrow & H \longrightarrow X \otimes Y \overset{h}{\longrightarrow} G \longrightarrow 0
\end{array}
\]

shows that the sequences are equivalent, and hence $$0 \longrightarrow H \longrightarrow X \otimes Y \overset{h}{\longrightarrow} G \longrightarrow 0$$
represents an element of \( p^{a \Ext(G,H)} \).

**Theorem 28.** If \( a \) is a limit ordinal, every \( p^a \)-projective group is a direct summand of a direct sum of \( p^\beta \)-projective groups for ordinals \( \beta < a \).

**Proof.** Let \( P \) be \( p^a \)-projective and for each \( \beta < a \), let

\[
0 \to K_\beta \to P_\beta \xrightarrow{f_\beta} P \to 0
\]

be \( p^\beta \)-pure with \( P_\beta \) \( p^\beta \)-projective. Let \( f = \sum_{\beta < a} f_\beta \), and \( K = \Ker f \), and consider the exact sequence

\[
(1) \quad 0 \to K \to \sum_{\beta < a} P_\beta \xrightarrow{f} P \to 0.
\]

If \( \delta < a \), let \( g_\delta = \sum_{\beta \neq \delta} f_\beta \), and apply Lemma 8 to the exact sequences

\[
0 \to \Ker f_\delta \to P_\delta \xrightarrow{f_\delta} P \to 0
\]

\[
0 \to \Ker g_\delta \to \sum_{\beta \neq \delta} P_\beta \xrightarrow{g_\delta} P \to 0
\]

to get (1) \( p^\delta \)-pure. It follows that (1) is \( p^a \)-pure, hence it splits and \( P \) is isomorphic to a summand of \( \sum_{\beta < a} P_\beta \).

**Corollary 9.** Every \( p^{\omega+\omega} \)-projective is an extension of a direct sum of cyclic groups by a direct sum of cyclic groups.

**Proof.** Let \( P \) be \( p^{\omega+\omega} \)-projective, and let

\[
0 \to K_\beta \to P_\beta \to P \to 0
\]

be a \( p^\beta \)-projective resolution of \( P \) for each \( \beta < \omega + \omega \). By Theorem 28 \( P \) is isomorphic to a summand of \( \sum_{\beta < \omega + \omega} P_\beta \). By Theorem 27 there is an exact sequence

\[
0 \to S_\beta \to P_\beta \to P_\beta/S_\beta \to 0
\]

for each \( \beta < \omega + \omega \) with \( \beta \) not a limit ordinal, where \( pS_\beta = 0 \), and \( P_\beta/S_\beta \) is \( p^{\beta-1} \)-projective. Inductively we may get sequences

\[
0 \to T_\beta \to P_\beta \to P_\beta/T_\beta \to 0
\]

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with $T_\beta$ bounded and $P_\beta/T_\beta$ a direct sum of cyclic groups.
(Let $T_\beta = 0$ if $\beta = 0$ or $\beta = \omega$.) Let $T = \sum_{\beta < \omega + \omega} T_\beta$. Then the sequence

$$0 \rightarrow T \cap P \rightarrow P \rightarrow \frac{P \cap T}{P} \rightarrow 0$$

is exact, and $T \cap P$ is a subgroup of a direct sum of cyclic groups, hence is a direct sum of cyclic groups. Also

$$\frac{P}{(P \cap T)} \cong \left( \frac{P + T}{T} \right) \subset \left( \sum_{\beta < \omega + \omega} P_\beta \right)/T \cong \sum_{\beta < \omega + \omega} \left( \frac{P_\beta}{T_\beta} \right)$$

implies $\frac{P}{(P \cap T)}$ is a direct sum of cyclic groups.

We now turn to the $p^\alpha$-injectives. The proofs of the following lemma and theorem are similar to those of Lemma 6 and Theorem 28, and are omitted.

**Lemma 7.** If

$$0 \rightarrow A \overset{f}{\rightarrow} B \rightarrow \frac{B}{A} \rightarrow 0$$

represents an element of $p^\alpha \text{Ext}(B/A, A)$, and

$$0 \rightarrow A \overset{g}{\rightarrow} C \rightarrow \frac{C}{A} \rightarrow 0$$

is any exact sequence, then

$$0 \rightarrow A \overset{h}{\rightarrow} B \oplus C \rightarrow \frac{(B \oplus C)}{h(A)} \rightarrow 0$$

is $p^\alpha$-pure, where $h(a) = (f(a), g(a))$.

**Theorem 29.** If $\alpha$ is a limit ordinal, then every $p^\alpha$-injective is a summand of a direct product of $p^\beta$-injectives for ordinals $\beta < \alpha$.

The following theorems give explicit constructions of $p^\alpha$-projective resolutions for ordinals $\alpha$ with $\omega \not\leq \alpha < \omega + \omega$. Given a group $G$, let $C(G)$ denote the direct sum of cyclic groups with the elements of $G$ as generators, these elements having the same order in $C(G)$ as in $G$. Theorem 30 appears in [7].

**Theorem 30.** Given a group $G$, the sequence

$$0 \rightarrow K \rightarrow C(G) \overset{g}{\rightarrow} G \rightarrow 0$$

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where \( g([n_a]_{a \in G}) = \sum_{a \in G} (n_a) \) for \([n_a]_{a \in G}\) in \(C(G) = \sum_{a \in G} 2a\), and \( K = \text{Ker} \, g \), is a pure-projective resolution of \( G \).

**Theorem 31.** Let \( F \xrightarrow{f} G \longrightarrow O \) be exact with \( F \) free. Then the sequence

\[
0 \longrightarrow K \longrightarrow F \otimes C(G_p) \xrightarrow{g} G \longrightarrow O
\]

is a \( p^\omega \)-projective resolution of \( G \), where \( g((x, [n_a]_{a \in G})) = f(x) + \sum_{a \in G} (n_a) \) and \( K = \text{Ker} \, g \).

**Proof.** The subgroup \( K \) of \( F \otimes C(G_p) \) is a direct sum of cyclic groups, and \( K_p = K_p \), thus \( K \) is \( p^\omega \)-projective. The sequence is \( p^\omega \)-pure if and only if the sequence

\[
0 \longrightarrow K \longrightarrow (F \otimes C(G_p))(p) \longrightarrow G_p \longrightarrow 0
\]

is \( p^\omega \)-pure. Since \((F \otimes C(G_p))(p) = F(p) \otimes C(G_p)\), the \( p^\omega \)-purity of this sequence follows immediately from Theorem 30 and Lemma 6.

The following theorem, together with Theorem 31, gives a method of constructing a \( p^{\omega + n} \)-projective resolution for a group \( G \). Note that the resolution can be constructed so that the kernel \( K \) is a direct sum of cyclic groups.

**Theorem 32.** Let \( G \subset D \) with \( D \) divisible and let

\[
0 \longrightarrow K \longrightarrow P \xrightarrow{f} D \longrightarrow O
\]

be a \( p^\omega \)-projective resolution of \( D \). Then the sequence

\[
0 \longrightarrow K \xrightarrow{h} (K \oplus P)/M \xrightarrow{g} D \longrightarrow O
\]

is a \( p^{\omega + n} \)-projective resolution of \( D \), where \( M = \{(p^k, -k) : k \in K\} \), \( h(k) = (k, 0) + M \) and \( g((k, x) + M) = f(x) \). Hence the sequence

\[
0 \longrightarrow K \xrightarrow{h} g^{-1}(G) \xrightarrow{g} G \longrightarrow O
\]

is a \( p^{\omega + n} \)-projective resolution of \( G \).

**Proof.** This result is contained in part of the proof of Theorem 27.
Theorem 33. Every group $A$ has a pure injective resolution
\[ 0 \rightarrow A \rightarrow I \rightarrow D \rightarrow 0 \]
with $D$ divisible.

Proof. We may assume $A$ is reduced. Then the sequence
\[ 0 \rightarrow Z \rightarrow Q \rightarrow Q/Z \rightarrow 0 \]
yields the exact sequence
\[ 0 \rightarrow \text{Hom}(Z,A) \rightarrow A \rightarrow \text{Ext}(Q/Z,A) \rightarrow \text{Ext}(Q,A) \rightarrow 0, \]
which is pure exact since $\text{Ext}(Q,A)$ is torsion free. Let $E \in \text{Ext}(Q/Z,A)$, divisible, and define $E \oplus (\text{Ext}(Q/Z,A)/\text{Pext}(Q/Z,A))$ by $f(a) = (a, g(a) + \text{Pext}(Q/Z,A))$.

Clearly $f$ is a monomorphism. It follows from Theorem 26 that $E \oplus (\text{Ext}(Q/Z,A)/\text{Pext}(Q/Z,A))$ is pure injective. Let $n$ be a positive integer, and let $(e,x)$ be in $E \oplus \text{Ext}(Q/Z,A)$. Since $\text{Ext}(Q/Z,A)/\text{Pext}(Q/Z,A)$ is divisible, there is an $a$ in $A$ and a $y$ in $\text{Ext}(Q/Z,A)$ with $x = ny + g(a)$. Let $d$ be in $E$ such that $d - a = nd$. Then $(e,x) = n(d,y) + (a,g(a))$. It follows that $\text{Cok} f$ is divisible.

It remains to show that $f(A)$ is pure in $E \oplus (\text{Ext}(Q/Z,A)/\text{Pext}(Q/Z,A))$. Suppose $f(a) = n((e,x) + \text{Pext}(Q/Z,A))$. Then $g(a) + \text{Pext}(Q/Z,A) = nx + \text{Pext}(Q/Z,A)$, and $g(a) = nx + ny$ for some $y$ in $\text{Ext}(Q/Z,A)$. Now since $g(A)$ is pure in $\text{Ext}(Q/Z,A)$, $g(a) = ng(b)$ for some $b$ in $A$. Thus $f(a) = (a,g(a)) = n(b,g(b))$ is in $nf(A)$, and the proof is complete.

Lemma 8. Let the exact sequence
\[ (1) \quad 0 \rightarrow A \rightarrow B \rightarrow f \rightarrow C \rightarrow 0 \]
represent an element of $p^\alpha \text{Ext}(C,A)$. If $H$ is a subgroup of $B$ with $H \subset p^\alpha B$ and $H \cap A = 0$, then the exact sequence
\[ (2) \quad 0 \rightarrow A \rightarrow B/H \rightarrow C/J \rightarrow 0 \]
where $J = f(H)$, represents an element of $p^\alpha \text{Ext}(C/J,A)$.  

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Proof. Let

$$O \rightarrow A \rightarrow E \rightarrow E/A \rightarrow O$$

be $p^a$-pure exact with $E$ $p^a$-injective. This yields the commutative diagram

$$\begin{array}{c}
\text{Hom}(C/J, E/A) \\
\downarrow \\
\text{Hom}(C, E/A) \rightarrow \text{Hom}(C, E/A) \rightarrow p^a \text{Ext}(C, A) \rightarrow O
\end{array}$$

with exact rows. There is a $g$ in $\text{Hom}(C, E/A)$ mapping onto the element of $p^a \text{Ext}(C, A)$ represented by $(1)$, and we may assume $B = \{(e, c) | e \in E, c \in C, g(c) = e + A\}$. Let $\delta : B \rightarrow E$ be the homomorphism defined by $\delta(e, c) = e$ for $(e, c)$ in $B$. Then $t = \delta f^{-1}$ maps $J$ into $E$, and for $j$ in $J$, $t(j) = e$ such that $(e, j)$ is in $H$. Write $E = D \oplus R$ where $D$ is divisible and $R$ is reduced. Since $E$ is $p^a$-injective, $p^a R = 0$ (Theorem 25). The homomorphism $\delta$ carries $H$ into $p^a E = D$, since $H \subseteq p^a B$, and hence $t(J) \subseteq D$. Thus $t$ can be extended to a map $t : C \rightarrow D \subseteq E$. Let $s$ be the natural map $E \rightarrow E/A$. Then $g - st$ is in $\text{Hom}(C, E/A)$ and maps onto the element of $p^a \text{Ext}(C, A)$ represented by $(1)$. If $j$ is in $J$, $g(j) - st(j) = g(j) - (t(j) + A) = 0$. We may now assume $g$ was chosen with $g(J) = 0$. Hence there is a homomorphism $h : C/J \rightarrow E/A$ with the diagram

$$\begin{array}{ccc}
C & \xrightarrow{g} & E/A \\
\downarrow & & \downarrow h \\
C/J & \rightarrow &
\end{array}$$

commutative. Now $h$ maps onto the element of $p^a \text{Ext}(C/J, A)$ represented by

$$O \rightarrow A \rightarrow Y \rightarrow C/J \rightarrow O$$

where $Y = \{(e, c + J) | e \in E, c + J \in C/J, h(c + J) = e + A\}$, with the maps the obvious ones. Since $h(c + J) = g(c) + J$ for $c + J$ in $C/J$, and $B = \{(e, c) | g(c) = e + A\}$, it is clear that the map $B \rightarrow Y : (e, c) \mapsto (e, c + J)$ has kernel $H$ and induces an isomorphism $B/H \rightarrow Y$ with the diagram.
Theorem 34. Given a group $G$ and $n < \omega$, let $f: G \to \text{Ext}(Z(p^\infty), G)/p^{\omega+n}\text{Ext}(Z(p^\infty), G)$ be the map induced by $G \cong \text{Hom}(Z, G) \to \text{Ext}(Z(p^\infty), G)$. and let $G \subset D$ with $D$ divisible. Then the sequence

$$0 \to G \xrightarrow{f} D \oplus (\text{Ext}(Z(p^\infty), G)/p^{\omega+n}\text{Ext}(Z(p^\infty), G)) \to C \to 0$$

where $g(x) = (x, f(x))$ for $x$ in $G$ and $C = \text{Cokernel } g$, is a $p^{\omega+n}$-injective resolution of $G$, and $C$ is divisible.

Proof. Let $I = \text{Ext}(Z(p^\infty), G)$ and $a = \omega + n$. The sequence is clearly exact. By Theorem 26, $D \oplus (I/p^aI)$ is $p^a$-injective. The proof that $C$ is divisible is similar to the proof of Theorem 33, since $\text{Ext}(Z(p^\infty), G)/f'(G) \cong \text{Ext}(Q_p, G)$ is divisible. It remains to show that the sequence is $p^a$-pure. First assume $p^aG = 0$. Then

$$\text{Hom}(Q_p, G) = 0 \to G \cong \text{Hom}(Z, G) \to \text{Ext}(Z(p^\infty), G)$$

$$\to \text{Ext}(Q_p, G) \to 0$$

is exact, that is,

$$0 \to G \xrightarrow{f'} I \to \text{Ext}(Q_p, G) \to 0$$

is exact. Since $Q_p$ is $p$-divisible, $\text{Ext}(Q_p, G)_p = 0$. Also $I_p = f'(G)_p$, so by Theorem 6 the sequence above is $p^\infty$-pure, and in particular, it is $p^a$-pure. Then $G \cap p^aI = p^aG = 0$, and by Lemma 8 the sequence

$$0 \to G \to I/p^aI \to I/(f'(G) \oplus p^aI) \to 0$$

is $p^a$-pure exact. Then by Lemma 6 the sequence

$$0 \to G \xrightarrow{f} D \oplus (I/p^aI) \to C \to 0$$

is $p^a$-pure exact, as desired.
Now let $G$ be any group, and let $J = \text{Ext}(Z(p^\infty), G/p^aG)$. By the paragraph above, the sequence

$$0 \longrightarrow G/p^aG \longrightarrow (D/p^aG) \oplus (J/p^aJ) \longrightarrow C' \longrightarrow 0$$

is $p^a$-pure exact. The sequence

$$0 \longrightarrow p^aG \longrightarrow G \longrightarrow G/p^aG \longrightarrow 0$$

yields the exact sequence

$$\text{Hom}(Z(p^\infty), G/p^aG) = 0 \longrightarrow \text{Ext}(Z(p^\infty), p^aG) \longrightarrow p^a\text{Ext}(Z(p^\infty), G/p^aG) \longrightarrow 0$$

by Theorem 21. Let $H$ be the image of $\text{Ext}(Z(p^\infty), p^aG)$ in the above sequence. Then $p^aI/H \cong p^aJ$. Also $I/H \cong J$, and combining these results,

$$I/p^aI \cong (I/H)/(p^aI/H) \cong J/p^aJ$$

under the natural maps. Since $f(G) \cap D = p^aG$, there is a commutative diagram

$$0 \longrightarrow G/p^aG \longrightarrow (D/p^aG) \oplus (I/p^aI) \longrightarrow C \longrightarrow 0$$

$$0 \longrightarrow G/p^aG \longrightarrow (D/p^aG) \oplus (J/p^aJ) \longrightarrow C' \longrightarrow 0$$

where $G/p^aG \longrightarrow (D/p^aG) \oplus (I/p^aI)$ is the map induced by $f$, and hence these sequences are equivalent. Under the map $\text{Ext}(C, G) \longrightarrow \text{Ext}(C, G/p^aG)$ the element represented by $(1)$ maps onto the element of $p^a\text{Ext}(C, G/p^aG)$ represented by the sequence

$$0 \longrightarrow G/p^aG \longrightarrow (D/p^aG) \oplus (I/p^aI) \longrightarrow C \longrightarrow 0.$$ 

By Theorem 21, the sequence

$$0 \longrightarrow \text{Ext}(C, p^aG) \longrightarrow p^a\text{Ext}(C, G) \longrightarrow p^a\text{Ext}(C, G/p^aG) \longrightarrow 0$$

is exact. It follows that $(1)$ represents an element of $p^a\text{Ext}(C, G)$, since the inverse image of $p^a\text{Ext}(C, G/p^aG)$ in $\text{Ext}(C, G)$ is exactly $p^a\text{Ext}(C, G)$.

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