Vector bundles near a negative curve: moduli and local Euler characteristic

E. Ballico       E. Gasparim

Abstract
We study moduli spaces of vector bundles on a two dimensional neighborhood \( Z_k \) of an irreducible curve \( C \cong \mathbb{P}^1 \) with \( C^2 = -k \) and give an explicit construction of these moduli as stratified spaces. We give sharp bounds for the local Euler characteristic of bundles on \( Z_k \) and prove existence of families of bundles with prescribed numerical invariants.

1 Introduction
We study moduli spaces of rank 2 bundles on a two dimensional neighborhood of an irreducible curve \( C \cong \mathbb{P}^1 \) with negative self-intersection \( C^2 = -k \neq 0 \). We are interested in the behavior of bundles over a small analytic neighborhood of \( C \) inside a smooth surface \( X_k \), and in coherent sheaves near the singular point of the surface \( X_k \) obtained from \( Z_k \) by contracting the curve \( C \). For this “local” problem of bundles near \( C \) it is enough to focus on vector bundles over the total space of \( \mathcal{O}_{\mathbb{P}^1}(−k) \). Hence we take \( Z_k := \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(−k)) \) where \( C \subset Z_k \) is the zero section. We write \( \pi: Z_k \to X_k \) for the map that contracts \( C \) to a point.

We give an explicit construction and a stratification of the moduli of rank two bundles on \( Z_k \).

A bundle \( E \) over \( Z_k \) has splitting type \((j_1, \ldots, j_r)\) if \( E\mid_C \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(j_i) \) with \( j_1 \leq \cdots \leq j_r \). For the moduli problem we concentrate on the case of rank 2 bundles \( E \) with vanishing first Chern class; in this case the splitting type of \( E \) must be \((-j, j)\) for some \( j \geq 0 \), and for short we say that \( E \) has splitting type \( j \). We then define

\[ \mathcal{M}_j(k) = \{ E \to Z_k \mid E|_C \cong \mathcal{O}(j) \oplus \mathcal{O}(-j) \}/\sim. \]

for the set of isomorphism classes of bundles over \( Z_k \) of splitting type \( j \). In this paper we focus on the cases when \( j \geq k \). We then prove:

**Theorem 4.2** The generic set of \( \mathcal{M}_j(k) \) is a complex projective space \( \mathbb{P}^{2j−k−2} \) minus a closed subvariety of codimension at least 2.

However, \( \mathcal{M}_j(k) \) is empty if \( k > 2j−2 \), and is non-Hausdorff for \( j > k \). To stratify \( \mathcal{M}_j(k) \) into Hausdorff components, we then need numerical invariants.
For $E$ a vector bundle over $Z_k$, we define the Artinian sheaf $Q_E$ on $X_k$ by the exact sequence

$$0 \to \pi_* E \to (\pi_* E)^{\vee \vee} \to Q_E \to 0. \quad (1)$$

$E$ has two independent numerical invariants, which we call the height and the width, by analogy with the case of instanton bundles, cf. [Ga4].

**Definition 1.1** We define the height and width of $E$ by

$$h_k(E) := \text{length } R^1 \pi_* E \quad \text{and} \quad w_k(E) := \text{length } Q_E.$$

We then show:

**Theorem 4.5** The pair $(h_k, w_k)$ stratifies $M_j(k)$ into Hausdorff components.

We remark that this pair of invariants gives the coarsest stratification of $M_j(k)$ into Hausdorff components. One can also decompose $M_j(k)$ by local second Chern classes $c := c_2(E) - c_2((\pi_* E)^{\vee \vee})$. However, $M_j(k)$ has non-Hausdorff subspaces with fixed $c$. See [BG1] for the case of an exceptional curve, i.e $k = 1$.

We explicitly calculate sharp bounds for the invariants $(h, w)$, as follows:

**Theorem 2.6** Let $E$ be a rank 2 bundle over $Z_k$ of splitting type $j$ with $j > k$. Set $n = \lfloor \frac{j - 2}{k} \rfloor$, and $\nu = j \mod k$. The following bounds are sharp:

$$j - 1 \leq h_k(E) \leq (j - 1)(n + 1) - k \binom{n}{2}$$

and

$$\binom{k}{2} \leq w_k(E) \leq \binom{j}{2} + \binom{\nu}{2}.$$

**Definition 1.2** ([Bl] def. 3.9) Let $\sigma(\tilde{X}, C) \to (X, x)$ be a resolution of an isolated quotient singularity. Let $\tilde{F}$ be a reflexive sheaf on $\tilde{X}$, set $\mathcal{F} := (\sigma_* \tilde{F})^{\vee \vee}$; notice that there is $\sigma_* \tilde{F} \hookrightarrow \mathcal{F}$. Then the local holomorphic Euler characteristic of $E$ is

$$\chi(x, \tilde{F}) := \chi((\tilde{X}, C), \tilde{F}) := h^0(X, \mathcal{F}/\sigma_* \tilde{F}) + \sum_{1 \leq i \leq n-1} (-1)^{i-1} h^0(X, R^i \sigma_* \tilde{F}).$$

**Corollary 2.7** Let $E$ be a rank 2 bundle over $Z_k$ of splitting type $j$ with $j > k$. Set $n = \lfloor \frac{j - 2}{k} \rfloor$, and $\nu = j \mod k$. The following are sharp bounds for the local holomorphic Euler characteristic of $E$:

$$j - 1 + \binom{k}{2} \leq \chi(E, C) \leq (j - 1)(n + 1) - k \binom{n}{2} + \binom{j}{2} + \binom{\nu}{2}.$$

We then consider the question of existence of vector bundles. We recall the concept of admissible sequence (definition 3.1) and prove the following existence result.
Theorem 3.2 Fix an admissible sequence \( \{a(i,l)\}_{1 \leq i \leq t} \) and let \( E \) and \( F \) be rank \( r \) vector bundles on \( \hat{C} \) with \( \{a(i,l)\}_{1 \leq i \leq t} \) as an associated admissible sequence. Then there exists a flat family \( \{E_s\}_{s \in T} \) of rank \( r \) vector bundles on \( \hat{C} \) parametrized by an integral variety \( T \) and \( s_0, s_1 \in T \) with \( E_{s_0} \cong E \) and \( E_{s_1} \cong F \) such that \( E_s \) has \( \{a(i,l)\}_{1 \leq i \leq t} \) as admissible sequence for every \( s \in T \).

In section 1, we calculate numerical invariants for bundles near negative curves. In section 2 we describe the method of balancing and show existence of moduli of rank 2 bundles on \( Z_k \).

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2 Bounds

Let \( Z_k \) be the total space of \( \mathcal{O}_{\mathbb{P}^1}(-k) \) and \( C \cong \mathbb{P}^1 \) the zero section, so that \( C^2 = -k \); we write \( C_N \) for the \( N \)-th infinitesimal neighborhood of \( C \), \( \hat{C} \) for the formal neighborhood of \( C \) in \( Z_k \), and \( \mathcal{O}(j) \) for the line bundle on \( Z_k \) or on \( \hat{C} \) that restricts to \( \mathcal{O}_{\mathbb{P}^1}(j) \) on \( C \). A vector bundle \( E \) has splitting type \( (j_1, \ldots, j_r) \) if \( E|_C \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(j_i) \) with \( j_1 \geq \cdots \geq j_r \). A result of Griffiths [Gr] implies that \( E \) splits on \( \hat{C} \) if \( j_1 - j_r \leq k + 1 \).

Lemma 2.1 Let \( Z \) be a smooth surface containing a curve \( C \cong \mathbb{P}^1 \) with \( C^2 = -k \). Let \( E \) be a rank \( r \) vector bundle on \( Z \) of splitting type \( j_1 \geq j_2 \geq \cdots \geq j_r \) and assume that \( j_1 - j_r \leq k + 1 \). Then \( E \) splits on the formal neighborhood \( \hat{C} \) of \( C \), that is, \( E|_{\hat{C}} \cong \bigoplus_{i=1}^r \mathcal{O}_{\hat{C}}(j_i) \).

Proof Apply [Gr], Proposition 1.1 and Proposition 1.4. The point is that \( \mathcal{H}om(E,E)|_C \) is a direct sum of line bundles of degree \( \geq -(j_1 - j_r) \), so the first order infinitesimal extensions \( H^1(\hat{C}, \mathcal{H}om(E,E) \otimes \mathcal{O}(-C)) \) vanish. \( \square \)

It is known that a reflexive sheaf on a surface quotient singularity is a direct sum of the tautological sheaves obtained from the irreducible representations of \( G \). For cyclic quotient singularities \( (\frac{1}{n}, a) \) these are just the eigensheaves \( \mathcal{O}(i) \) of the group action, such that \( \pi_* \mathcal{O} = \bigoplus_{i=1}^n \mathcal{O}(i) \). We now study the holomorphic invariants \( h_k(E) = R^k \pi_* E \) and \( w_k(E) = l(Q_E) \) of \( E \) as defined in 1.1.

Remark 2.2 In principle, we need the Theorem on Formal Functions [Ha], p. 276 to calculate \( w \) and \( h \). However, since holomorphic bundles on \( Z_k \) are algebraic (see [Ga3], lemmas 2.1, 2.3) the limit stabilizes at a finite order and it is enough to compute the cohomology on a fixed infinitesimal extension of \( D \) of order \( N \) denoted, where \( N \) is not too small, roughly, twice the degree of the extension class.
We fix once and for all coordinate charts $U = \mathbb{C}^2_u$ and $V = \mathbb{C}^2_v$ on $Z_k$, glued by $\zeta = z^{-1}, v = z^k u$. In these charts, the bundle $O(j)$ has the transition matrix $z^{-1}$. For $a \in \mathbb{Z}$ we denote $\binom{a}{2} := a(a+1)/2$.

Lemma 2.3  
(1) Set $n = \lfloor -\frac{j+2}{k} \rfloor$, then

$$h_k(O(j)) = \begin{cases} \binom{j}{2} & \text{if } j \leq -2, \\ 0 & \text{otherwise.} \end{cases}$$

(II) Set $\nu = j \mod k$, then

$$w_k(O(j)) = \begin{cases} \binom{j}{2} & \text{if } j \geq 0, \\ \binom{j}{2} & \text{if } j \leq 0. \end{cases}$$

**Proof**  
(1) $R^1 \pi_* O(j) = \lim H^1(O_{\nu C}(j))$, with surjective restriction maps. The first result comes from the exact sequences

$$0 \to H^1(O_{(n+1)C}(j)) \to H^1(O_{nC}(j)) \to H^1(O_{nC}(j)) \to 0$$

together with $H^1(O_{C}(j)) = H^1(\mathbb{P}^1, O(j+nz))$ which gives $h_k(O(j)) = \sum_{n=0}^\infty (-j - nk)^+ \text{ (here } ^+ \text{ means the sum of positive terms only).}$

(II) $h(Q)$ equals the dimension of $Q_k$ as a $\mathbb{C}$-vector space, where $o \in X_k$ is the singular point. Since $Q$ is defined by the sequence (1) we need to study the map $\pi_* O(j) \to (\pi_* O(j))^{\vee \vee}$ and compute the dimension of its cokernel as a $\mathbb{C}$-vector space. We first compute the $k_o$-module structure on $M := (\pi_* O(j))_o$. By [GA3, lemma 2.3] it suffices to compute $H^0(C_N, O(j)|_{C_N})$ for large $N$. Note that here $k_o \simeq \mathbb{C}[[x_0, x_1, \cdots, x_k]]/\{x_i x_{i+2} - x_{i+1}^2\}$, for $i = 0, 1, \cdots k-2$ and the contraction map $\pi: Z_k \to X_k$ is given in $z, u$ coordinates by $x_i = z^i u$.

First assume $j \geq 0$. In this case $M$ is generated as a $k_o$-module by the monomials $\beta_i = z^i$ for $0 \leq i \leq j$ with relations $\beta_i x_0 - \beta_{i-1} x_1 = 0$ for $1 \leq i \leq j$. The dual module $M^{\vee}$ has as its single generator the map $M \to k_o$ given by $\beta_i \mapsto x_i^{j+i-1} x_1^{i-1}$. Therefore the cokernel of the evaluation map $\rho: M \to M^{\vee}$ is generated by the monomials $x_0^r x_1^i$ with $s + i < j$. There are $\binom{j}{2}$ such elements, hence in this case $w_k(O(j)) = \binom{j}{2}$.

Second assume $j < 0$. Set $\nu = j \mod k$, hence $j = qk + \nu$. Then for $N$ large, $H^0(C_N, O(j)|_{C_N})$ and hence $M$ is generated by the set of monomials $\alpha_i = z^i u^{q|i|$ for $0 \leq i \leq \nu$ with relations $\alpha_i x_0 - \alpha_{i-1} x_1 = 0$ for $1 \leq i \leq \nu$. Consequently, $M^{\vee}$ has as its single generator the map $M \to k_o$ given by $\alpha_i \mapsto x_0^{j+i} x_1$. The cokernel of the evaluation map $\rho: M \to M^{\vee}$ in this case is generated by the monomials of the form $x_0^r x_1^i$ with $s + i < r$ and there are $\binom{r}{2}$ such elements, hence $w_k(O(j)) = \binom{r}{2}$. \hfill $\square$

Proposition 2.4  
If $E|_C \simeq \bigoplus_{i=1}^r O(j_i)$ then

$$w_k(E) = \sum_{i=1}^r w_k(O(j_i)) \quad \text{and} \quad h_k(E) = \sum_{i=1}^r h_k(O(j_i)). \quad \square$$
Corollary 2.5 Let $E$ be a rank $r$ holomorphic bundle over $Z_k$ such that $E_{|C} \simeq \bigoplus_{i=1}^{r} O_C(j_i)$ with $j_1 \geq j_2 \geq \cdots \geq j_r$ and $j_r - j_1 \geq -k - 1$. Then

$$w_k(E) = \sum_{i=1}^{r} w_k(O(j_i)) \quad \text{and} \quad h_k(E) = \sum_{i=1}^{r} h_k(O(j_i)).$$

**Proof** By Lemma 2.1, $E_{|C} \simeq \bigoplus_{i=1}^{r} O_C(j_i)$. Hence $E$ has the same invariants as the split bundle, and the result follows from Proposition 2.4. □

Bundles with vanishing first Chern class correspond to instantons under the Kobayashi–Hitchin correspondence and are specially interesting for applications to physics, see [LT]. For the case $c_1 = 0$, we also calculate the lower bounds for the numerical invariants $h$ and $w$. The second author [Ga1] proved that holomorphic bundles on $Z_k$ are algebraic extensions of line bundles. By [GA1, Theorem 3.3] a bundle that is an extension

$$0 \to O(j_1) \to E \to O(j_2) \to 0 \quad (2)$$

(with $j_1 \leq j_2$) has transition matrix $\begin{pmatrix} z^{-j_1} & p \\ 0 & z^{-j_2} \end{pmatrix}$, in canonical coordinates, where

$$p = \sum_{i=1}^{[j_2 - j_1 - 2]/k} \sum_{s=ki+j_1+1}^{j_2-1} p_{ik} z^s u^i. \quad (3)$$

**Theorem 2.6** Let $E$ be a rank 2 bundle over $Z_k$ of splitting type $j$ with $j > k$. Set $n = \lfloor \frac{j - 2}{k} \rfloor$, and $\nu = j \mod k$. The following bounds are sharp

$$j - 1 \leq h_k(E) \leq (j - 1)(n + 1) - k \binom{n}{2}$$

and

$$\binom{k}{2} \leq w_k(E) \leq \binom{j}{2} + \binom{\nu}{2}.$$

**Proof** The upper bounds are always attained by the split bundles. In this case, just apply Lemma 2.3 and Proposition 2.4.

Computation of the lower bounds goes as follows. Since $E_{|C}$ splits as $O_C(j) \oplus O_C(-j)$, $R^1 \pi_* E$ surjects to $H^1(O_C(-j))$. This gives the lower bound for $h_k(E)$. Sharpness of this lower bound is proven as in [Ga2, Lemma 4.3].

To calculate the lower bound for $w(E)$ we use the bundle given in canonical coordinates by transition matrix $\begin{pmatrix} z^j & u \\ 0 & z^{-j} \end{pmatrix}$. This bundle is generic and therefore attains the lower bound. As in part (II) of the proof of Lemma 2.3, to calculate $w_k$ we find the generators of $H^0(C_N, E_{|C_N})$ and those will be the generators of $M := (\pi_* E)_0^\wedge$. Set $\beta_i = \begin{pmatrix} 0 \\ z^i \end{pmatrix}$ for $i = 0, \ldots, k$ and $\beta_j^u = \begin{pmatrix} -u \\ z^j \end{pmatrix}$. Then a presentation for $M$ is given by $M = \langle \beta_0, \ldots, \beta_k, \beta_j^u \rangle/R$, where $R$ is the set of
relations $\beta_i x_0 - \beta_{i-1} x_1 = 0$ for $i = 1, \ldots, k$. Standard computations then show that if $\rho: M \to M^{\vee\vee}$ is the inclusion into the bidual then $\text{coker } \rho$ has dimension $\binom{k}{2}$. Hence $w_k(E) = \binom{k}{2}$. □

**Corollary 2.7** Let $E$ be a rank 2 bundle over $Z_k$ of splitting type $j$ with $j > k$. Set $n = \lfloor \frac{j}{2} \rfloor$, and $\nu = j \mod k$. The following are sharp bounds for the local holomorphic Euler characteristic of $E$:

$$j - 1 + \binom{k}{2} \leq \chi(E, C) \leq (j - 1)(n + 1) - k \binom{n}{2} + \binom{j}{2} + \binom{\nu}{2}.$$  

**Proof** It follows directly from the definitions that for $C^2 = -k$ inside a smooth surface, $h^0(X, E/\sigma \ast E) = w_k(E)$ and $h^0(X, R^1 \sigma \ast E) = h_k(E)$. □

We intend to study numerical invariants of bundles over more general exceptional loci in future papers.

### 3 Balancing

We now consider the question of constructing vector bundles with specified numerical invariants. We use the technique of balancing bundles to prove the existence of bundles over $Z_k$ with certain prescribed numerical invariants. These techniques were used in [BG2] to prove the existence of bundles over $Z_1$ with any prescribed numerically admissible invariants, and in [BG3] some properties of balancing on $Z_2$ were given.

Given two bundles $E$ and $E'$ of splitting type $(j_1, \ldots, j_r)$ and $(j_1', \ldots, j_r')$, we say that $E$ is more balanced than $E'$ if $j_1 - j_r \leq j_1' - j_r'$. The advantage of balancing a bundle is that we control the numerical invariants at each step, and we only need to compute numerical invariants for a smaller range of bundles.

The simplest case of balancing is for rank 2 bundles and goes as follows. If $j_1 - j_2 \leq k - 1$, we have won and we stop. If $j_2 \leq j_1 - k$ we make the construction of [BG3], namely an elementary transformation with respect to $O_C(j_2)$ and obtain a new more balanced bundle with splitting type $(j_1, j_2 + k)$. We may also compare the invariants of the two bundles; at the end we reduce to a case with $j_1 - j_2 \leq k - 1$. We now describe how to balance bundles of rank $r \geq 2$.

Let $E$ be a rank $r$ vector bundle over $Z_k$ of splitting type $j_1 \geq j_2 \geq \cdots \geq j_r$. We say that $E$ is balanced if $j_1 \leq j_r + k - 1$. The objective is to balance $E$.

Balancing associates to $E$ the following data:

1. a positive integer $t$ (the number of steps);
2. a finite sequence of $r$-tuples of nonincreasing integers $\{j(i, l)\}$ with $1 \leq i \leq t$ and $1 \leq j \leq r$ (the splitting types) satisfying:
   (a) $j(1, l) = j_l$ for $1 \leq l \leq r$ (the splitting of the bundle $E$)
   (b) $\sum_{1 \leq i \leq r} j(i, l) = \sum_{1 \leq i \leq r} j(1, l) + ki - k$ for $2 \leq i \leq t$ (change of splitting produced by an elementary transformation)
in the latter case we have a sequence. Then there exists a flat family of vector bundles on \( \hat{C} \) with \( \{j(i,l)\}_{1 \leq i \leq t} \) as an associated admissible sequence. Then there exists a flat family \( \{E_s\}_{s \in T} \) of rank \( r \) vector bundles on \( \hat{C} \).
parametrized by an integral variety $T$ and $s_0, s_1 \in T$ with $E_{s_0} \simeq E$ and $E_{s_1} \simeq F$ such that $E_s$ has $\{j(i, l)\}_{1 \leq i \leq t}$ as admissible sequence for every $s \in T$.

**Proof** We use induction on $t$. If $t = 1$ the result is obvious because $E$ and $F$ are split vector bundles with the same splitting type and are hence isomorphic. Assume that $t > 1$ and that the result is true for $t - 1$. Let $E_2, F_2$ be the second bundles associated to $E, F$ respectively. Hence $E_2$ and $F_2$ have $\{j(2, l)\}_{2 \leq i \leq t}$ as an associated admissible sequence. By induction, there is a flat family $\{E'_s\}_{s \in S}$ of rank $r$ vector bundles on $\hat{C}$ and $m_0, m_1 \in S$ with $E_{m_0}' \simeq E_2, E_{m_1}' \simeq F_2$ and such that $E'_s$ has $\{j(i, l)\}_{2 \leq i \leq t}$ as an associated admissible sequence for every $s \in S$. We write $j'_i = j(2, i)$ to simplify notation. By the balancing construction, the bundles $E$ and $F$ fit into exact sequences

$$
0 \rightarrow E \rightarrow E_{2}(C) \rightarrow \mathcal{O}_C(j_1 - k) \rightarrow 0
$$

$$
0 \rightarrow F \rightarrow F_{2}(C) \rightarrow \mathcal{O}_C(j_1 - k) \rightarrow 0.
$$

For every bundle $M$ on $\hat{C}$ having $\{j(i, l)\}_{2 \leq i \leq t}$ as an associated admissible sequence, the set of surjective homomorphisms $t: M(C) \rightarrow \mathcal{O}_C(j_1 - k)$ is parametrized by an integral variety whose dimension depends only on $j_1, j_2$ and $j'_2 = j_1 + j_2 - j'_1 + k$. The kernel $\text{ker} t|_C$ is an extension of $\mathcal{O}_C(j'_1)$ by $\mathcal{O}_C(j_1)$. This extension splits since $j_1 \geq j'_1 + k$, and hence the bundle $\text{ker} t$ has $\{j(i, l)\}_{1 \leq i \leq t}$ as an admissible sequence. Varying $M$ among bundles $E'_s$ for $s \in S$ we get that the set of all such surjections is parametrized by an irreducible nonempty variety $T$. For any fixed ample line bundle $H$ on the $n$th neighborhood of $C$, it follows from the exact sequences in the balancing construction, that the bundles in this family have the same Hilbert polynomial with respect to $H$ and therefore the family is flat. 

\[\square\]

### 4 Moduli

**Definition 4.1** We write

$$
\mathcal{M}_j(k) = \{ E \rightarrow Z_k \mid E|_C \simeq \mathcal{O}(j) \oplus \mathcal{O}(-j) \}/\sim.
$$

for the set of isomorphism classes of bundles over $Z_k$ of splitting type $j$.

Recall from (2) that a bundle $E$ on $Z_k$ of splitting type $j$ is an extension of $\mathcal{O}(j)$ by $\mathcal{O}(-j)$ and is therefore determined by its extension class. In our choice of coordinates, this amounts to saying that $E$ is determined by the pair $(j, p)$ where $j$ is the splitting type and $p$ is a polynomial as in (3). Let $s = \left\lfloor \frac{2j - 2}{t} \right\rfloor$; then $p$ has $N := s(2j - 1) - k(s + 1)$ coefficients. We identify $p$ as an element in $\mathbb{C}^N$ by writing its coefficients in lexicographical order. We then define the equivalence relation $p \sim p'$ if $(j, p)$ and $(j, p')$ define isomorphic bundles over $Z_k$. We give $\mathbb{C}^N$ the quotient topology. There is a bijection $\phi: \mathcal{M}_j(k) \rightarrow \mathbb{C}^N/\sim$. We give $\mathcal{M}_j(k)$ the topology induced by this bijection. Here are some examples.
Example 1 For each $k$, $M_0(k)$ contains only one point, corresponding to the trivial bundle over $\mathbb{Z}_k$. In other words, if a bundle over $\mathbb{Z}_k$ is trivial over the zero section, it is globally trivial.

Example 2 For each $k$, $M_1(k)$ contains only one point. In other words, a holomorphic bundle over $\mathbb{Z}_k$ of splitting type 1 splits. This can be verified directly from formula (3).

Example 3 $M_2(2)$ contains exactly 2 points ([So], Theorem 6.24).

Example 4 $M_2(1) \cong \mathbb{P}^1 \cup \{A, B\}$, where $A$ and $B$ are points, with open sets $U \subset \mathbb{P}^1$ open in the usual topology, $\mathbb{P}^1 \cup \{A\}$, and the whole space ([Ga2], Theorem 4.2).

Example 5 $M_3(2) \cong \mathbb{P}^2 \cup \{A, B\}$, where $A$ and $B$ are points, with open sets $U \subset \mathbb{P}^1$ open in the usual topology, $\mathbb{P}^2 \cup \{A\}$, and the whole space ([So], Theorem 6.35).

Example 6 $M_j(k)$ is non-Hausdorff for $j > k$. This uses Theorem 4.3 below.

Theorem 4.2 $M_j(k)$ has an open dense subspace homeomorphic to a complex projective space $\mathbb{P}^{2j-k-2}$ minus a closed subvariety of codimension at least 2.

Proof [Ga2], Theorem 3.5 showed that the generic set of $M_j(1)$ is a projective space $\mathbb{P}^{2j-3}$ minus a closed subvariety of codimension $\geq 2$. The only modification needed to generalize the proof to $k > 1$ is the calculation of dimension of the generic set. Generic points correspond to bundles that do not split on the first formal neighborhood and for such bundles the only equivalence relation is projectivization, cf. [Ga3] Prop.3.2. The dimension count follows from formula (3) which shows that the $u$-coefficients are $\sum_{s=k-j+1}^{j-1} p_{1s}$. There are $2j-k-1$ coefficients, and after projectivizing, we obtain $\mathbb{P}^{2j-k-2}$. The closed subvariety to be removed contains all points having coefficients of $u$ and of $zu$ both zero; such points are not generic, see [BG1]. □

Theorem 4.3 There is a topological embedding $\Phi: M_j(k) \to M_{j+k}(k)$. The image of $\Phi$ consists of all bundles in $M_{j+k}(k)$ that split on the second formal neighborhood of $C$.

Proof Using the identification $\phi: M_j(k) \to \mathbb{C}^N/\sim$ we define a map

$$\Phi: M_j(k) \to M_{j+k}(k) \quad \text{by} \quad (j, p) \mapsto (j + k, z^k u^2 p).$$

We want to show that $\Phi$ defines an embedding. We first show that the map is well defined. Suppose $(z_j, p_j)$ and $(z'_j, p'_j)$ represent isomorphic bundles. Then
there are coordinate changes \((\frac{a}{c} \frac{b}{d})\) holomorphic in \(z, u\) and \((\frac{\alpha}{\gamma} \frac{\beta}{\delta})\) holomorphic in \(z^{-1}, z^ku\) such that

\[
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta
\end{pmatrix} = \begin{pmatrix}
z^j & p' \\
0 & z^{-j}
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
z^{-j} & -p \\
0 & z^j
\end{pmatrix}.
\]

Therefore these two bundles are isomorphic exactly when the system of equations

\[
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta
\end{pmatrix} = \begin{pmatrix}
a + z^{-j}p'c & z^{2j}b + z^j(p'd - ap) - pp'c \\
0 & d - z^{-j}pc
\end{pmatrix}
\]

\(*\)
can be solved by a matrix \((\frac{a}{c} \frac{b}{d})\) holomorphic in \(z, u\) which makes \((\frac{\alpha}{\gamma} \frac{\beta}{\delta})\) holomorphic in \(z^{-1}, z^k u\).

On the other hand, the images of these two bundles are given by transition matrices \((z^{j+k} \nu'z')\) and \((z^{j+k} \nu''z')\), which represent isomorphic bundles if and only if there are coordinate changes \((\frac{a}{c} \frac{b}{d})\) holomorphic in \(z, u\) and \((\frac{\alpha}{\gamma} \frac{\beta}{\delta})\) holomorphic in \(z^{-1}, zu\) satisfying the equality

\[
\begin{pmatrix}
\bar{\alpha} \\
\bar{\beta} \\
\bar{\gamma} \\
\bar{\delta}
\end{pmatrix} = \begin{pmatrix}
z^{j+k} & z^ku^2p' \\
0 & z^{-j-k}
\end{pmatrix}
\begin{pmatrix}
\bar{a} & \bar{b} \\
\bar{c} & \bar{d}
\end{pmatrix}
\begin{pmatrix}
z^{-j-k} & -z^k u^2p \\
0 & z^{j+k}
\end{pmatrix}.
\]

That is, the images represent isomorphic bundles if the system

\[
\begin{pmatrix}
\bar{\alpha} \\
\bar{\beta} \\
\bar{\gamma} \\
\bar{\delta}
\end{pmatrix} = \begin{pmatrix}
\bar{a} + z^{-j}u^2 p' \bar{c} & z^{2k}(z^j \bar{b} + z^j u^2 (p'd - ap) - u^4 pp' \bar{c}) \\
z^{-2j-2k} \bar{c} & d - z^{-j}u^2 p \bar{c}
\end{pmatrix}
\]

\(**\)
has a solution.

Write \(x = \sum x_iu^i\) for \(x \in \{a, b, c, d, \bar{a}, \bar{b}, \bar{c}, \bar{d}\}\) and choose \(\bar{a}_i = a_{i+k}, \bar{b}_i = b_{i+k}u^2, \bar{c}_i = c_{i+k}u^{-2}, \bar{d}_i = d_{i+k}\). Then if \((\frac{a}{c} \frac{b}{d})\) solves \((*)\), one verifies that \((\frac{\alpha}{\gamma} \frac{\beta}{\delta})\) solves \((***)\), which implies that the images represent isomorphic bundles and therefore \(\phi\) is well defined. To show that the map is injective just reverse the previous argument. Continuity is obvious. Now we observe also that the image \(\phi(M_j)\) is a saturated set in \(M_{j+k}\) (meaning that if \(y \sim x\) and \(x \in \phi(M_j)\) then \(y \in \phi(M_j)\)). In fact, if \(E \in \phi(M_j)\) then \(E\) splits on the 2nd formal neighborhood. Now if \(E' \sim E\) than \(E'\) must also split in the 2nd formal neighborhood therefore the polynomial corresponding to \(E'\) is of the form \(u^2 p'\) and hence \(\phi(z^{-k} p')\) gives \(E'\). Note also that \(\phi(M_j)\) is a closed subset of \(M_{j+k}\) given by the equations \(p_{il} = 0\) for \(i = 1, 2\). Now the fact that \(\phi\) is a homeomorphism over its image follows from the following easy lemma. □

**Remark 4.4** R. Moraru gave us the following coordinate free expression of the map \(\Phi: M_j(k) \rightarrow M_{j+k}(k)\).

\[
\Phi(E) = \otimes O(-k) \circ Elm_{\sigma_c((j+k)} \circ Elm_{\sigma_c(j)}(E),
\]

where \(Elm_L\) denotes the elementary transformation with respect to the line bundle \(L\). Using this coordinate free expression it becomes obvious that \(\Phi\) is well defined.
Lemma 4.5 Let $X \subset Y$ be a closed subset and $\sim$ an equivalence relation in $Y$ such that $X$ is $\sim$ saturated. Then the map $I: X/\sim \to Y/\sim$ induced by the inclusion is a homeomorphism over the image.

Proof Denote the projections by $\pi_X: X \to X/\sim$ and $\pi_Y: Y \to Y/\sim$. Let $F$ be a closed subset of $X/\sim$. Then $\pi_X^{-1}(F)$ is closed and saturated in $X$ and therefore $\pi_Y(\pi_X^{-1}(F))$ is also closed and saturated in $Y$. It follows that $\pi_Y(\pi_X^{-1}(F))$ is closed in $Y/\sim$. □

Theorem 4.6 The pair $(h_k, w_k)$ stratifies $M_j(k)$ into Hausdorff components.

Proof This proof uses the same techniques as that of [BG1], Thm. 4.1. On the first formal neighborhood, we have two possibilities: in the first case we have bundles belonging to the open dense subset $\mathbb{P}^{2j-k-1}$, singled out by having the lowest possible values of the numerical invariants; in the second possibility, at least one of the invariants is strictly higher than the lower bound and such bundles are separated away from the most generic stratum. On the second formal neighborhood, the problem is solved by first separating the most generic stratum from the other ones. For the remaining part of the second neighborhood, one divides the polynomial by $u$ falling back to the same analysis done for the first neighborhood. We are then left only with bundles which split on the second neighborhood. We use induction $j$, assuming that the invariants stratify $M_{j-1}(k)$ into Hausdorff components together with the embedding Theorem 4.3, stating that $\text{Im } \Psi(M_{j-1}(k))$ is the set of bundles on $M_j(k)$ that split on the second neighborhood. □

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Edoardo Ballico,
University of Trento, Department of Mathematics,
I–38050 Povo (Trento), Italia
e-mail: ballico@science.unitn.it

Elizabeth Gasparim,
New Mexico State University, Department of Mathematics,
P.O. Box 30001, Las Cruces, New Mexico 88003, USA
e-mail: gasparim@nmsu.edu