MATH 541- Part I - summary

Material covered before the first exam. Write examples for each of the concepts defined below.

I. POINT SET TOPOLOGY

1. Topological space. A topological space consists of a pair \((X, \tau)\) where \(X\) is a set and \(\tau \subset P(X)\) is a family of subsets of \(X\) satisfying the 3 properties below:
   i) \(\emptyset \in \tau\) and \(X \in \tau\),
   ii) \(U_1, \ldots, U_n \in \tau\) implies \(\cap_{i=1}^{n} U_i \in \tau\),
   iii) \(U_i \in \tau \forall i \in I\) implies \(\cup_{i \in I} U_i \in \tau\).

2. Continuity. Let \((X, \tau_X)\) and \((Y, \tau_Y)\) be topological spaces. A map \(f: X \to Y\) is called continuous if \(A \in \tau_Y \Rightarrow f^{-1}(A) \in \tau_X\).

3. Connectedness. \(X\) is disconnected is there exist \(A, B \in \tau_X\) such that \(X = A \cup B\) and \(A \cap B = \emptyset\). \(X\) is connected if it is not disconnected.

4. Hausdorff. \(X\) is Hausdorff if for every pair of distinct points \(x, y \in X\) there exist open sets \(U_x, U_y \in \tau\) such that \(x \in U_x, y \in U_y\) and \(U_x \cap U_y = \emptyset\).

5. Compactness. \(X\) is compact if every open cover of \(X\) has a finite subcover.

   SOME IMPORTANT RESULTS

6. In \(\mathbb{R}^n\) continuity with epsilons and deltas is equivalent to topological continuity.

7. In \(\mathbb{R}^n\) the topologies defined by the \(\ell_1, \ell_2\) and \(\ell_\infty\) norms are equivalent.

8. If \(X\) is Compact and \(F\) is a closed subset of \(X\), then \(F\) is compact.

9. If \(X\) is Hausdorff and \(K\) is a compact subset of \(X\), then \(K\) is closed.
II. Algebraic Topology

10. **Homotopy of paths.** Let $\alpha : I \rightarrow X$ and $\beta : I \rightarrow X$ be paths in $X$ with same end points, that is, $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. Then $\alpha$ is homotopic to $\beta$ relative $(0,1)$ if there exits a continuous map $H : I \times I \rightarrow X$ such that

   i. $H(s,0) = \alpha(s) \forall s$
   
   ii. $H(s,1) = \beta(s) \forall s$
   
   iii. $H(0,t) = \alpha(0) = \beta(0) \forall t$
   
   iv. $H(1,t) = \alpha(1) = \beta(1) \forall t$.

11. **Fundamental Group.** A loop in $X$ with base point $x$ is a closed path $\alpha$ in $X$ with end points $x$, that is, $\alpha(0) = \alpha(1) = x$. Homotopy equivalence relative $(0,1)$ defines an equivalence relation on the sets of loops in $X$ with base point $x$. This set of equivalence classes forms a group with the operation of *concatenation* of loops, defined by

   \[
   \alpha \beta(s) = \begin{cases} 
   \alpha(2s) \text{ if } 1 \leq s \leq 1/2 \\
   \beta(1-2s) \text{ if } 1/2 \leq s \leq 1
   \end{cases}. 
   \]

   This group is called the *fundamental* group of $X$ at $x$ denoted $\pi_1(X,x)$.

12. **Important result:**

   If $X$ is path-connected then for any $x_1, x_2 \in X$, $\pi_1(X, x_1) \simeq \pi_2(X, x_2)$, and in this case we omit the base point and denote simply $\pi_1(X)$.

13. **Some examples:**

   \[
   \pi_1(S^1) = \mathbb{Z}, \\
   \pi_1(\mathbb{R}^n) = 0, \forall n \geq 1.
   \]

14. **Homotopy equivalence of spaces.** $X$ and $Y$ are homotopy equivalent written $X \sim Y$ if there exist maps $f : X \rightarrow Y$ and $g : X \rightarrow Y$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$.

15. **Some examples:**

   \[
   \mathbb{R}^n \sim \{0\}. \\
   \mathbb{R}^n - \{0\} \sim S^{n-1}.
   \]

16. **Products.** $\pi_1(X \times Y) = \pi_1(X) \oplus \pi_1(Y)$. 

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III. Important Concepts

Write the definitions of:

17. Differentiable manifolds
   A differentiable manifold $M$ of class $C^r$ and dimension $n$ is a topological space satisfying:
   
   - $M$ is locally Euclidean, that is, for every point $x \in M$ there exists an open neighborhood $U$ of $x$ and a homeomorphism $\varphi: U \to \varphi(U) \subset \mathbb{R}^n$, called a local chart around $x$
   - for every pair of local charts $(U, \varphi)$ and $(V, \psi)$ of $M$ having $U \cap V \neq \emptyset$ the composite
     
     \[
     \psi \circ \varphi^{-1}|_{\varphi(U \cap V)}: \varphi(U \cap V) \to \psi(U \cap V)
     \]
     
     is of class $C^r$. This composite is called a transition function.

   Note 1: It is usual to require in the definition that $M$ have a maximal atlas. That is, one defines the notion of compatible charts and demands that all compatible charts belong to the collection of charts associated to $M$. This is convenient, as for instance, it follows that any restriction of a local chart to a subset is also part of the atlas. However, given a manifold as we defined above, with any atlas (= a collection of charts that cover $M$), an application of Zorn’s lemma shows that there exists a unique maximal atlas associated to $M$.

   Note 2: By smooth manifold we mean a manifold of class $C^\infty$. In this course, all our manifolds will be smooth manifolds.

18. Complex manifolds are manifolds with a finer structure, where instead of local charts to $\mathbb{R}^n$ we require local charts to $\mathbb{C}^n$ and instead of differentiable transition functions we require holomorphic (=analytic) transitions functions. Explicitly,

   A complex manifold $M$ of (complex) dimension $n$ is a topological space satisfying:

   - $M$ is locally complex, that is, for every point $x \in M$ there exists an open neighborhood $U$ of $x$ and a homeomorphism $\varphi: U \to \varphi(U) \subset \mathbb{C}^n$, called a local chart around $x$
• for every pair of local charts \((U, \varphi)\) and \((V, \psi)\) of \(M\) having \(U \cap V \neq \emptyset\) the composite
\[
\psi \circ \varphi^{-1}|_{\varphi(U \cap V)} : \varphi(U \cap V) \to \psi(U \cap V)
\]
is holomorphic.

19. **Lie Groups** are manifolds having a group structure. That is, a smooth manifold \(M\) is called a **Lie group** if it is endowed with a group operation \(*\) such that the *multiplication* map \((a, b) \to a * b\) and the inverse operation \(x \mapsto x^{-1}\) are continuous with respect to the manifold structure.