Arthur Cayley and the First Paper on Group Theory

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May 6, 2003

Introduction Arthur Cayley's 1854 paper *On the theory of groups, as depending on the symbolic equation* $\theta^n = 1$ inaugurated the abstract idea of a group [2]. I have used this very understandable paper several times for reading, discussion, and homework in teaching introductory abstract algebra, where students are first introduced to group theory. Many facets of this short paper provide wonderful pedagogical benefits, not least of which is tremendous motivation for the modern theory. More information on teaching with original historical sources is available at our web resource [8], which includes information on the efficacy of teaching with original sources [3, 4, 5, 9, 10], about courses based entirely on original sources [6, 7], and which provides access to books and other resources for teaching with original sources.

Cayley introduces groups as a unifying framework for several critical examples current in the early nineteenth century. These include "substitutions" (permutations, with reference to Galois on solutions of equations), "quaternion imaginaries", compositions arising in the theory of elliptic functions, and transposition and inversion of matrices. These phenomena are either familiar to or easily understandable for our students. Cayley observes that they all have common features, which he abstracts in a first attempt at defining a group axiomatically. He proceeds to develop initial steps of a theory, including stating and using what Lagrange's theorem tells us on orders of elements, which classifies groups of prime order. Then he provides a detailed classification of all groups of order up to six. Cayley gives a brief description of how to form what we call a group algebra, and comments that when applied to the six-element symmetric group on three letters, the group

ARTHUR CAYLEY
1821-1895

algebra does not appear to have anything analogous to the modulus for the quaternions. He ends with a description of two nonabelian groups of higher order.

The richness of Cayley's paper provides a highly-motivated foundation for the beginnings of the theory of groups in an abstract algebra course. A wealth of questions and contrasts arises with students as one studies the paper. First I will display some excerpts, and footnote features that I have focused on with students. Then I will discuss specific pedagogical aspects of incorporating Cayley's paper in the classroom, and the accruing benefits.

Cayley's paper I provide here selected extracts, with my own pedagogical comments as footnotes in order not to interrupt unduly the beautiful flow of Cayley's exposition. I will let his paper essentially speak for itself as an inspiring and powerful tool for learning.

On the theory of groups, as depending on the symbolic equation $\theta^n = 1$
Arthur Cayley

Let θ be a symbol of operation, which may, if we please, have for its operand, not a single quantity x, but a system $(x,\, y, \ldots)$, so that

$$\theta(x,\ y, \ldots) = (x',\ y', \ldots),$$

where x', $y', \ldots$ are any functions whatever of x, $y, \ldots$, it is not even necessary that x', $y', \ldots$ should be the same in number with x, $y, \ldots$. In particular, x', y', &c. may represent a permutation of x, y, &c., θ is in this case what is termed a substitution; and if, instead of a set x, $y, \ldots$, the operand is a single quantity x, so that $\theta x = x' = fx$, θ is an ordinary functional symbol[1]. It is not necessary (even if this could be done) to attach any meaning to a symbol such as $\theta \pm \phi$, or to the symbol 0, ... but the symbol 1 will naturally denote an operation which ... leaves the operand unaltered ... . A symbol $\theta\phi$ denotes the compound operation, the performance of which is equivalent to the performance, first of the operation ϕ, and then of the operation θ; $\theta\phi$ is of course in general different from $\phi\theta$. But the symbols θ, ϕ, ... are in general such that $\theta.\phi\chi = \theta\phi.\chi$, &c., so that $\theta\phi\chi$, $\theta\phi\chi\omega$, &c. have a definite signification independent of the particular mode of compounding the symbols[2]; this will be the case even if the functional operations involved in the symbols θ, ϕ, &c. contain parameters such as the quaternion imaginaries[3] i, j, k; but not if these functional operations contain parameters such as the imaginaries which enter into the theory of octaves[4], and for which, e.g. $\alpha.\beta\gamma$ is something different from $\alpha\beta.\gamma$, a supposition which is altogether excluded from the present paper[5].

[1] It is delightfully unclear just how Cayley's initial general notion of operation really differs from that of function, and this makes good classroom discussion.

[2] One can discuss here why we know this, e.g., for function composition, and whether this is exactly Cayley's actual context.

[3] The eight-element quaternion group is a wonderful example to motivate students.

[4] Also called the octonions, or Cayley numbers, these are an eight-dimensional non-associative real division algebra [1].

[5] This contrast offers opportunity for discussing the sense in which quaternions or octaves are or are not acting as functions, and how their multiplication is connected to function composition.

The order of the factors of a product $\theta\phi\chi\ldots$ must of course be attended to, since even in the case of a product of two factors the order is material; it is very convenient to speak of the symbols θ, $\phi\ldots$ as the first or furthest, second, third, &c., and last or nearest factor. What precedes may be almost entirely summed up in the remark, that the distributive law has no application to the symbols $\theta\phi\ldots$; and that these symbols are not in general convertible[6], but are associative. It is easy to see that $\theta^0 = 1$, and that the index law $\theta^m.\theta^n = \theta^{m+n}$, holds for all positive or negative integer values, not excluding zero[7]. ...

A set of symbols,

$$1,\ \alpha,\ \beta,\ldots$$

all of them different, and such that the product of any two of them (no matter what order), or the product of any one of them into itself, belongs to the set, is said to be a *group*[8] [Cayley's own footnote]. It follows that if the entire group is multiplied by any one of the symbols, either as further or nearer factor, the effect is simply to reproduce the group[9]; or what is the same thing, that if the symbols of the group are multiplied together so as to form a table[10], thus:

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[6] I.e., what we today call commutative.

[7] By indicating that negative exponents are always allowed, Cayley here implicitly seems to assume that his "operations", now metamorphosing seamlessly into "symbols", always have inverses.

[8] [Cayley's footnote]: The idea of a group as applied to permutations or substitutions is due to Galois, and the introduction of it may be considered as marking an epoch in the progress of the theory of algebraical equations.

[9] This raises the question of what Cayley is really requiring for something to be a group. Although not mentioned explicitly, it seems clear from above and below that he has in mind associativity and existence of inverses as implicit. Is this a good example of axiomatization? Will students agree, and think this necessary? All Cayley specifies literally is closure and the existence of an identity. Cayley does not justify his claim that translation always reproduces the group, but it clearly follows from existence of inverses and associativity.

[10] Here we see why we call such a "Cayley table" today.

Nearer factors \ Further factors	1	α	β	..
1	1	α	β	..
α	α	α^2	$\beta\alpha$	
β	β	$\alpha\beta$	β^2	
	:			

that as well each line as each column of the square will contain all the symbols $1,\ \alpha,\ \beta, \ldots$ . ... Suppose that the group

$$1,\ \alpha,\ \beta, \ldots$$

contains n symbols, it may be shown[11] that each of these symbols satisfies the equation

$$\theta^n = 1;$$

so that a group may be considered as representing a system of roots of this symbolic equation[12]. It is, moreover, easy to show that if any symbol α of the group satisfies the equation $\theta^r = 1$, where r is less than n, then that r must be a submultiple of n; it follows that when n is a prime number, the group is of necessity of the form[13]

$$1,\ \alpha,\ \alpha^2,\ \ldots \alpha^{n-1},\ (\alpha^n = 1);$$

and the same may be (but is not necessarily) the case, when n is a composite number. But whether n be prime or composite, the group, assumed to be of the form in question, is in every respect analogous to the system of the roots of the ordinary binomial equation[14] $x^n - 1 = 0$ ... .

[11] This fascinating claim of Cayley's is usually proven from Lagrange's theorem on orders of subgroups, which comes from an analysis of cosets. How did Cayley know this? Did he have a proof? Students can look ahead for this result in a text.

[12] Note Cayley's interesting point of view here, that this result on orders of elements can be interpreted as an "equation" for which the group is a solution set.

[13] Students may relate this form to the group of numbers on a clock, i.e., cyclic groups.

[14] By this Cayley means the group of n-th roots of unity in the complex plane. One can examine with students why these complex roots form a cyclic group.

The distinction between the theory of the symbolic equation $\theta^n = 1$, and that of the ordinary equation $x^n - 1 = 0$, presents itself in the very simplest case, $n = 4$. ...[15]

... and we have thus a group[16] essentially distinct from that of the system of roots of the ordinary equation $x^4 - 1 = 0$.

Systems of this form are of frequent occurrence in analysis, and it is only on account of their extreme simplicity that they have not been expressly remarked. For instance, in the theory of elliptic functions, if n be the parameter, and

$$\alpha(n) = \frac{c^2}{n}\ \beta(n) = -\frac{c^2+n}{1+n}\ \gamma(n) = -\frac{c^2(1+n)}{c^2+n},$$

then α, β, γ form a group of the species in question[17]. ...[18]

Again, in the theory of matrices, if I denote the operation of inversion, and tr that of transposition, ... we may write[19]

$$\alpha = I,\ \beta = \text{tr},\ \gamma = I.\text{tr} = \text{tr}.I.$$

I proceed to the case of a group of six symbols, ...[20]

An instance of a group of this kind is given by the permutation of three letters[21]; ...

It is, I think, worth noticing, that if, instead of considering α, β, &c. as symbols of operation, we consider them as quantities (or, to use a more abstract term, 'cogitables') such as the quaternion imaginaries; the equations expressing the existence of the group are, in fact, the equations defining the meaning of the product of two complex quantities of the form

$$w + a\alpha + b\beta + \dots;$$

thus, in the system just considered, ...[22]

[15]At this point Cayley makes a detailed logical analysis of all the possibilities for the nature of a group of order four, shows that there are only two essential types (i.e., up to isomorphism), and displays their multiplication tables. This is excellent material for a class to work through.

[16]The non-cyclic "Klein Viergruppe."

[17]Students may check that these functions compose to create the Viergruppe.

[18]Cayley also mentions how the theory of quadratic forms provides another example.

[19]And this, too, may be verified to create the Viergruppe.

[20]Cayley's exhaustive analysis now shows that there are exactly two types of groups of order six, the cyclic one and another kind.

[21]This is a nice segue to permutation groups for students.

[22]Cayley now shows how to multiply in the six-dimensional system based on his non-commutative group of order six. Here he is presenting the general idea of a group algebra,

It does not appear that there is in this system anything analogous to the modulus $w^2 + x^2 + y^2 + z^2$, so important in the theory of quaternions[23].

... I conclude for the present with the following two examples of groups of higher orders. The first of these is a group of eighteen, viz. ...; and the other a group of twenty-seven, ...

Pedagogy What are the challenges and rewards Cayley's paper offers in an abstract algebra course? The challenges include that, while Cayley's writing is very expansive and accessible, it is hard to understand in places, partly due to older terminology, but more due to its pioneering nature, in which his assumptions and contexts are not always crystal clear to us today. For instance, reading Cayley with students raises the following questions:

- Are all his group operations tacitly formed from composing functions or not, and what are the implications for associativity (e.g., quaternions, octonions)?
- Are inverses always assumed to exist? Does one need to assume this?
- Is Cayley tacitly assuming that his groups are finite? Why?
- How did Cayley know that every element in a (finite) group satisfies the "symbolic equation" for the order of the group? This is a nontrivial result, usually proved today by studying the theory of cosets of a subgroup.

But these very challenges are also precisely the pedagogical strengths of Cayley's paper, leading directly to rewards. These questions lead instructor and student to grapple with many of the key motivations and issues for group theory: What is the appropriate role of function composition, associativity, inverses, orders of elements? Through studying this paper and these questions, via the diverse and rich examples Cayley raises, students will quickly become intimately engaged by the most fundamental issues about groups, showcasing their unifying role in algebra and other mathematical contexts. Relevant assignments can include:

and he makes analogy and contrast with quaternions. In fact, while the quaternions are not actually a group algebra, the real group algebra of the eight-element quaternion group is a direct sum of two copies of the four-dimensional algebra of quaternions over the real numbers. This can be shown from the modern theories of semi-simple rings or of Clifford algebras. I thank Pat Morandi and Ray Mines for showing me this.

[23]Cayley's interest in a modulus is presumably related to the question of whether such a system supports division, since division in the complex numbers, quaternions, and octonions (Cayley numbers) goes hand in hand with a modulus.

- Find Cayley's claims in their context in a modern textbook, and contrast their place today with their role for Cayley. For instance, Cayley's claim about elements satisfying the "symbolic equation" for the group will lead students ahead to cosets and Lagrange's theorem on orders of subgroups.

- Contrast Cayley's "symbolic equation" $\theta^n = 1$ with his "ordinary equation" $x^n - 1 = 0$, with its complex n^{th} roots of unity as solutions, and consider which group it represents.

- Study and describe the general "classification" question in mathematics, algebra, and group theory, e.g., the classification results for finite simple groups, and finitely generated abelian groups, with Cayley's classification of groups of certain orders as inspiration.

- Develop the concepts of normal subgroups and quotient groups by examining Cayley's tables for groups of order six, including how his layout and heavily drawn lines begin to display the quotient group formed by certain cosets.

- Explore a new concept raised by Cayley, e.g., the notion of a group algebra, with examples.

In these ways, Cayley's paper produces group theory in a fully motivated context, emerging from important mathematical currents of the mid-nineteenth century. Working with the paper inserts students, as personal witnesses and even as vicarious participants, into the dynamic process of mathematical research, at a formative moment in the birth of one of the most essential concepts of modern mathematics. Students' own struggles with Cayley's ideas in comparison with their modern textbook will leave them with a more profound technical comprehension, while simultaneously initiating them into thinking and acting like mathematicians.

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