

# Early Writings on Graph Theory: Hamiltonian Circuits and The Icosian Game

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## Introduction

Problems that are today considered to be part of modern graph theory originally appeared in a variety of different connections and contexts. Some of these original questions appear little more than games or puzzles. In the instance of the ‘Icosian Game’, this observation seems quite literally true. Yet for the game’s inventor, the Icosian Game encapsulated deep mathematical ideas which we will explore in this project.

Sir William Rowan Hamilton (1805 - 1865) was a child prodigy with a gift for both languages and mathematics. His academic talents were fostered by his uncle James Hamilton, an Anglican clergyman with whom he lived from the age of 3. Under his uncle’s tutelage, Hamilton mastered a large number of languages — including Latin, Greek, Hebrew, Persian, Arabic and Sanskrit — by the age of 10. His early interest in languages was soon eclipsed by his interests in mathematics and physics, spurred in part by his contact with an American calculating prodigy. Hamilton entered Trinity College in Dublin in 1823, and quickly distinguished himself. He was appointed Astronomer Royal of Ireland at the age of 22 based on his early work in optics and dynamics. Highly regarded not only by his nineteenth century colleagues, Hamilton is today recognized as a leading mathematician and physicist of the nineteenth century.

In mathematics, Hamilton is best remembered for his creation of a new algebraic system known as the ‘quarternions’ in 1843. The system of quarternions consists of ‘numbers’ of the form  $Q = a + bi + cj + dk$  subject to certain basic ‘arithmetic’ rules. The project that led Hamilton to the discovery of quarternions was the search for an algebraic system that could be reasonably interpreted in the three-dimensional space of physics, in a manner analogous to the interpretation of the algebra of complex numbers  $a + bi$  in a two-dimensional plane. Although this geometrical interpretation of the complex numbers is now standard, it was discovered by mathematicians only in the early 1800’s and thus was relatively new in Hamilton’s time. Hamilton was one of several nineteenth century British mathematicians interested in developing a purely *algebraic* foundation for complex numbers that would capture the essence of this geometrical interpretation. His algebraic development of the complex numbers as ordered pairs of real numbers  $(a, b)$  subject to certain operations appeared in a landmark 1837 essay entitled *Theory of Conjugate Functions, or Algebraic Couples; with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time*.

Hamilton concluded his 1837 essay with a statement concerning his hope that he would soon publish a similar work on the algebra of triplets. After years of unsuccessful work on this problem, Hamilton was able to solve it in 1843 only by abandoning the property of commutativity. For example, two of the basic multiplication rules of the quarternion system are  $ij = k$  and  $ji = -k$ , so that  $ij \neq ji$ . Hamilton also replaced ‘triplets’ by the ‘four-dimensional’ quarternion  $a + bi + cj + dk$ .

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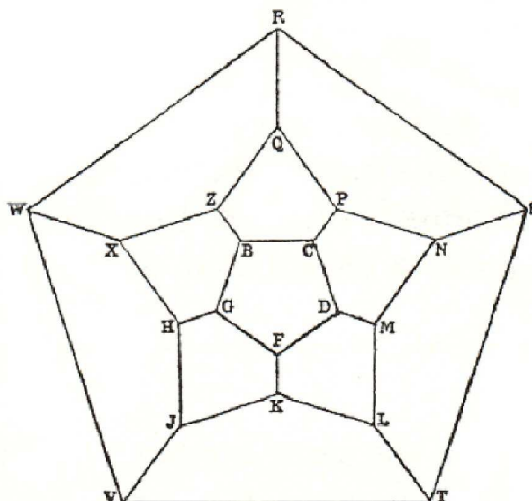
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Soon after Hamilton's discovery, physicists realized that only the 'vector part'  $bi + cj + dk$  of a quaternion was needed to represent three-dimensional space. Although vectors replaced the use of quaternions in physics by the end of the nineteenth century, the algebraic system of vectors retains the non-commutativity of quaternions.

Today's students of mathematics are familiar with a variety of non-commutative algebraic operations, including vector cross-product and matrix multiplication. In Hamilton's day, however, quaternions constituted a major breakthrough comparable to the discovery of non-Euclidean geometry. Immediately following Hamilton's 1843 announcement of his discovery, at least seven other new numbers systems were discovered by several other British algebraists. In the 'Icosian Game', Hamilton himself developed yet another example of a non-commutative algebraic system. In this project, we explore both the algebra of that system and the graph theoretical notion of 'Hamiltonian circuit' on which Hamilton's interpretation of this algebra is based. The idea for the game was first exhibited by Hamilton at an 1857 meeting of the British Association in Dublin [3], and later sold for 25 pounds to 'John Jacques and Son,' a wholesale dealer in games. We begin with the preface to the instructions pamphlet which Hamilton prepared for marketing of the game in 1859 [1, pp. 32 - 35].

## The Icosian Game and Hamiltonian Circuits

### THE ICOSIAN GAME



In this new Game (invented by Sir WILLIAM ROWAN HAMILTON, LL.D., &c., of Dublin, and by him named *Icosian* from a Greek word signifying 'twenty') a player is to place the whole or part of a set of twenty numbered pieces or men upon the points or in the holes of a board, represented by the diagram above drawn, in such a manner as always to proceed *along the lines* of the figure, and also to fulfil certain *other* conditions, which may in various ways be assigned by another player. Ingenuity and skill may thus be exercised in *proposing* as well as in *resolving* problems of the game. For example, the first of the two players may place the first five pieces in any five consecutive holes, and then require the second player to place the remaining fifteen men consecutively in such a manner that the succession may

be *cyclical*, that is, so that No. 20 may be adjacent to No. 1; and it is always possible to answer any question of this kind. Thus, if B C D F G be the five given initial points, it is allowed to complete the succession by following the alphabetical order of the twenty consonants, as suggested by the diagram itself; but after placing the piece No. 6 in hole H, as before, it is *also* allowed (by the supposed conditions) to put No. 7 in X instead of J, and then to conclude with the succession, W R S T V J K L M N P Q Z. Other Examples of Icosian Problems, with solutions of some of them, will be found in the following page.

In graph theoretic terminology, the holes of the game board are referred to as *vertices* (singular: *vertex*) and the lines that join two holes (vertices) are called *edges*. The collection of vertices and edges in a given relationship (as represented by a diagram such as the game board) is called a *graph*. Two vertices that are joined by an edge in the graph are said to be *adjacent*. Thus, the instruction ‘always to proceed *along the lines* of the figure’ requires the player to find a sequence of adjacent vertices; such a sequence is known as a *path*. In the case where no vertex is repeated in the sequence, the path is said to be a *simple path*. In the case where every vertex of the graph is used exactly once in the sequence, the path is said to be a *Hamiltonian path*. The term *cycle* is now used to describe what Hamilton referred to as a ‘cyclical’ path.

1. Explain why the rules of the Icosian Game require players to always find a simple path.
2. Use modern terminology to formally define the terms *cycle* and *Hamiltonian cycle*.

Following the preface, Hamilton includes several examples of Icosian Problems in the Instruction Pamphlet. We consider only the first two problems as a means of familiarizing ourselves with the concepts of Hamiltonian cycle and Hamiltonian path.

## EXAMPLES OF ICOSIAN PROBLEMS

### FIRST PROBLEM

*Five initial points are given; cover the board, and finish cyclically. (As hinted in the preceding page, a succession is said to be *cyclical* when the *last* piece is adjacent to the *first*.)*

[This problem is always possible in at least two, and sometimes in four, different ways. Two examples have been assigned: the following are a few others.]

*Example 3.* Given B C P N M as initial: two solutions exist; one is the succession, D F K L T S R Q Z X W V J H G; the other is D F G H X W V J K L T S R Q Z.

*Example 4.* Five initials, L T S R Q. Four solutions.

*Example 5.* Five initials, J V T S R. Two solutions.

3. Explain why Hamilton’s first problem is equivalent to the problem of finding a Hamilton circuit beginning from a given initial sequence of five vertices.
4. In *Example 3*, Hamilton specifies *BCPNM* as the first five vertices in the desired circuit. He then claims that the two solutions listed in the example are the *only* two solutions of this particular problem. Prove that these are in fact the only two solutions by completing the details of the following argument. Include copies of the diagram illustrating each step of the argument in a different color as part of your proof.

- (a) Explain why the initial conditions for this example imply that the solution to the problem must include either the sequence  $RST$  or the sequence  $TSR$ .
  - (b) Explain why the initial conditions for this example imply that the solution to the problem must include either the sequence  $RQZ$  or the sequence  $ZQR$ .
  - (c) Explain why we can now conclude that the solution to this problem must include either the sequence  $XWV$  or the sequence  $VWX$ .
  - (d) Explain why the initial conditions for this example imply that the solution to the problem must include either the sequence  $FDM$  or the sequence  $MDF$ .
  - (e) Explain why we can now conclude that the solution to this problem must include either the sequence  $KLT$  or the sequence  $TLK$ .
  - (f) Use the information from above concerning which edges and vertices we know must be part of the solution to prove that the two circuits Hamilton lists are the only to solutions to the problem.
5. In *Example 4*, Hamilton claims there are four Hamiltonian circuits that begin with the vertices  $LTSRQ$ . Find them. (You do not need to prove these are the only four.)

Although Hamilton claims that every initial sequence of five vertices will lead to at least two solutions (and possibly four) within the Icosian Game, he does not offer a proof of this claim. Nor does he claim that this is true of all graphs.

6. Show that it is not true of every graph that any initial sequence of five vertices will lead to at least one Hamiltonian circuit by finding an example of a graph with at least 5 vertices that has no Hamiltonian circuit. Prove that your graph does not contain a Hamiltonian circuit.

The next question pertains to Hamilton's second problem, which Hamilton describes in the pamphlet as follows:

### EXAMPLES OF ICOSIAN PROBLEMS (continued)

#### SECOND PROBLEM

*Three* initial points are given; cover the board *non-cyclically*. (A succession is said to be *non-cyclical* when the *last* piece is *not* adjacent to the first.)

[This problem is sometimes soluble in only *one* way; sometimes in only *two* ways; sometimes in *four* ways; and sometimes it is not soluble at all, as will be seen in the following examples.]

*Example 6.* Three initials, B C D; cover, and end with T. There is in this case only one solution, namely, F G H X Z Q P N M L K J V W R S T.

*Example 7.* Same initials; cover, and end with W. Two solutions.

*Example 8.* Same initials; cover, and end with J. Two solutions.

[The same number of solutions exists, if it be required, having the same three initials, to end with K, or L, or N, or V.]

*Example 9.* Same initials; cover, and end with R. Four solutions.

*Example 10.* Same initials; cover, and end with M. Impossible.

[The same result, if it be required to end with F, or H, or Q, or S, or X.]

7. In *Example 10* , Hamilton claims the problem of finding a ‘non-cyclical’ path that uses all vertices beginning with  $BCD$  and ending with  $M$  is impossible. Prove that he is correct.

## The Icosian Game and Non-Commutative Algebra

We now turn to the portion of Hamilton’s pamphlet which links the Icosian Game to a non-commutative algebra.

### HINTS ON THE ICOSIAN CALCULUS, OF WHICH THE ICOSIAN GAME IS DESIGNED TO BE AN ILLUSTRATION.

I. In a "MEMORANDUM respecting a New System of Roots of Unity," which appeared in the *Philosophical magazine* for December 1856, Sir W. R. Hamilton expressed himself nearly as follows (a few words only being here omitted):

‘I have lately been led to the conception of a new system, or rather *family of systems*, of *non-commutative roots of unity*, which are entirely distinct from the  $i j k$  of quaternions, though having some general analogy thereto; and which admit, even more easily than the quaternion symbols do, of geometrical interpretation. In the system which seems at present to be the most interesting one among those included in this new family, I assume three symbols,  $\iota, \kappa, \lambda$ , such that  $\iota^2 = 1, \kappa^3 = 1, \lambda^5 = 1, \lambda = \iota\kappa$ ; where  $\iota\kappa$  must be *distinguished* from  $\kappa\iota$ , since otherwise we should have  $\lambda^6 = 1, \lambda = 1$ . As a very simple *specimen* of the symbolical conclusions deduced from these fundamental assumptions I may mention that if we make  $\mu = \iota\kappa^2 = \lambda\iota\lambda$ , we shall have also  $\mu^5 = 1, \lambda = \mu\iota\mu$ ; so that  $\mu$  is a new fifth root of reciprocity. A long train of such symbolical deductions is found to follow; and every one of the results may be *interpreted* as having reference to the passage from *face to face* (or from corner to corner) of the *icosahedron* (or of the dodecahedron): on which account, I am at present disposed to give the name of ‘Icosian Calculus’ to this new system of symbols, and of rules for their operations.’

The system of ‘*non-commutative roots of unity*’ described above employs three symbols  $\iota, \kappa, \lambda$  subject to the following (non-commutative) rules:

$$\iota^2 = 1, \kappa^3 = 1, \lambda^5 = 1, \lambda = \iota\kappa, \iota\kappa \neq \kappa\iota$$

The symbol ‘1’ represents the identity, so that  $1\iota = \iota 1 = \iota$ ,  $1\kappa = \kappa 1 = \kappa$ , and  $1\lambda = \lambda 1 = \lambda$ . In Part II of *Hints on the Icosian Calculus*, Hamilton describes in detail how to interpret his system of ‘non-commutative roots of unity’ within the Icosian Game. First, consider only the symbolic action of  $\iota, \kappa, \lambda$  as defined by the above multiplication rules to to complete Questions 8 and 9.

8. Prove symbolically that  $\kappa = \iota\lambda$ .

9. Prove symbolically that  $\iota\kappa^2 = \lambda\iota\lambda$ .

(This shows that it makes sense to define the new symbol  $\mu$  by  $\mu = \iota\kappa^2 = \lambda\iota\lambda$ .)

Extra Credit Question: Show symbolically that  $\mu^5 = 1$ .

We now consider Hamilton's interpretation of this algebraic system within the Icosian Game. This interpretation provides a concrete method for deriving new symbolic equations such as those mentioned in the following excerpt.

### HINTS ON THE ICOSIAN CALCULUS (continued)

II. In a LITHOGRAPH, which was distributed in Section A of the British Association, during its Meeting at Dublin in 1857, Sir W. R. H. pointed out a few other symbolical results of the same kind: especially the equations  $\lambda\mu^2\lambda = \mu\lambda\mu$ ,  $\mu\lambda^2\mu = \lambda\mu\lambda$ ,  $\lambda\mu^3\lambda = \mu^2$ ,  $\mu\lambda^3\mu = \lambda^2$ ; and the formula  $(\lambda^3\mu^3(\lambda\mu)^2)^2 = 1$ , which serves as a *common mathematical type* for the solution of *all cases* of the First Problem of the Game. He also gave at the same time an oral (and hitherto unprinted) account of his rules of *interpretation* of the principal symbols; which rules, with reference to the present Icosian Diagram (or ICOSIAN), may be briefly stated as follows:

1. The operation  $\iota$  reverses (or reads backwards) a *line* of the figure; changing, for example, BC to CB.
2. The operation  $\kappa$  causes a line to *turn* in a particular direction round its final point; changing, for instance, BC to DC.
3. The operation  $\lambda$  changes a line considered as a *side* of a pentagon to the *following side* thereof, proceeding always *right-handedly* for every pentagon except the large or outer one; thus  $\lambda$  changes BC to CD, but SR to RW.
4. The operation  $\mu$  is *contrasted* with  $\lambda$ , and changes a line considered as a side of a *different pentagon*, and in the *opposite order* or rotation, to the consecutive side of that *other* pentagon; thus  $\mu$  changes BC to CP, and SR to RQ; but it changes also RS to ST, whereas  $\lambda$  would change RS to SN.
5. The only operations employed in the *game* are those marked  $\lambda$  and  $\mu$ ; but another operation,  $\omega = \lambda\mu\lambda\mu\lambda = \mu\lambda\mu\lambda\mu$ , having the property that  $\omega^2 = 1$ , was also mentioned in the Lithograph above referred to; and to complete the present statement of *interpretations*, it may be added that the effect of this operation  $\omega$  is to change an *edge* of a pentagonal *dodecahedron* to the *opposite edge* of that *solid*; for example, in the diagram, BC to TV.

Note that proceeding 'right handedly' may be described as moving clockwise around the appropriate pentagon, so that action which is 'in the opposite order' of proceeding 'right handedly' may be described as moving counter-clockwise.

10. Use the interpretation of  $\iota$  in (1) to explain why  $\iota^2 = 1$ .

Begin by looking at the effect of applying the operation  $\iota$  *twice in succession*, beginning with the edge *BC*. Then explain in general.

11. Use the interpretation of  $\kappa$  in (2) to explain why  $\kappa^3 = 1$ .

Begin by looking at the effect of applying the operation  $\kappa$  *three times in succession*, beginning with the edge *BC*. Then look at the effect of applying the operation  $\kappa$  *three times in succession*, beginning with the edge *PN*. Finally, explain in general.

12. Use the interpretation of  $\lambda$  in (3) to explain why  $\lambda^5 = 1$ .

Begin by looking at the effect of applying the operation  $\lambda$  *five times in succession*, beginning with the edge  $BC$ . Then look at the effect of applying the operation  $\lambda$  *five times in succession*, beginning with the edge  $SR$ . Finally, explain in general.

13. Use the interpretation of  $\mu$  in (4) to explain why  $\mu^5 = 1$ .

Begin by looking at the effect of applying the operation  $\mu$  *five times in succession*, beginning with the edge  $BC$ . Then look at the effect of applying the operation  $\mu$  *five times in succession*, beginning with the edge  $RS$ . Finally, explain in general. (Note that this provides a geometric solution of the extra credit question stated above, immediately following question 9.)

14. Beginning with the edge  $BC$ , use the interpretations given for the four symbols  $\iota, \kappa, \lambda, \mu$  to illustrate that  $\mu = \iota\kappa^2 = \lambda\iota\lambda$ .

Extra Credit Question: Establish the equation  $\lambda\mu^2\lambda = \mu\lambda\mu$  symbolically; then illustrate this equation within the Icosian Game, first beginning with the edge  $BC$  and then beginning with the edge  $RS$ .

### Some Closing Remarks

Notice Hamilton's claim that an *algebraic proof* using equations of the type  $(\lambda^3\mu^3(\lambda\mu)^2)^2 = 1$  can be used to find all Hamiltonian cycles beginning with a specified initial sequence of five vertices. Notice also the contrast between the graph theoretical proof that you completed in question 4 above, and the proof that we would get of this same result by interpreting this degree twenty equation within the context of the Icosian Game Board. In general, however, it is not viable to associate an algebraic system with an arbitrary graph as a means to find all Hamiltonian circuits within that graph. In fact, as you demonstrated in question 6 above, a graph may contain no Hamiltonian circuits at all. Unlike the known situation for other kinds of circuits (e.g., Euler circuits), there is no known simple condition on a graph which allows one to determine in all cases whether a Hamiltonian circuit exists or not. In the case that a graph does contain a Hamiltonian circuit, we say the graph is *Hamiltonian*.

The more general question of determining a condition under which a graph is Hamiltonian was first studied by Thomas Penyngton Kirkman (1840 - 1892). Unlike Hamilton, who was primarily interested in the algebraic connections of one specific graph, Kirkman was interested in the general study of 'Hamiltonian circuits' in arbitrary graphs. The rector of a small and isolated English parish, Kirkman presented a paper on this subject to the Royal Society on 6 August 1855. Regrettably, his solution of the problem was incorrect. He did, however, present a second paper in 1856 in which he described a general class of graphs which do not contain such a circuit. Kirkman also studied the existence of Hamiltonian circuits on the dodecahedron, a variation of the Icosian Game which Hamilton also studied. In fact, the two men met once in 1861 when Hamilton visited Kirkman at his rectory. That Hamilton's name became associated with the circuits, and not Kirkman's, appears to be one of the accidents of history, or perhaps a credit to the fame of Hamilton's quaternions and work in mathematical physics.

### Notes to the Instructor

This project contains two sub-sections "The Icosian Game and Hamiltonian Circuits" and "The Icosian Game and Non-Commutative Algebra," both of which were developed specifically for use

in an introductory undergraduate course in discrete mathematics. Because no prior background in graph theory is assumed, the connection to symbolic algebra makes the project suitable for use in a junior-level abstract algebra course as well. In a discrete mathematics course, the project could be assigned independently or in conjunction with one or both of the projects “Early Writings on Graph Theory: Euler Circuits and The Königsberg Bridge Problem” and “Early Writings on Graph Theory: Topological Connections,” both of which appear in this volume. For students with no prior knowledge of non-commutative algebras, the instructor may wish to provide more explicit directions for Questions 8 and 9, or work these together as a whole class. Otherwise, the project may be completed by students working in small groups over 2–3 in-class days, or assigned as a week-long individual project outside of class. Multiple copies of the Icosian Game diagram will be needed for each student; use of color pencils or markers is also highly recommended.

## References

- [1] Biggs, N., Lloyd, E., Wilson, R., *Graph Theory: 1736–1936*, Clarendon Press, Oxford, 1976.
- [2] Crowe, M. J., *A History of Vector Analysis: The Evolution of the Idea of a Vectorial System*, Dover Publications, New York, 1994.
- [3] Hamilton, W. R., “Account of the Icosian Game,” *Proc. Roy. Irish. Acad.* **6** (1853-7), 415–416.
- [4] Katz, V., *A History of Mathematics: An Introduction*, Second Edition, Addison-Wesley, New York, 1998.