

**Universal Weyl-type bounds for eigenvalues of the
Dirichlet Laplacian** \diamond

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\diamond a selection of results based on joint work with Evans
Harrell, J. Func. Analysis (to appear)

- Problem and Historical Background
- Definitions and Properties of Riesz Means
- Laptev-Weidl and Berezin-Li-Yau
- A Monotonicity Principle in Disguise
- From Universal Lower Bounds for Riesz Means to Universal Upper Bounds for Averages

Problem and Historical Background

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Consider the two classical problems:

- Fixed Membrane Problem:

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

Eigenmodes: $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$

Eigenfunctions: u_1, u_2, u_3, \dots

- Free Membrane Problem

$$\begin{aligned} -\Delta v &= \mu v \quad \text{in } \Omega \subset \mathbb{R}^d \\ \frac{\partial v}{\partial n} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2}$$

Eigenmodes: $0 = \mu_1 < \mu_2 \leq \mu_3 \leq \dots$

One can characterize these eigenvalues using the Rayleigh-Ritz principle:

$$\lambda_{k+1} \leq \frac{\int_{\Omega} |\nabla \phi|^2 dx}{\int_{\Omega} \phi^2 dx}$$

subject to

$$\int_{\Omega} \phi u_i dx = 0$$

for $i = 1, 2, \dots, k$.

Weyl (1910, 1911)

$$\lambda_k \sim \frac{4\pi^2 k^{2/d}}{(C_d |\Omega|)^{2/d}} = \frac{k^{2/d}}{(L_{0,d}^{cl} |\Omega|)^{2/d}} \quad \text{as } k \rightarrow \infty, \quad (3)$$

$$\mu_{k+1} \sim \frac{4\pi^2 k^{2/d}}{(C_d |\Omega|)^{2/d}} = \frac{k^{2/d}}{(L_{0,d}^{cl} |\Omega|)^{2/d}} \quad \text{as } k \rightarrow \infty,$$

where $C_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}$ = volume of the d -Ball.

Pólya Conjecture: (Patterns of Plausible Reasoning, 1954): For $k = 0, 1, 2, \dots$

$$\mu_{k+1} \leq \frac{k^{2/d}}{\left(L_{0,d}^{cl} |\Omega|\right)^{2/d}} \leq \lambda_k \quad (4)$$

Proved by Pólya (in a paper dedicated to Weyl, 1961), for tiling domains.

Friedlander (1991) settled conjecture of Payne (1954): For $\Omega \subset \mathbb{R}^d$ with smooth boundary;

$$\mu_{k+1} \leq \lambda_k \quad (5)$$

Focus today: Inequalities for $\overline{\lambda}_k$, λ_{k+1}/λ_1 , $\overline{\lambda}_k/\overline{\lambda}_j$, and the like.

We are also interested in inequalities for $R_\sigma(z)$, when $z \in \mathbb{R}$.

The *ramp* function $x_+ = \max(0, x)$

The Counting function:

$$N(z) = \sum_{\lambda_k \leq z} 1 = \sup_{\lambda_k \leq z} k.$$

By convention, this is sometimes written as

$$N(z) = \sum_k (z - \lambda_k)_+^0$$

This parallels the definition of the **Riesz mean of order $\sigma > 0$**

$$R_\sigma(z) = \sum_k (z - \lambda_k)_+^\sigma.$$

In fact the two are related by the *Riesz iteration* (also called *Aizenman-Lieb procedure*)

$$R_\sigma(z) = \sigma \int_0^\infty (z - t)_+^{\sigma-1} N(t) dt.$$

In fact:

$$R_{\sigma+\delta}(z) = \frac{\Gamma(\sigma + \delta + 1)}{\Gamma(\sigma + 1) \Gamma(\delta)} \int_0^\infty (z - t)_+^{\delta-1} R_\sigma(t) dt.$$

See: **“Typical Means”** by Chandrasekharan & Minakshisundaram (1954).

Weyl Asymptotic

$$N(z) \sim \frac{C_d}{(2\pi)^d} |\Omega| z^{d/2} = L_{0,d}^{cl} |\Omega| z^{d/2}$$

as $z \rightarrow \infty$.

Note that the Riesz iteration of the Weyl asymptotic immediately gives the statement, also as $z \rightarrow \infty$.

$$R_\sigma(z) \sim L_{\sigma,d}^{cl} |\Omega| z^{\sigma+d/2}, \quad (6)$$

where the classical constant is now given by

$$L_{\sigma,d}^{cl} = \frac{\Gamma(1 + \sigma)}{(4\pi)^{d/2} \Gamma(1 + \sigma + d/2)}.$$

Applying (heuristically) the Laplace transform to (6)

$$\mathcal{L}\{f\}(t) = \int_0^\infty f(z)e^{-zt}dz.$$

one obtains

$$Z(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \sim \frac{|\Omega|}{(4\pi t)^{d/2}}.$$

$z \rightarrow \infty$ corresponds to $t \rightarrow 0+$

The Berezin-Li-Yau inequality

$$\overline{\lambda}_k := \frac{1}{k} \sum_{\ell \leq k} \lambda_\ell \geq 4\pi \frac{\left(\Gamma\left(1 + \frac{d}{2}\right)\right)^{\frac{2}{d}} k^{\frac{2}{d}}}{|\Omega|^{\frac{2}{d}} \left(1 + \frac{2}{d}\right)}; \quad (7)$$

Theorem (Laptev-Weidl). For $\sigma \geq 1$, one has

$$R_\sigma(z) \leq L_{\sigma,d}^{cl} |\Omega| z^{\sigma+d/2}, \quad (8)$$

Berezin-Li-Yau is a consequence of the $\sigma = 1$ case of the Laptev-Weidl Theorem:

The Legendre Transform

$$\Lambda\{f\}(w) := \sup_{\lambda \geq 0} (w \lambda - f(\lambda)).$$

$$\Lambda\left\{\sum_j (z - \lambda_j)_+\right\} = (w - [w]) \lambda_{[w]+1} + \sum_{j=1}^{[w]} \lambda_j.$$

$$\Lambda\left(c z^{1+d/2}\right) = \frac{d}{2} c^{-2/d} \frac{w^{1+2/d}}{(1 + d/2)^{1+2/d}}$$

$$f(x) \leq g(x) \Rightarrow \Lambda\{f\}(w) \geq \Lambda\{g\}(w)$$

Theorem (Harrell-H.). For $\sigma \geq 2$ and $z \geq \lambda_1$,

$$R_{\sigma-1}(z) \geq \left(1 + \frac{d}{2\sigma}\right) \frac{1}{z} R_{\sigma}(z). \quad (9)$$

$$R'_{\sigma}(z) \geq \left(\sigma + \frac{d}{2}\right) \frac{1}{z} R_{\sigma}(z). \quad (10)$$

and consequently

$$\frac{R_{\sigma}(z)}{z^{\sigma + \frac{d}{2}}}$$

is a nondecreasing function of z .

Note that $R'_{\sigma}(z) = \sigma R_{\sigma-1}(z)$.

Theorem (Harrell-H.). For $0 < \sigma \leq 2$ and $z \geq \lambda_1$,

$$R_{\sigma-1}(z) \geq \left(1 + \frac{d}{4}\right) \frac{1}{z} R_{\sigma}(z). \quad (11)$$

$$R'_{\sigma}(z) \geq \left(1 + \frac{d}{4}\right) \frac{\sigma}{z} R_{\sigma}(z). \quad (12)$$

and consequently

$$\frac{R_{\sigma}(z)}{z^{\sigma + \frac{d\sigma}{4}}}$$

is a nondecreasing function of z .

Immediate consequence of the new monotonicity principle (for $\sigma \geq 2$): **Weyl Asymptotic + Monotonicity \Rightarrow Laptev-Weidl Inequality.**

- We know from the Weyl law that

$$\frac{R_\sigma(z)}{z^{\sigma+\frac{d}{2}}} \rightarrow L_{\sigma,d}^{cl} |\Omega|$$

as $z \rightarrow \infty$. Since $\frac{R_\sigma(z)}{z^{\sigma+\frac{d}{2}}}$ is a nondecreasing function, it is less than its limit for all finite z .

- The difference inequality can be rewritten in the form

$$\frac{R_{\sigma-1}(z)}{L_{\sigma-1,d}^{cl} z^{\sigma-1+\frac{d}{2}}} \geq \frac{R_\sigma(z)}{L_{\sigma,d}^{cl} z^{\sigma+\frac{d}{2}}}. \quad (13)$$

This follows from

$$L_{\sigma-1,d}^{cl} = \left(1 + \frac{d}{2\sigma}\right) L_{\sigma,d}^{cl}.$$

Inequality (13) is a refinement of the the Aizenman-Lieb inequality, at the level of ratios rather than their suprema.

So, for $z \geq z_0 > 0$

$$\frac{R_{\sigma-1}(z)}{L_{\sigma-1,d}^{cl} z^{\sigma-1+\frac{d}{2}}} \geq \frac{R_{\sigma}(z)}{L_{\sigma,d}^{cl} z^{\sigma+\frac{n}{2}}} \geq \frac{R_{\sigma}(z_0)}{L_{\sigma,d}^{cl} z_0^{\sigma+\frac{d}{2}}}.$$

Send $z \rightarrow \infty$, and apply Weyl to obtain Berezin-Li-Yau, once more.

Observe:

- Statement (9) can be reworded in the following terms:

$$\sum_k (z - \lambda_k)_+^\sigma \leq \frac{2\sigma}{d} \sum_k \lambda_k (z - \lambda_k)_+^{\sigma-1}.$$

For (11), i.e., for $0 \leq \sigma \leq 2$,

$$\sum_k (z - \lambda_k)_+^\sigma \leq \frac{4}{d} \sum_k \lambda_k (z - \lambda_k)_+^{\sigma-1}.$$

- $\sigma = 2 \Rightarrow \sigma \geq 2$ (This is Riesz iteration as described above, but I will describe a proof solely based on the trace identity (or sum rule) of Harrell-Stubbe with which we obtain the above theorems all at once.)

- Case $\sigma = 2$ was proved by Harrell-Stubbe (1998) (see also Levitin-Parnovski, 2001)
- Note that these statements contain the inequality of H.C. Yang (1991)

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{d} \sum_{i=1}^k \lambda_i (\lambda_{k+1} - \lambda_i)$$

- For $\sigma \geq 2$, the statement reduces to (see Ashbaugh-H., Harrell-Stubbe)

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\sigma \leq \frac{2\sigma}{d} \sum_{i=1}^k \lambda_i (\lambda_{k+1} - \lambda_i)^{\sigma-1}$$

- Note also that for $0 \leq \sigma \leq 2$

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\sigma \leq \frac{4}{d} \sum_{i=1}^k \lambda_i (\lambda_{k+1} - \lambda_i)^{\sigma-1}$$

The above statements contain the following classical inequalities as corollaries.

- PPW (1956)

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{dk} \sum_{j=1}^k \lambda_j \quad (14)$$

- Hile-Protter (1981)

$$\sum_{i=1}^k \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{dk}{4}$$

- H.C. Yang 2 (1991/1995)

$$\lambda_{k+1} \leq \left(1 + \frac{4}{d}\right) \bar{\lambda}_k$$

- PPW

$$\frac{\lambda_{k+1}}{\lambda_k} \leq \left(1 + \frac{4}{d}\right) \quad (15)$$

- Ashbaugh-Benguria (91, 92, 93)

$$\frac{\lambda_{2^m}}{\lambda_1} \leq \left(\frac{j_{d/2,1}^2}{j_{d/2-1,1}^2} \right)^m. \quad (16)$$

While optimal for low-lying eigenvalues, (16) is not of Weyl-type since the right side of the inequality behaves like $k^{5.77078/d}$ as $k \rightarrow \infty$.

- Ashbaugh-Benguria (89): The eigenvalues of the Schrödinger operator $-d^2/dx^2 + V(x)$ acting on $L^2(I)$, for a closed interval $I \subset \mathbb{R}$ for Dirichlet boundary conditions at the endpoints of I , and for $V \in L^1(I)$ and $V \geq 0$ a.e. on I , satisfy

$$\frac{\lambda_k}{\lambda_1} \leq k^2.$$

This bound is optimal, and for $k > 1$, equality obtains iff $V \equiv 0$ a.e. on I . For $j \leq k$,

$$\frac{\lambda_k}{\lambda_j} \leq \left[\frac{k}{j} \right]^2.$$

- Weyl-type bound (H., 2004, TAMS to appear)

$$\frac{\lambda_{k+1}}{\lambda_1} \leq 1 + \left(1 + \frac{d}{2}\right)^{2/d} H_d^{2/n} k^{2/d}, \quad (17)$$

and

$$\frac{\bar{\lambda}_k}{\lambda_1} \leq 1 + \frac{H_d^{2/d}}{1 + \frac{2}{d}} k^{2/d}, \quad (18)$$

where

$$H_d = \frac{2d}{j_{d/2-1,1}^2 J_{d/2}^2(j_{d/2-1,1})}. \quad (19)$$

As usual, $j_{\alpha,p}$ denotes the p -th positive zero of the Bessel function $J_\alpha(x)$. Generalization and new proof by Frank, Laptev and Molchanov (2006). We have yet another independent proof based on the Bethe sum rule

$$\sum_k (\lambda_k - \lambda_j) \left| \int_\Omega u_k u_j e^{ix \cdot \xi} dx \right|^2 = |\xi|^2.$$

- Cheng and Yang (2007)

$$\frac{\lambda_{k+1}}{\lambda_1} \leq \left(1 + \frac{4}{d}\right) k^{\frac{2}{d}}. \quad (20)$$

More generally,

$$\frac{\lambda_{k+1}}{\lambda_1} \leq C_0(d, k) k^{\frac{2}{d}}. \quad (21)$$

For instance, for $k \geq d+1$, Cheng and Yang improve (20) to

$$\frac{\lambda_{k+1}}{\lambda_1} \leq \left(1 + \frac{4}{d}\right) \left(1 + \frac{8}{d+1} + \frac{8}{(d+1)^2}\right)^{\frac{1}{2}} (d+1)^{-\frac{2}{d}} k^{\frac{2}{d}}.$$

New results (Harrell-H.)

- For $k \geq j \geq 1$,

$$\lambda_{k+1}/\bar{\lambda}_j \leq \left(1 + \frac{4}{d}\right) \left(\frac{k}{j}\right)^{\frac{2}{d}}. \quad (22)$$

- For $k \geq j \frac{1+\frac{d}{2}}{1+\frac{d}{4}}$, the means of the eigenvalues of the Dirichlet Laplacian satisfy a universal Weyl-type bound,

$$\bar{\lambda}_k/\bar{\lambda}_j \leq 2 \left(\frac{1 + \frac{d}{4}}{1 + \frac{d}{2}}\right)^{1+\frac{2}{d}} \left(\frac{k}{j}\right)^{\frac{2}{d}}. \quad (23)$$

- For $k \geq \frac{(d+1)(1+\frac{d}{2})}{1+\frac{d}{4}}$,

$$\bar{\lambda}_k/\lambda_1 \leq \frac{d+5}{2^{\frac{2}{d}}} \left(\frac{(d+4)}{(d+1)(d+2)}\right)^{1+\frac{2}{d}} k^{\frac{2}{d}}. \quad (24)$$

Where do these come from?

All are consequences of the monotonicity principle via direct comparison, or through a Legendre transform, or through a refinement, using another inequality of Ashbaugh and Benguria which is tied to the dimension d :

$$\overline{\lambda_{d+1}}/\lambda_1 \leq 1 + \frac{4}{d+1}.$$

Cor.

- For $1 \leq \sigma < 2$ and $z \geq \left(1 + \frac{2\sigma+2}{d}\right) \lambda_1$,

$$R_\sigma(z) \geq \frac{(2\sigma+2)^\sigma d^{\frac{d}{2}}}{(d+2\sigma+2)^{\sigma+\frac{d}{2}}} \lambda_1^{-\frac{d}{2}} z^{\sigma+\frac{d}{2}}, \quad (25)$$

- For $0 \leq \sigma < 1$, $z \geq \left(1 + \frac{2\sigma+4}{d}\right) \lambda_1$,

$$R_\sigma(z) \geq \frac{\left(1 + \frac{d}{4}\right) (2\sigma+4)^{\sigma+1} d^{\frac{d}{2}}}{(d+2\sigma+4)^{\sigma+1+\frac{d}{2}}} \lambda_1^{-\frac{d}{2}} z^{\sigma+\frac{d}{2}}. \quad (26)$$

For convenience we collect the most important cases $\sigma = 0, 1$, which simplify to

$$R_1(z) \geq \left(1 + \frac{d}{4}\right) \frac{1}{z} R_2(z) \geq \frac{4 d^{\frac{d}{2}}}{(d+4)^{1+\frac{d}{2}}} \lambda_1^{-\frac{d}{2}} z^{1+\frac{d}{2}},$$

and,

$$N(z) = R_0(z) \geq \left(1 + \frac{d}{4}\right)^2 \frac{1}{z^2} R_2(z) \geq \left(\frac{z}{\left(1 + \frac{4}{d}\right) \lambda_1}\right)^{\frac{d}{2}}.$$

To obtain the Cheng-Yang bound, simply let z approach λ_{k+1} from below.

These bounds compete with (Harrell-H.)

$$R_\sigma(z) \geq H_d^{-1} \lambda_1^{-d/2} \frac{\Gamma(1+\sigma)\Gamma(1+d/2)}{\Gamma(1+\sigma+d/2)} (z - \lambda_1)_+^{\sigma+d/2},$$

valid for $\sigma \geq 1$, where H_d is defined as above.

Cor. Suppose that $z \geq \left(1 + \frac{4}{d}\right) \bar{\lambda}_j$. Then

$$R_2(z) \geq \frac{jz^{2+\frac{d}{2}}}{\left(1 + \frac{d}{4}\right)^2 \left(\left(1 + \frac{4}{d}\right) \bar{\lambda}_j\right)^{\frac{d}{2}}}. \quad (27)$$

Therefore,

$$R_1(z) \geq \frac{jz^{1+\frac{d}{2}}}{\left(1 + \frac{d}{4}\right) \left(\left(1 + \frac{4}{d}\right) \bar{\lambda}_j\right)^{\frac{d}{2}}} \quad (28)$$

and

$$N(z) \geq j \left(\frac{z}{\left(1 + \frac{4}{d}\right) \bar{\lambda}_j} \right)^{\frac{d}{2}}. \quad (29)$$

Proof of the first statement. Let j be the largest such that $\lambda_j \leq z < \lambda_{j+1}$, then

$$R_2(z) = j \left(z^2 - 2z\bar{\lambda}_j + \overline{\lambda_j^2} \right). \quad (30)$$

For any integer j and all $z \geq \lambda_j$,

$$R_2(z) \geq Q(z, j) := j \left(z^2 - 2z\bar{\lambda}_j + \overline{\lambda_j^2} \right).$$

By monotonicity, for $z \geq z_j \geq \lambda_j$,

$$R_2(z) \geq Q(z_j, j) \left(\frac{z}{z_j} \right)^{2+\frac{d}{2}}. \quad (31)$$

Also, by Cauchy-Schwarz $\bar{\lambda}_j^2 \leq \overline{\lambda_j^2}$, so

$$\begin{aligned} Q(z, j) &= j \left((z - \bar{\lambda}_j)^2 + \overline{\lambda_j^2} - \bar{\lambda}_j^2 \right) \\ &\geq j (z - \bar{\lambda}_j)^2. \end{aligned} \quad (32)$$

Choose $z_j = \left(1 + \frac{4}{d}\right) \bar{\lambda}_j$

The statement which interpolates between average-Yang and Cheng-Yang, i.e.,

$$\lambda_{k+1}/\bar{\lambda}_j \leq \left(1 + \frac{4}{d}\right) \left(\frac{k}{j}\right)^{\frac{2}{d}}$$

is automatic if $\lambda_{k+1} \leq \left(1 + \frac{4}{d}\right) \bar{\lambda}_j$. Suppose to the contrary that $\lambda_{k+1} > \left(1 + \frac{4}{d}\right) \bar{\lambda}_j$. As z increases to λ_{k+1} , (29), i.e.,

$$N(z) \geq j \left(\frac{z}{\left(1 + \frac{4}{d}\right) \bar{\lambda}_j} \right)^{\frac{d}{2}}$$

becomes valid and in the limit reads

$$k \geq j \left(\frac{\lambda_{k+1}}{\left(1 + \frac{4}{d}\right) \bar{\lambda}_j} \right)^{\frac{d}{2}},$$

which when solved for $\lambda_{k+1}/\bar{\lambda}_j$ yields the claim.

Applying Legendre

For $z \geq \left(1 + \frac{4}{d}\right) \overline{\lambda}_j$, we have proved

$$R_1(z) \geq \frac{jz^{1+\frac{d}{2}}}{\left(1 + \frac{d}{4}\right) \left(\left(1 + \frac{4}{d}\right) \overline{\lambda}_j\right)^{\frac{d}{2}}}.$$

From which one gets, upon transforming

$$(w - [w]) \lambda_{[w]+1} + [w] \overline{\lambda}_{[w]} \leq \frac{2}{j^{\frac{2}{d}}} \left(\frac{1 + \frac{d}{4}}{1 + \frac{d}{2}}\right)^{1+\frac{2}{d}} \overline{\lambda}_j w^{1+\frac{2}{d}}$$

For any w , let $k - 1 \leq w < k$. If $k > j \frac{1+\frac{d}{2}}{1+\frac{d}{4}}$ and if we let w approach k from below, we obtain from (33)

$$\lambda_k + (k - 1) \overline{\lambda}_{k-1} \leq \frac{2}{j^{\frac{2}{d}}} \left(\frac{1 + \frac{d}{4}}{1 + \frac{d}{2}}\right)^{1+\frac{2}{d}} \overline{\lambda}_j k^{1+\frac{2}{d}}.$$

Where is the condition $k > j \frac{1+\frac{d}{2}}{1+\frac{d}{4}}$ coming from?

Legendre transform is about maximizing

$$\Lambda[f](w) = wz - f(z)$$

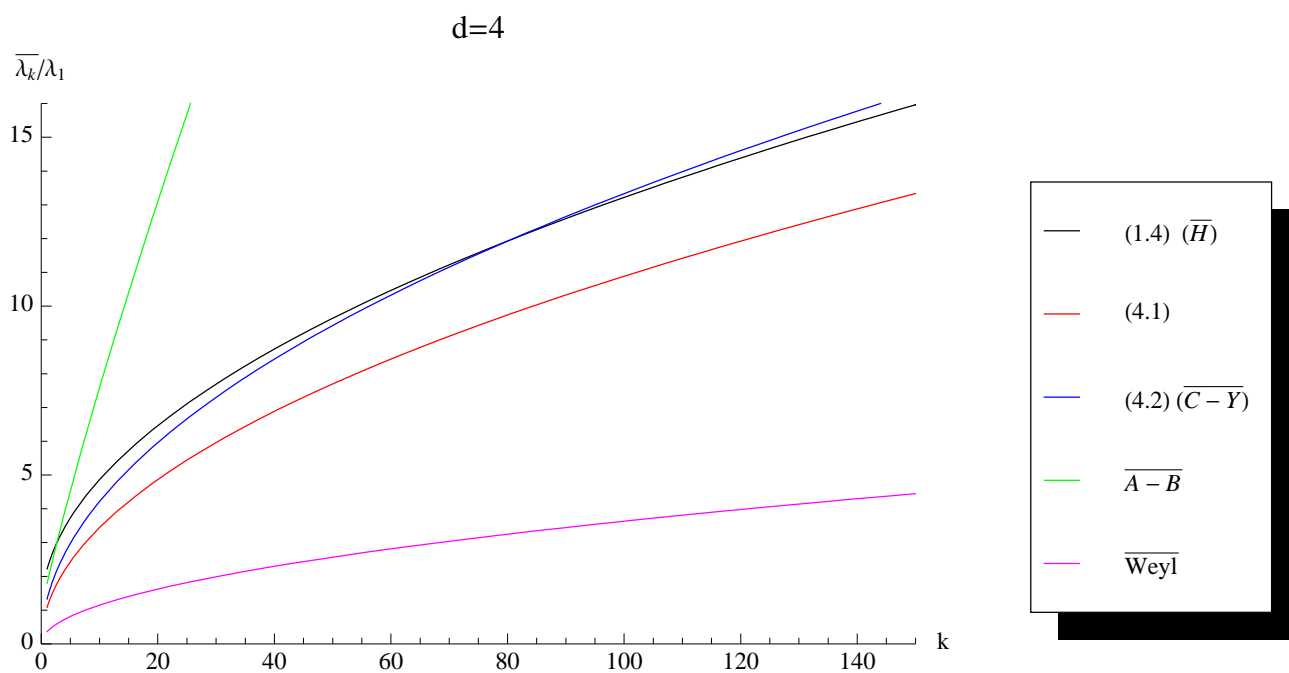
for

$$f(z) = \frac{jz^{1+\frac{d}{2}}}{\left(1 + \frac{d}{4}\right) \left(\left(1 + \frac{4}{d}\right) \bar{\lambda}_j\right)^{\frac{d}{2}}}$$

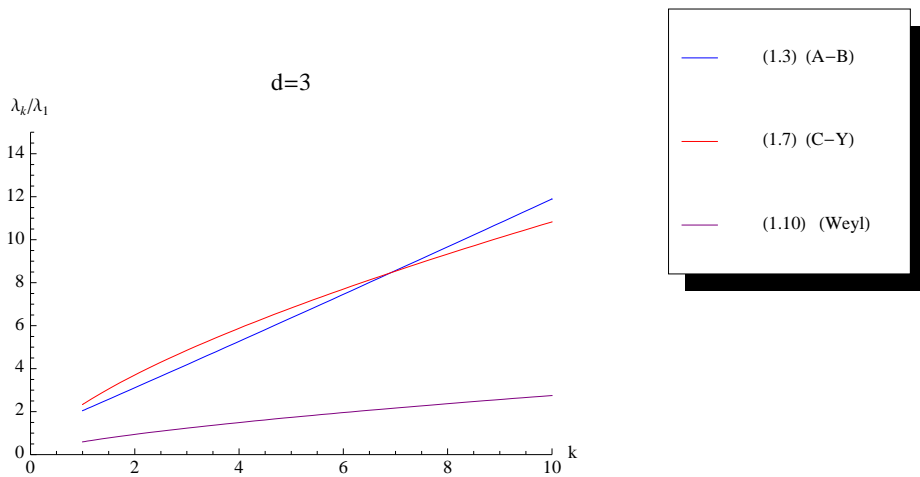
The maximizer z_* is such that

$$w = j \left(\frac{1 + \frac{d}{2}}{1 + \frac{d}{4}} \right) \left(\frac{z_*}{\left(1 + \frac{4}{d}\right) \bar{\lambda}_j} \right)^{\frac{d}{2}}, \quad (33)$$

Comparison with Previous Work



Bound for $\bar{\lambda}_k/\lambda_1$ as a function of k ($d = 4$).



Ashbaugh-Benguria vs Cheng-Yang for low k values ($d = 3$).

Sketch of Proof of Main Theorem

Lemma 1. (Trace Identity of Harrell-Stubbe)

Denoting the L^2 -normalized eigenfunctions of the Laplace operator $\{u_j\}$, let

$$T_{\alpha jm} := \left| \left(\frac{\partial u_j}{\partial x_\alpha}, u_m \right) \right|^2$$

for $j, m = 1, \dots$ and $\alpha = 1, \dots, d$. Then for each fixed α ,

$$\begin{aligned} R_\sigma(z) = & 2 \sum_{j,m:\lambda_j \neq \lambda_m} \frac{(z - \lambda_j)_+^\sigma - (z - \lambda_m)_+^\sigma}{\lambda_m - \lambda_j} T_{\alpha jm} \\ & + 4 \sum_{j,q:\lambda_j \leq z < \lambda_q} \frac{(z - \lambda_j)^\sigma}{\lambda_q - \lambda_j} T_{\alpha jq}. \end{aligned} \quad (34)$$

Lemma 2. Let $0 < x < y$ and $\sigma \geq 0$. Then

$$\frac{y^\sigma - x^\sigma}{y - x} \leq C_\sigma (y^{\sigma-1} + x^{\sigma-1}), \quad (35)$$

where

$$C_\sigma := \begin{cases} \frac{\sigma}{2}, & \text{if } 0 \leq \sigma \leq 1 \\ 1, & \text{if } 1 < \sigma \leq 2 \\ \frac{\sigma}{2}, & \text{if } 2 \leq \sigma < \infty. \end{cases}$$

Let the first term on the right of (34) be

$$G(\sigma, z, \alpha) := 2 \sum_{j,m:\lambda_j \neq \lambda_m} \frac{(z - \lambda_j)_+^\sigma - (z - \lambda_m)_+^\sigma}{\lambda_m - \lambda_j} T_{\alpha jm}.$$

$$\begin{aligned} G(\sigma, z, \alpha) &= 2 \sum_{j,m:\lambda_{j,m} \leq z, \lambda_j \neq \lambda_m} \frac{(z - \lambda_j)^\sigma - (z - \lambda_m)^\sigma}{(z - \lambda_j) - (z - \lambda_m)} T_{\alpha jm} \\ &\leq 2C_\sigma \sum_{j,m:\lambda_{j,m} \leq z} ((z - \lambda_j)^{\sigma-1} + (z - \lambda_m)^{\sigma-1}) T_{\alpha jm} \\ &= 4C_\sigma \sum_{j,m:\lambda_{j,m} \leq z} (z - \lambda_j)_+^{\sigma-1} T_{\alpha jm} \end{aligned}$$

by symmetry in $j \leftrightarrow m$.

Now, extend the sum to all m , subtracting the same quantity from the final term in (34), we find

$$R_\sigma(z) \leq 4C_\sigma \sum_{j:\lambda_j \leq z, \text{ all } m} (z - \lambda_j)_+^{\sigma-1} T_{\alpha jm} + 4H(\sigma, z, \alpha), \quad (36)$$

where

$$H(\sigma, z, \alpha) := \sum_{j,q:\lambda_j \leq z < \lambda_q} T_{\alpha jq} (z - \lambda_j)^{\sigma-1} \left(\frac{(z - \lambda_j) - C_\sigma (\lambda_q - \lambda_j)}{\lambda_q - \lambda_j} \right).$$

Next observe that because $\{u_m\}$ is a complete orthonormal set,

$$\sum_m T_{\alpha jm} = \left\| \frac{\partial u_j}{\partial x_\alpha} \right\|_2^2$$

and thus

$$\sum_{m,\alpha} T_{\alpha jm} = \|\nabla u_j\|_2^2 = \lambda_j.$$

Therefore, averaging over $\alpha = 1, \dots, d$ in (36), one obtains

$$R_\sigma(z) \leq \frac{4C_\sigma}{d} \sum_j (z - \lambda_j)_+^{\sigma-1} \lambda_j + \frac{4}{d} \sum_{\alpha=1}^n H(\sigma, z, \alpha),$$

or, since

$$\sum_j (z - \lambda_j)_+^{\sigma-1} \lambda_j = zR_{\sigma-1}(z) - R_\sigma(z),$$

$$\left(1 + \frac{4C_\sigma}{d}\right) R_\sigma(z) - \frac{4zC_\sigma}{d} R_{\sigma-1}(z) \leq \frac{4}{d} \sum_{\alpha=1}^d H(\sigma, z, \alpha). \quad (37)$$

We consider three cases:

Case I. $1 \leq \sigma \leq 2$. In this case $C_\sigma = 1$ and $H(\sigma, z, \alpha) \leq 0$, establishing that

$$\left(1 + \frac{4}{d}\right) R_\sigma - \frac{4z}{d} R_{\sigma-1}(z) \leq 0,$$

Case II. $0 < \sigma < 1$.

Since the sum defining H runs over $\lambda_q > z$,

$$\frac{(z - \lambda_j) - C_\sigma (\lambda_q - \lambda_j)}{\lambda_q - \lambda_j} \leq \frac{(\lambda_q - \lambda_j) - C_\sigma (\lambda_q - \lambda_j)}{\lambda_q - \lambda_j} = 1 - C_\sigma.$$

Therefore

$$H(\sigma, z, \alpha) \leq (1 - C_\sigma) \sum_{j, q: \lambda_q \geq z} T_{\alpha j q} (z - \lambda_j)_+^{\sigma-1},$$

and since in this case $1 - C_\sigma = 1 - \sigma/2 > 0$, we may extend the sum over all q , obtaining

$$\sum_{\alpha=1}^d H(\sigma, z, \alpha) \leq (1 - C_\sigma) \sum_j \lambda_j (z - \lambda_j)_+^{\sigma-1}.$$

Substituting this into (37), there is a cancellation of the C_σ 's, and we again obtain

$$\left(1 + \frac{4}{d}\right) R_\sigma - \frac{4z}{d} R_{\sigma-1}(z) \leq 0,$$

Case III. $\sigma > 2$. Arguing as in Case II, we come as far as

$$H(\sigma, z, \alpha) \leq (1 - C_\sigma) \sum_{j,q:\lambda_j \leq z < \lambda_q} T_{\alpha j q} (z - \lambda_j)^{\sigma-1},$$

but since now $(1 - C_\sigma) = 1 - \sigma/2 < 0$, we cannot extend the sum in q to simplify $T_{\alpha j q}$, and only conclude that

$$\left(1 + \frac{4C_\sigma}{d}\right) R_\sigma - \frac{4zC_\sigma}{d} R_{\sigma-1}(z) \leq 0,$$

Trace identities of the type derived by Harrell-Stubbe, and later generalized by Levitin-Parnovski, proved to be a very efficient procedure to produce universal Yang-type bounds for eigenvalues of the Dirichlet Laplacian.

In this talk we show how these identities can be used to produce new universal Weyl-type bounds for averages of eigenvalues, and provide alternative routes to the Berezin-Li-Yau inequality as viewed by Laptev and Weidl.

This is joint work with Professor Evans Harrell of Georgia Tech.

A preprint is available at <http://lanl.arxiv.org/abs/0705.3673>