Evolution of Graphs in Carnot Groups by Horizontal Gauss Curvature

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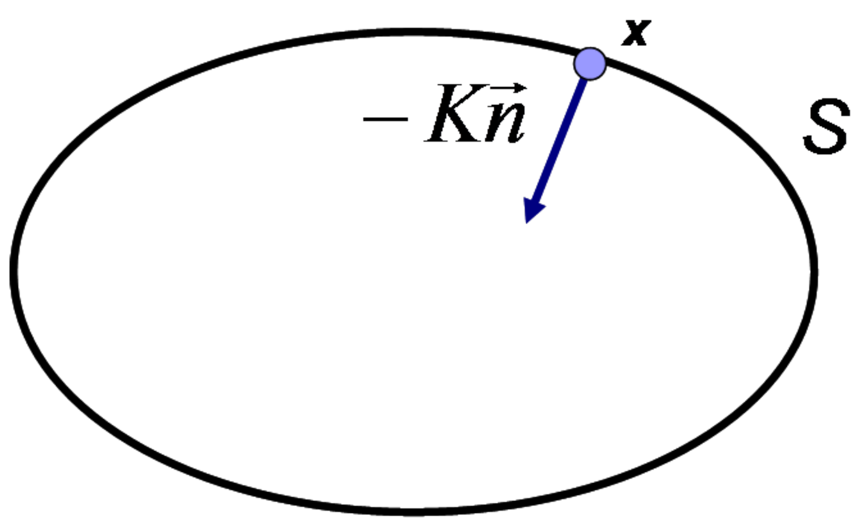
Euclidean Case

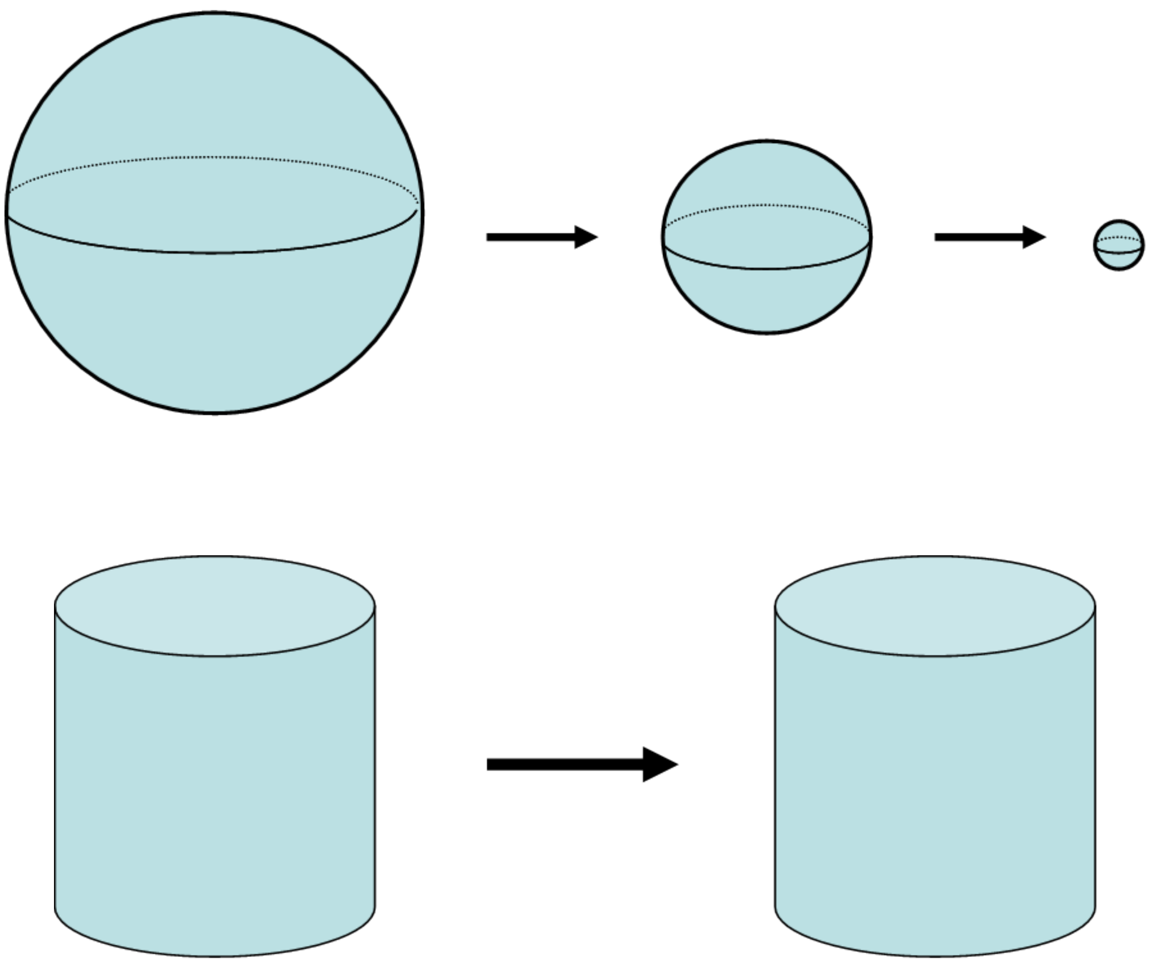
The Big Picture



Goal: Given a hypersurface $S \subset \mathbb{R}^n$, determine how the surface evolves when each point x moves in the direction of its inner unit normal $-\vec{n}$ with speed given by its Gauss curvature K.

Euclidean Case

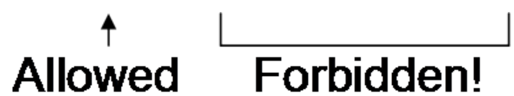
Examples



Carnot Groups

- A Carnot group $\mathbb{G}$ of step $r \geq 1$ is a simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ admits a vector space decomposition in r layers

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_r$$



  having the properties that $[V_1, V_j] = V_{j+1}$, $j = 1, \ldots, r-1$ and $[V_j, V_r] = 0$, $j = 1, \ldots, r$ and which is equipped with a Carnot-Carathéadory metric, d_{CC}, which measures the distance between points as the minimum length of the horizontal paths connecting the points.

- Let $m_j = \dim(V_j)$ and let $X_{i,j}$ denote a left-invariant basis of V_j where $1 \leq j \leq r$ and $1 \leq i \leq m_j$.

Carnot Groups

- Exponential coordinates:

$$exp(xX) = (x_{1,1}, \ldots, x_{m_1,1}, x_{1,2}, \ldots, x_{m_r,r})$$

- Given d_{CC} we can define an equivalent gauge norm on $\mathbb{G}$ given by

$$|x|_g = \left(\sum_{j=1}^{r} \left(\sum_{i=1}^{m_j} |x_{i,j}|^2 \right)^{\frac{r!}{j}} \right)^{\frac{1}{2r!}}.$$

- For each $X_{i,j}$, there exist polynomials $a_{k,(i,j)}(x)$ s.t. we can write

$$X_{i,j}(x) = \sum_{k=1}^{m} a_{k,(i,j)}(x) \frac{\partial}{\partial x_{i,j}}.$$

Carnot Groups - Notation

- The horizontal gradient of u is given by

$$D_0 u = \sum_{i=1}^{m_1} (X_{i,j} u) X_{i,j} \in \mathbb{R}^{m_1}.$$

- The horizontal Hessian of u is given by

$$D_0^2 u = \Big(X_{i,1} X_{j,1} u \Big)_{1 \leq i,j \leq m_1} \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_1}.$$

- The symmetrized horizontal Hessian of u is given by

$$(D_0^2 u)^* = \frac{1}{2} \Big(D_0^2 u + (D_0^2 u)^T \Big).$$

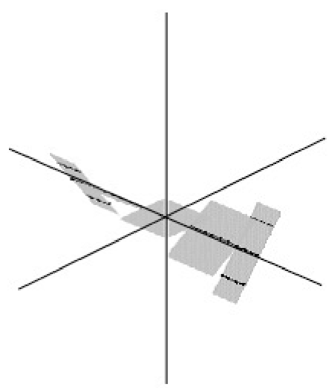
An Example - $\mathbb{H}^1$

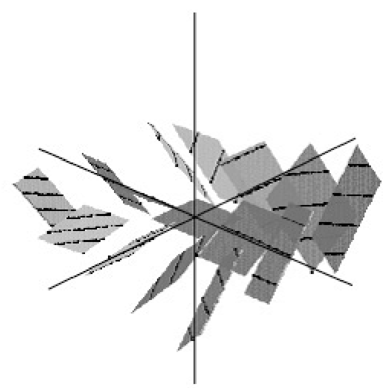
$$X_{1,1} = \frac{\partial}{\partial x_1} - \frac{1}{2} x_2 \frac{\partial}{\partial x_3}$$

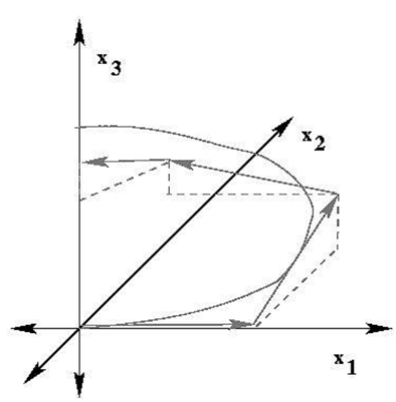
$$X_{2,1} = \frac{\partial}{\partial x_2} + \frac{1}{2} x_1 \frac{\partial}{\partial x_3}$$

$$X_{1,2} = \frac{\partial}{\partial x_3}$$

- $V_1 = \text{Span}\{X_{1,1}, X_{2,1}\}$ and $V_2 = \text{Span}\{X_{1,2}\}$

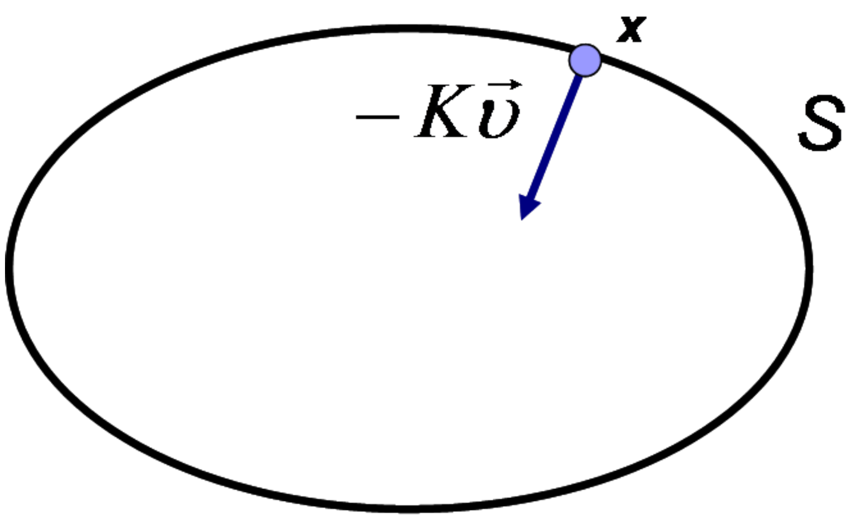






[Reference: *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem* by L. Capogna, D. Danielli, S. Pauls, and J. Tyson]

The Big Picture



Goal: Given a hypersurface $S \subset \mathbb{G}$, determine how the surface evolves when each point x moves in the direction of its inner unit horizontal normal $-\vec{\nu}$ with speed given by K, the sub-Riemannian analog of the Gauss curvature.

Horizontal Gauss Curvature

- $K = \det((II_0)^*)$

- For a sequence II_L of Riemannian second fundamental forms corresponding to a sequence of approximating metrics, II_L restricted to the horizontal tangent space converges to $(II_0)^*$ as $L \to \infty$.

[Reference: *Minimal surfaces, surfaces of constant mean curvature and isoperimetry in Carnot groups* by D. Danielli, N. Garofalo, and D.-M. Nhieu [2001] and *Convexity and a Horizontal Second Fundamental Form for Hypersurfaces in Carnot Groups* by L. Capogna, S. Pauls, and J. Tyson[2006]]

The Level Set Method for Graphs

$$S_0 = \{(x,s) \in \mathbb{G} \times \mathbb{R} : u_0(x) - s = 0\}$$

$$S_t = \{(x,s) \in \mathbb{G} \times \mathbb{R} : u(x,t) - s = 0\}$$

$$\vec{\nu} = \frac{\sum_{i=1}^{m_1} X_{i,1} u X_{i,1} - \frac{\partial}{\partial s}}{\sqrt{1 + |D_0 u|^2}}$$

$$K(x,s,t) \;=\; \det((II_0)^*)$$

The function $u : \mathbb{G} \times [0,\infty) \to \mathbb{R}$ used to describe the surface S_t is the solution to the problem:

$$u_t = \frac{\det\left((D_0^2 u)^*\right)}{\left(\sqrt{1 + |D_0 u|^2}\right)^{m_1+1}} \qquad (1)$$

$$u(x,0) = u_0(x) \qquad (2)$$

Viscosity Solutions

$$u_t = F(D_0 u, (D_0^2 u)^*)$$

Definition. An upper semi-continuous function u is a viscosity subsolution if for all $\varphi \in \Gamma^{2,1}(\mathbb{G} \times [0,\infty))$ such that $u - \varphi$ has a local maximum at $(x_0, t_0) \in \mathbb{G} \times [0,\infty)$ we have

$$\varphi_t(x_0, t_0) \leq F(D_0\varphi(x_0, t_0), (D_0^2\varphi(x_0, t_0))^*).$$

Definition. A lower semi-continuous function u is a viscosity supersolution if for all $\varphi \in \Gamma^{2,1}(\mathbb{R}^n \times [0,\infty))$ such that $u - \varphi$ has a local minimum at $(x_0, t_0) \in \mathbb{G} \times [0,\infty)$ we have

$$\varphi_t(x_0, t_0) \geq F(D_0\varphi(x_0, t_0), (D_0^2\varphi(x_0, t_0))^*).$$

Definition. A function u is a viscosity solution if

$$u^*(x,t) := \lim_{r\downarrow 0} \sup \left\{ u(y,s) : |y^{-1}x|_g + |s-t| \leq r \right\}$$

is a viscosity subsolution and

$$u_*(x,t) := \lim_{r\downarrow 0} \inf \left\{ u(y,s) : |y^{-1}x|_g + |s-t| \leq r \right\}$$

is a viscosity supersolution.

Modified Problem

Our PDE is

$$u_t = \frac{\det\left((D_0^2 u)^*\right)}{\left(\sqrt{1+|D_0 u|^2}\right)^{m_1+1}}$$

Only if M, N are positive semi-definite and $M \leq N$ do we have that

$$F(p, M) \leq F(p, N)$$

Instead we consider,

$$u_t = F(D_0 u, (D_0^2 u)^*) = \frac{\det_+((D_0^2 u)^*)}{\left(\sqrt{1+|D_0 u|^2}\right)^{m_1+1}} \quad (3)$$

where

$$\det_+((D_0^2 u)^*) = \prod_{i=1}^{m_1} \max\{\lambda_i, 0\}.$$

Why don't the Euclidean theorems apply?

$F(D_0 u, (D_0^2 u)^*) = F(x, Du, D^2 u)$ is not degenerate parabolic in the Euclidean sense.

Development of New Viscosity Theory

Main Theorem. Let $h_0 \in C(\mathbb{G})$ be such that

$$h_0(x) \geq \varepsilon_0 |x|_g^{2r!} \, \forall x \in \mathbb{G}$$

for some $\varepsilon_0 > 0$.Let $u_0 \in C(\mathbb{G})$ be such that

$$\sup_{\mathbb{G}} |u_0(x) - h_0(x)| < \infty$$

and for each $\varepsilon \in (0,1)$ there exists a constant $B_\varepsilon > 0$ such that

$$|u_0(x) - u_0(\xi)| \leq \varepsilon + B_\varepsilon h_0(\xi^{-1} x)$$

where $h_0 \in C^2$ is a function satisfying

$$C \geq F(D_0 h_0, (D_0^2 h_0)^*)$$

for some constant $C > 0$ and $h_0(0) = 0$. Then there is a viscosity solution $u \in C(\mathbb{G} \times [0,\infty))$ of

$$\begin{cases} u_t = \dfrac{\det_+((D_0^2 u)^*)}{\left(\sqrt{1+|D_0 u|^2}\right)^{m_1+1}} & \text{in } \mathbb{G} \times (0,\infty) \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{G} \end{cases}$$

Method of Proof

- Perron's Method
  - Given a subsolution u and a supersolution v such that $u \leq v$ on $\mathbb{G} \times (0, \infty)$, Perron's method yields that

    $$w(x,t) := \sup \Big\{ h_-(x,t) : h_- \text{ is a subsolution of (3) such that } h_- \leq v \Big\}$$

    is a solution to (3).
  - The idea of the proof is the same as in the Euclidean case; the technical details are slightly different.
  - Replace the Euclidean derivatives with horizontal derivatives and the Euclidean norm with the gauge norm.

[References for Euclidean proof: *User's Guide to Viscosity Solutions of Second Order Partial Differential Equations* by M. Crandall, H. Ishii, and P. L. Lions, *Surface Evolution Equations: A Level Set Approach* by Y. Giga, and *Uniqueness and Existence of Viscosity Solutions of Generalized Mean Curvature Flow Equations* by Y.-G. Chen, Y. Giga, and S. Goto]

Method of Proof

- Comparison Principle
  - Suppose u is a subsolution and v is a supersolution of (3) such that u and v satisfy appropriate growth conditions on $\mathbb{G} \times [0, \infty)$. If

    $$u(x, 0) \leq v(x, 0) \text{ for all } x \in \mathbb{G}$$

    then

    $$u(x, t) \leq v(x, t) \text{ for all } (x, t) \in \mathbb{G} \times [0, \infty)$$

  - The challenges for the Carnot group case lie here!

- Combining these we are able to obtain the desired continuous solution.

Comparison Principle - What has been done?

- Existence of viscosity solutions to parabolic equations in Carnot groups has been shown for bounded domains and functions F that satisfy the property that there exists a function $\omega : [0, \infty] \to [0, \infty]$ with $\omega(0+) = 0$ so that

  $$|F(p, M) - F(p, N)| \leq \omega(\|M - N\|)$$

  where

  $\|M-N\| = \sup\{|\lambda| : \lambda$ is an eigenvalue of $M-N\}$

  [Thomas Bieske for $\mathbb{H}^1$, dissertation and Juan Manfredi, 2004] .

- Existence of viscosity solutions to degenerate elliptic equations in Carnot groups has been shown for bounded domains [Changyou Wang, 2003].

Summary of Difficulties

- Our domain $\mathbb{G}$ is unbounded.

- $F(D_0 u, (D_0^2 u)^*) = \frac{\det_+((D_0^2 u)^*)}{\left(\sqrt{1+|D_0 u|^2}\right)^{m_1+1}}$ does not satisfy the property that there exists a function $\omega : [0, \infty] \to [0, \infty]$ with $\omega(0+) = 0$ so that $|F(p, M) - F(p, N)| \leq \omega(\|M - N\|)$

Comparison Principle

Theorem. Let $h_0 \in C(\mathbb{G})$ be such that

$$h_0(x) \geq \varepsilon_0 |x|_g^{2r!} \; \forall x \in \mathbb{G}$$

for some $\varepsilon_0 > 0$. Suppose u is a viscosity subsolution and v is a viscosity supersolution to

$$u_t = \frac{\det_+((D_0^2 u)^*)}{\left(\sqrt{1 + |D_0 u|^2}\right)^{m_1+1}}$$

such that

$$u(x, 0) \leq v(x, 0) \; \forall x \in \mathbb{G}$$

and that for each $T > 0$

$$\sup_{(x,t) \in \mathbb{G} \times [0,T]} (|u(x,t) - h_0(x)| + |v(x,t) - h_0(x)|) < \infty. \quad (4)$$

Then for any $\theta \in (0, 1)$ the inequality $u(x, \theta t) \leq v(x, t)$ holds for all $(x, t) \in \mathbb{G} \times (0, \infty)$

Further,

1. if we assume that u is continuous in t then $u \le v$ on $\mathbb{G} \times [0, \infty)$.

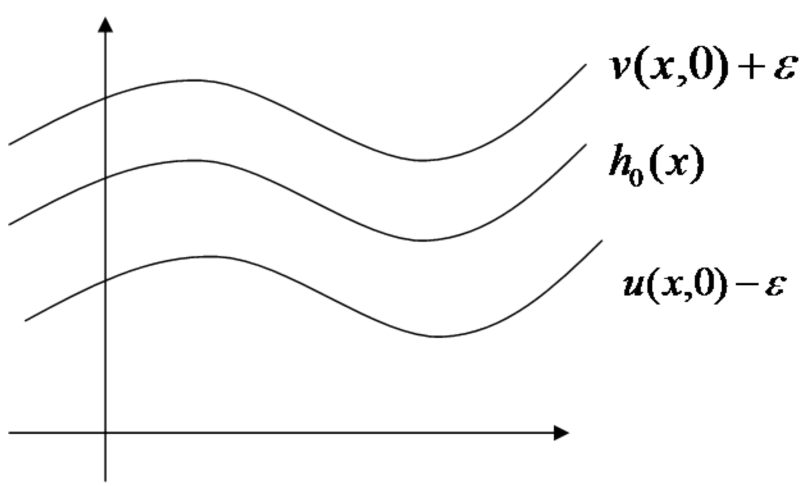
2. for more general viscosity solutions, if we assume that $h_0 \in C^2(\mathbb{G})$ and

   $\det_+ D_0^2 h_0(x) \le C(1+|D_0 h_0(x)|^2)^{(m_1+1)/2}\ \forall x \in \mathbb{G}$

   for some constant $C > 0$ and that for each $\varepsilon > 0$ there exists a constant $R = R(\varepsilon) > 0$ such that for all $x \in \mathbb{G}$, if $|x|_g \ge R$ then

   $$u(x, 0) - \varepsilon \le h_0(x) \le v(x, 0) + \varepsilon,$$

   then $u \le v$ on $\mathbb{G} \times [0, \infty)$



The Comparison Principle Simplified

Let $\Omega \subset \mathbb{G}$ be a bounded domain and $\tilde{u}, v \in C^{2,1}(\Omega)$ such that $\tilde{u}$ is a subsolution and v is a supersolution to $w_t = F(D_0 w, (D_0^2 w)^*)$. Suppose $\tilde{u} \leq v$ on $\Omega \times \{0\}$ and on $\partial\Omega \times [0, T)$. Then $\tilde{u}(x,t) \leq v(x,t)$ on $\Omega \times [0, T)$.

Proof.

Since $\tilde{u} \in C^{2,1}$ is a subsolution, for $\varepsilon > 0$ $u = \tilde{u} - \frac{\varepsilon}{T-t} \in C^{2,1}$ is a subsolution and satisfies

$$u_t \leq F(D_0 u, (D_0^2 u)^*) - \frac{\varepsilon}{T^2}.$$

Since $v \in \Gamma^{2,1}$ is a supersolution, we have

$$v_t \geq F(D_0 v, (D_0^2 v)^*).$$

Suppose

$$\sup_{\Omega \times (0,T)} u(x,t) - v(x,t) = u(\hat{x}, \hat{t}) - v(\hat{x}, \hat{t}) > 0.$$

Then

$$Du(\hat{x},\hat{t}) = Dv(\hat{x},\hat{t})$$

$$u_t(\hat{x},\hat{t}) = v_t(\hat{x},\hat{t})$$

$$D^2u(\hat{x},\hat{t}) \le D^2v(\hat{x},\hat{t})$$

$\Longrightarrow$

$$D_0u(\hat{x},\hat{t}) = D_0v(\hat{x},\hat{t})$$

$$(D_0^2u)^*(\hat{x},\hat{t}) \le (D_0^2)^*v(\hat{x},\hat{t})$$

This, combined with the degenerate parabolicity of F gives the following at $(\hat{x},\hat{t})$

$$\begin{aligned} 0 < \frac{\varepsilon}{T^2} &\le F(D_0u,(D_0^2u)^*) - F(D_0v,(D_0^2v)^*) - u_t + v_t \\ &= F(D_0u,(D_0^2u)^*) - F(D_0v,(D_0^2v)^*) \\ &\le F(D_0v,(D_0^2v)^*) - F(D_0v,(D_0^2v)^*) = 0 \end{aligned}$$

Comparison Principle - Outline

- Require growth conditions on u and v.
- Use sup/inf convolutions of u and v.
- Apply Jensen's Maximum Principle.

Comparison Principle - Proof

- Fix $\theta \in (0,1)$ and $T > 0$. Then we need to show

$$u(x,\theta t) \leq v(x,t)\, \forall (x,t) \in \mathbb{G} \times [0,T).$$

- Let $\mu \in (0,1)$ such that $\theta\mu^{-(m_1-1)} \leq 1$. Then

$$\tilde{w} = \mu u(x,\theta t)$$

is a viscosity subsolution and WLOG we just need to show

$$\tilde{w}(x,t) \leq v(x,t)\, \forall (x,t) \in \mathbb{G} \times [0,T).$$

- By the growth constraints, there exists $C_0 > 0$ such that

$$\tilde{w}(x,t) \leq v(x,t) - (1-\mu)\varepsilon_0 |x|_g^{2r!} + 2C_0.$$

Therefore, there exists $R > 0$ such that

$$\tilde{w}(x,t) \leq v(x,t) \text{ on } (\mathbb{G} \backslash B(0,R)) \times [0,T).$$

- Defining

$$w = \tilde{w} - \frac{\varepsilon}{T-t}$$

we see that w is a subsolution to

$$w_t \le F(D_0 w, (D_0^2 w)^*) - \frac{\varepsilon}{T^2}$$

with the property that

$$\lim_{t \to T} w(x,t) = -\infty \text{ uniformly on } \overline{B(0,R)}.$$

Thus showing our claim for w and letting $\varepsilon \to 0$ will yield the desired result.

- Suppose, to obtain a contradiction, that

$$\sup_{\overline{B(0,R)} \times [0,T)} (w - v) > 0.$$

By construction, this means such a supremum must occur inside the cylinder $B(0,R) \times (0,T)$.

- Let $\varepsilon > 0$ and define for $(x,t) \in \overline{B(0,R)} \times [0,T)$

$$w^\varepsilon(x,t) = \sup_{(y,s)\in\overline{B(0,R)}\times[0,T)} \left\{ w(y,s) - \frac{1}{2\varepsilon}(|y^{-1}x|_g^{2r!} + |t-s|^2) \right\}$$

and

$$v_\varepsilon(x,t) = \inf_{(y,s)\in\overline{B(0,R)}\times[0,T)} \left\{ v(y,s) + \frac{1}{2\varepsilon}(|y^{-1}x|_g^{2r!} + |t-s|^2) \right\}$$

- Wang showed that $w^\varepsilon, -v_\varepsilon$ are semi-convex, $w^\varepsilon, -v_\varepsilon$ are Lipschitz with respect to $|\cdot|_g + |\cdot|$, $w^\varepsilon \to w$, $v_\varepsilon \to v$ pointwise as $\varepsilon \to 0$, and that w^ε is a subsolution and v_ε is a supersolution on $B_\varepsilon \times [0,T)_\varepsilon$ where

$$B_\varepsilon = \{x \in B(0,R) : \inf_{y\in\partial B(0,R)} : |y^{-1}x|_g^{2r!} \geq (1+2R_0)\varepsilon\}$$

$$[0,T)_\varepsilon = \{t \in [0,T) : \inf_{s\in\{0,T\}} : |t-s|^2 \geq (1+2R_0)\varepsilon\}$$

with

$$R_0 = \max\{\|w\|_{L^\infty(B(0,R)\times(0,T))}, \|v\|_{L^\infty(B(0,R)\times(0,T))}\}.$$

- For ε small enough and k large enough,

$$\sup_{\overline{B(0,R)}\times[0,T)} (w^{\varepsilon} - v_{\varepsilon} - \frac{1}{k}t) > 0.$$

occurs inside the cylinder $B(0,R) \times (0,T)$ at a point (x_k, t_k).

- It follows that for all $\sigma > 0$

$$\max_{\overline{B(0,R)}\times[0,T)} \left(w^{\varepsilon}(x,t) - v_{\varepsilon}(x,t) - \frac{1}{\sigma}\left(|x_k^{-1}x|_g^{2r!} + |t-t_k|^2\right) - \frac{1}{k}t \right)$$

occurs at $(x_k^{\sigma}, t_k^{\sigma}) \in B_{\varepsilon} \times [0,T)_{\varepsilon}$ for ε small enough.

- Noticing that this function is semi-convex for each σ and each ε small enough, we may apply Jensen's maximum principle to obtain a sequence

$$\{(y_l^{\sigma,k}, s_l^{\sigma,k})\} \text{ such that}$$

$$(y_l^{\sigma,k}, s_l^{\sigma,k}) \to (x_k^{\sigma}, t_k^{\sigma}) \text{ as } l \to \infty$$

and a sequence

$$\{(a_l^{\sigma,k}, b_l^{\sigma,k})\} \text{ satisfying } |a_l^{\sigma,k}|_g \leq \frac{1}{l}, |b_l^{\sigma,k}| \leq \frac{1}{l}$$

such that

$$w^{\varepsilon}(x,t) - v_{\varepsilon}(x,t) - \frac{1}{\sigma}(|x_k^{-1}x|_g^{2r!} + |t-t_k|^2) + < a_l^{\sigma,k}, x >_g + b_l^{\sigma,k} t - \frac{1}{k} t$$

attains a maximum at $(y_l^{\sigma,k}, s_l^{\sigma,k})$ and is twice differentiable (in the Euclidean sense) at $(y_l^{\sigma,k}, s_l^{\sigma,k})$.

- Computing derivatives and utilizing the parabolic nature of F and the boundedness of our cylinder, we obtain

$$
\begin{aligned}
0<\delta \;\; & \leq \;\; F\Bigg(D_0 w^{\varepsilon}(y_l^{\sigma,k}, s_l^{\sigma,k}) - \frac{1}{\sigma} D_0(|x_k^{-1} y_l^{\sigma,k}|^{2r!}) - (a_l^{\sigma,k})_H, \\
& \qquad \Big(D_0^2 w^{\varepsilon}(y_l^{\sigma,k}, s_l^{\sigma,k})\Big)^* - \frac{1}{\sigma}\Big(D_0^2(|x_k^{-1} y_l^{\sigma,k}|^{2r!})\Big)^*\Bigg) \\
& - \;\; F\left(D_0 w^{\varepsilon}(y_l^{\sigma,k}, s_l^{\sigma,k}), \Big(D_0^2 w^{\varepsilon}(y_l^{\sigma,k}, s_l^{\sigma,k})\Big)^*\right) \\
& - \;\; \frac{1}{k} + b_l^{\sigma,k} - \frac{2}{\sigma}(s_l^{\sigma,k} - t_k)
\end{aligned}
$$

- Taking the limits as $k, l, \sigma \to \infty$, we obtain a contradiction.

Open Problems

- Regularity of solutions (for graphs)

- Comparison principle, existence, etc. for general level sets

- Gauss curvature flow for more general sub-Riemannian structures

  - Boundaries of strictly pseudoconvex domains in $\mathbb{C}^n$

  - Roto-translation group