One-dimensional dynamics of gel swelling

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Outline

1. The model
2. The Cauchy problem
3. From free boundary problem to fixed boundary problem
4. Local wellposedness
5. Classical solutions in large time

Governing equation of gels

$$
\begin{cases}
\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = \nabla\cdot\mathcal{T}, \\
\frac{\partial \mathbf{U}}{\partial t} + (1-2\phi_1)(\nabla\mathbf{U})\mathbf{U} - (\mathbf{U}\otimes\mathbf{U})\nabla\phi_1 + (\nabla\mathbf{V})\mathbf{U} + (\nabla\mathbf{U})\mathbf{V} \\
\quad = \frac{1}{\phi_1}\nabla\cdot\mathcal{T}_1 - \frac{1}{1-\phi_1}\nabla\cdot\mathcal{T}_2 - \frac{\beta}{\phi_1(1-\phi_1)}\mathbf{U} + \frac{\lambda\nabla\phi_1}{\phi_1(1-\phi_1)}, \\
F_t + (\mathbf{V}+(1-\phi_1)\mathbf{U})\cdot\nabla F = \nabla(\mathbf{V}+(1-\phi_1)\mathbf{U})F, \\
\frac{\partial\phi_1}{\partial t} + ((\mathbf{V}+(1-\phi_1)\mathbf{U})\cdot\nabla)\phi_1 + \phi_1\nabla\cdot(\mathbf{V}+(1-\phi_1)\mathbf{U}) = 0, \\
\nabla\cdot\mathbf{V} = 0.
\end{cases}
$$

- ϕ_1: volume fraction of the polymer,
- $\mathbf{V}$: center of mass velocity, $\mathbf{U}$: diffusion velocity,
- $\mathcal{T}$: total stress,
- F: deformation gradient,
- λ: Lagrange multiplier.

1D geometry

$$\Omega = \{(x, y, z) : \; -L \leq x \leq L\}.$$

$$\mathbf{V} = (V(x,t), 0, 0), \; \mathbf{U} = (U(x,t), 0, 0), \; \phi_1 = \phi_1(x,t), \; \lambda = \lambda(x,y,z,t),$$

$$F = \mathrm{diag}(\det F(x,t), 1, 1)$$

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$$F = \mathrm{diag}(\det F(x,t),1,1)$$

The equation for ϕ_1 and U:

$$\begin{cases} \partial_t \phi_1 + \partial_x(\phi_1(1-\phi_1)U) = 0, \\ \partial_t U + \partial_x\Big(\frac{1}{2}U^2(1-2\phi_1) - G(\phi_1)\Big) = -\frac{\beta U}{\phi_1(1-\phi_1)}, \end{cases}$$

where

$$\begin{aligned} G(\phi) =& \frac{K_B T}{V_m N_x}\Big(-\frac{1}{2}\alpha^{2/3}\phi^{-2/3} - (\frac{1}{2}+\frac{N_x}{N_1})\log\phi \\ & + \mu\alpha^2\phi^{-2} - \frac{K_B T\chi}{V_m}\phi + \frac{K_B T}{N_2 V_m}\log(1-\phi)\Big). \end{aligned}$$

$$\begin{cases} \partial_t \phi_1 + \partial_x(\phi_1(1-\phi_1)U) = 0, \\ \partial_t U + \partial_x\left(\frac{1}{2}U^2(1-2\phi_1) - G(\phi_1)\right) = -\frac{\beta U}{\phi_1(1-\phi_1)}, \\ t = 0: \quad \phi_1 = \phi_0, \quad U = U_0. \end{cases} \tag{1}$$

Denote

$$\mathbf{u} = [\phi_1, U]^T,$$

$$\mathbf{F} = [\phi_1(1-\phi_1)U, \frac{1}{2}U^2(1-2\phi_1) - G(\phi_1)]^T$$

$$\mathbf{G} = [0, \frac{\beta U}{\phi_1(1-\phi_1)}]^T.$$

The Cauchy problem is now

$$\begin{cases} \mathbf{u}_t + \mathbf{F}(\mathbf{u})_x + \mathbf{G}(\mathbf{u}) = 0, \\ t = 0: \quad \mathbf{u} = \mathbf{u}_0. \end{cases}$$

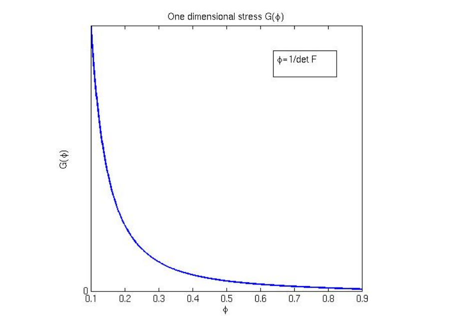
The gradient matrix is

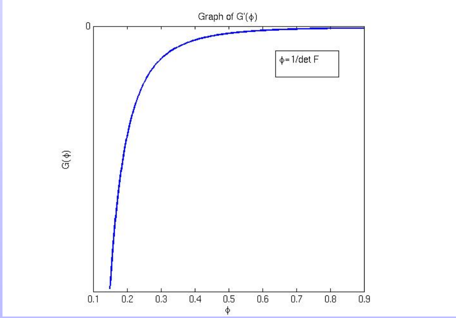
$$D\mathbf{F} = \begin{pmatrix} (1-2\phi_1)U & \phi_1(1-\phi_1) \\ -U^2 - G'(\phi_1) & (1-2\phi_1)U \end{pmatrix}$$

The system is ▸ strictly hyperbolic if

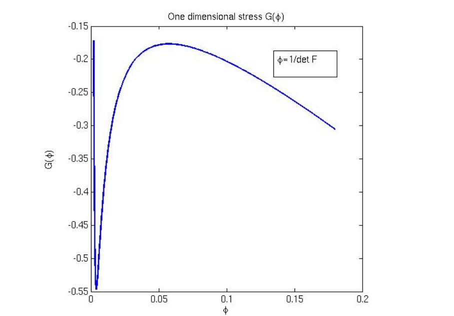
$$U^2 + G'(\phi_1) < 0.$$

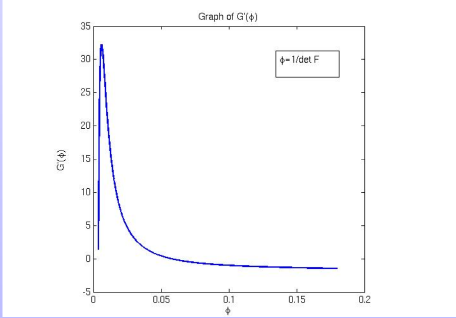
Graphs of G and G' for polymer data





Graphs of G and G' for polysaccharide data





Entropy-entropy flux pair

Entropy: $\eta(\mathbf{u})$, Entropy flux: $q(\mathbf{u})$

$$Dq(\mathbf{u}) = D\eta(\mathbf{u})D\mathbf{F}(\mathbf{u}).$$

Convexity of $\eta(\mathbf{u})$ $\implies$ local wellposedness of Cauchy problem.

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$$\eta = \frac{1}{2}\phi_1(1-\phi_1)U^2 - \int_{\phi_1} G(s)ds$$
$$q = \phi_1(1-\phi_1)U\left[\frac{1}{2}(1-2\phi_1)U^2 - G(\phi_1)\right].$$

$$D^2\eta = \begin{pmatrix} -(G'(\phi_1)+U^2) & (1-2\phi_1)U \\ (1-2\phi_1)U & \phi_1(1-\phi_1) \end{pmatrix} > 0 \text{ for } (\phi_1, U) \text{ near } (\phi^*, 0),$$

$\phi^* \in (0,1)$ any constant.

Global existence of admissible BV solutions

General theory for hyperbolic system of balance laws in 1D.

$$\begin{cases} \mathbf{u}_t + \mathbf{F}(\mathbf{u})_x + \mathbf{G}(\mathbf{u}) = 0, \\ t = 0 : \quad \mathbf{u} = \mathbf{u}_0, \end{cases}$$

with $(\eta(\mathbf{u}), q(\mathbf{u}))$ an entropy-entropy flux pair and η convex.

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Entropy admissibility criterion

$$\partial_t \eta(\mathbf{u}) + \partial_x q(\mathbf{u}) + D\eta(\mathbf{u})\mathbf{G}(\mathbf{u}) \le 0.$$

Heuristics

Conservation laws ($\mathbf{G} \equiv 0$)

Global existence

- random choice (Glimm, 1965).
- front tracking (Bressan, 2000).
- vanishing viscosity (Bianchini-Bressan, 2005).

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The stability estimate

$$TV_{(-\infty,\infty)}\mathbf{u}(\cdot,t) \leq a TV_{(-\infty,\infty)}\mathbf{u}_0(\cdot).$$

- blow-up solutions with initial data of large total variation (Jenssen, 2005).

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- local-in-time existence (Dafermos-Hsiao, 1982).

▸ Strongly dissipative source $\mathbf{G}(\mathbf{u}) \implies$ Global existence.

- random choice (Dafermos-Hsiao, 1982).
- front tracking (Amadori-Guerra, 1999, 2001).
- vanishing viscosity (Christoforou, 2006).

Gel model

$\mathbf{u}_e = (\phi^*, 0)^T$ with $0 < \phi^* < 1$ a constant.

$$DG(\mathbf{u}_e) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\beta}{\phi^*(1-\phi^*)} \end{pmatrix},$$

$$R(\mathbf{u}_e) = \begin{pmatrix} \sqrt{\frac{\phi^*(1-\phi^*)}{-G'(\phi^*)}} & -\sqrt{\frac{\phi^*(1-\phi^*)}{-G'(\phi^*)}} \\ 1 & 1 \end{pmatrix},$$

$$A = \frac{\beta}{2\phi^*(1-\phi^*)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

G is weakly dissipative.

L^1-stability of $\mathbf{u}_e$

$\exists r, b > 0$ such that any admissible BV solution $\mathbf{u}$, defined on $[0, T), 0 < T \leq \infty$, and taking values in $B_r(\mathbf{u}_e)$ satisfies

$$\|\mathbf{u}(\cdot, t) - \mathbf{u}_e(\cdot, t)\|_{L^1} \leq b\|\mathbf{u}_0(\cdot) - \mathbf{u}_e(\cdot, t)\|_{L^1}, \quad 0 < t \leq T.$$

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Theorem (Dafermos, 2006)

L^1-stability of $\mathbf{u}_e$ + weak dissipativeness of $\mathbf{G}$ $\Longrightarrow$
If $\mathbf{u}_0 \in B_r(\mathbf{u}_e)$, $\|\mathbf{u}_0 - \mathbf{u}_e\|_{L^1}$, $TV_{(-\infty,\infty)}\mathbf{u}_0 < \infty$ small enough, then there exists an admissible global BV solution to the Cauchy problem.

Proposition

The Cauchy problem (1) is L^1-stable at the equilibrium state $(\phi^*, 0)$.

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Theorem (Calderer-Zhang, 2008)

Let r, b be defined to ensure the L^1-stability. Consider initial data (ϕ_0, U_0) taking values in $B_r(\phi^*, 0)$, and

$$\int_{-\infty}^{\infty} |\phi_0(x) - \phi^*| + |U_0(x)| dx,\ TV_{(-\infty,\infty)}|\phi_0(x)| + TV_{(-\infty,\infty)}|U_0(x)| < \infty$$

small enough. The Cauchy problem (1) has an admissible global BV solution.

The free boundary problem

Polymer domain $\Omega(t) = [-S(t), S(t)]$. $S(t)$: free boundary, $S(0) = L$.

Boundary conditions are needed to characterize the degree of permeability of the membrane. There are three types of membrane boundaries (Yamaue-Doi)

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where

$\Pi_2 = \Pi_2(\phi_1, \phi_2)$: mixing components of osmotic pressure

P : pressure of the external solvent

α : permeability constant

$\mathbf{n}$: unit outward normal vector to $\partial\Omega$

1D Gel subject to prescribed boundary force/pressure

On the boundary $\partial\Omega$

$$-\lambda \mathbf{n} + (\sigma_1 + \sigma_2)\mathbf{n} = -P\mathbf{n},$$

where $\sigma_{1,2} = \sigma_{1,2}(\phi_1, \phi_2)$ are stresses.

In the fully permeable case

$$\Pi_2 \mathbf{n} + (\sigma_1 + \sigma_2)\mathbf{n} = 0.$$

Hence can solve for $\phi_1 = \phi^*$ on $S(t)$. That ϕ^* is known as the *saturation volume fraction*.
The constitutive condtion on $S(t)$: $S'(t) = (1 - \phi_1)U$ at $x = S(t)$.

Denote $\phi = \phi_1, u = U$. The free boundary problem for 1D gel-swelling

$$\begin{cases} \phi_t + \Big[\phi(1-\phi)u\Big]_x = 0, \\ u_t + \Big[\frac{1}{2}u^2(1-2\phi) - G(\phi)\Big]_x = -\frac{\beta u}{\phi(1-\phi)}, \\ \phi(x,t) = \phi^*, \text{ at } x = \pm S(t) \\ S(0) = L, S'(t) = [1-\phi(S(t),t)]u(S(t),t) \\ \phi(x,0) = \phi^0, \quad u(x,0) = u^0, \text{ for } -L < x < L. \end{cases}$$

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This system is *strictly hyperbolic* in a vicinity of the equilibrium state $(\phi^*, 0)$ provided that

$$G'(\phi) < 0.$$

Prior results

For the linearized system with compatiblity conditions.

1. Local wellposedness of classical solutions, (Yang-Yi, 2001)
2. Global wellposedness of classical solutions, (Calderer-Zhang, 2008)

From free boundary problem to fixed boundary problem

Perform the following change of coordinates. Let

$$y = \int_{-S(t)}^{x} \phi(z,t)\, dz, \quad \tau = t.$$

$\int_{-S(t)}^{S(t)} \phi\, dz$ gives the total mass of the polymer (normalized to be 1). Then

$$(-S(t), S(t)) \to (0, 1)$$

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Still denote the space-time variables (x, t), the system transformes into

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Let $\psi = 1/\phi$ (hence $\psi > 1$) and let $f(s)$ satisfy that $f'(s) = sG'(s)$. Denote $F(s) = f(1/s)$. The the system becomes

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$$\begin{cases} \begin{pmatrix} \psi \\ u \end{pmatrix}_t + \begin{pmatrix} -(1-\frac{1}{\psi})u \\ -\frac{u^2}{2\psi^2} - F(\psi) \end{pmatrix}_x = \begin{pmatrix} 0 \\ \frac{-\beta u \psi^2}{\psi - 1} \end{pmatrix}, \text{ in } (0,1)\times(0,T) \\ B\begin{pmatrix} \psi \\ u \end{pmatrix} = \begin{pmatrix} \psi^* \\ 0 \end{pmatrix}, \text{ at } x = 0,1 \\ \begin{pmatrix} \psi \\ u \end{pmatrix}\Big|_{t=0} = \begin{pmatrix} \psi^0 \\ u^0 \end{pmatrix}, \text{ for } 0 < x < 1, \end{cases} \tag{2}$$

where $\psi^* = 1/\phi^*$ and $\psi^0 = 1/\phi^0$ and B is the boundary operator.

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note: no condition on u at the boundary!

▸ Existence theorem

The gradient matrix is

$$Q(\psi, u) = \begin{pmatrix} -\frac{u}{\psi^2} & \frac{1-\psi}{\psi} \\ \frac{u^2+G'(1/\psi)}{\psi^3} & -\frac{u}{\psi^2} \end{pmatrix}$$

with eigenvalues

$$\lambda_{1,2}(\psi, u) = \frac{-u \mp \sqrt{\left[u^2 + G'(1/\psi)\right](1-\psi)}}{\psi^2}.$$

and the corresponding right and left eigenvectors

$$R_{1,2}(\psi, u) = \begin{pmatrix} \mp\psi\sqrt{\frac{1-\psi}{u^2+G'(1/\psi)}} \\ 1 \end{pmatrix}, \quad L_{1,2}(\psi, u) = \left(\mp \frac{1}{\psi}\sqrt{\frac{u^2 + G'(1/\psi)}{1-\psi}}, 1\right).$$

Back

Local wellposedness

Conditions that ensure the local wellposedness

(NC) Non-characteristic condition: the matrix $Q(\psi, u)$ is non-singular.

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(N) Normality condition: The boundary matrix B is of constant, maximal rank and

$$\begin{aligned} \mathbb{R}^2 &= \ker B \oplus E^s(Q(\psi, u)) \qquad \text{at } x = 1 \\ &= \ker B \oplus E^u(Q(\psi, u)) \qquad \text{at } x = 0, \end{aligned}$$

where $E^s(Q(\psi, u))$ is the stable subspace of $Q(\psi, u)$ and $E^u(Q(\psi, u))$ is the unstable subspace of $Q(\psi, u)$.

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(UKL) Uniform Kreiss-Lopatinskiĭ condition: There exists $C > 0$ so that

$$\|V\| \leq C\|BV\|$$

for all V in the unstable subspace of $Q^{-1}(\psi, u)$ at $x = 0$ and for all V in the stable of $Q^{-1}(\psi, u)$ at $x = 1$.

Conditions (**NC**) and (**N**) together require that $\lambda_1(\psi, u) < 0 < \lambda_2(\psi, u)$, i.e.

$$u^2 < \frac{1-\psi}{\psi} G'(1/\psi). \tag{3}$$

The (**UKL**) condition is satisfied when there exists a $\gamma > 0$ such that

$$\psi \sqrt{\frac{1-\psi}{u^2 + G'(1/\psi)}} \geq \gamma. \tag{4}$$

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Theorem (Calderer-C)

If $m > 2$ is an integer, then for all $(\psi^0, u^0) \in H^{m+1/2}([0,1]) \times H^{m+1/2}([0,1])$ near $(\psi^*, 0)$, satisfying the compatibility conditions, (3) and (4), then there exists $T > 0$ such that the ▸ IBVP problem (2) *admits a unique solution $u \in H^m([0,1] \times [0,T])$.*

Polymers: $G' < 0$ holds. graph

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Polysaccharides: $G'(\phi_c) = 0$. ▸ graph
If $\phi^* > \phi_c$, interface evolves $\Longrightarrow$ swelling
until ϕ reaches ϕ_c $\Longrightarrow$ ϕ_c: onset of de-swelling

Large-time C^1 solutions

Let $\eta = \psi - \psi^*$. Then system (2) becomes

$$
\begin{cases}
\begin{pmatrix} \eta \\ u \end{pmatrix}_t + Q(\eta, u) \begin{pmatrix} \eta \\ u \end{pmatrix}_x + P(\eta, u) = 0, \text{ in } (0,1) \times (0, T) \\
B \begin{pmatrix} \eta \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ at } x = 0, 1, \\
\begin{pmatrix} \eta \\ u \end{pmatrix}\Big|_{t=0} = \begin{pmatrix} \eta^0 \\ u^0 \end{pmatrix} = \begin{pmatrix} \psi^0 - \psi^* \\ u^0 \end{pmatrix}, \text{ for } 0 < x < 1,
\end{cases}
\tag{5}
$$

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where

$$Q(\eta, u) = \begin{pmatrix} -\frac{u}{(\eta+\psi^*)^2} & \frac{1-(\eta+\psi^*)}{\eta+\psi^*} \\ \frac{u^2+G'(1/(\eta+\psi^*))}{(\eta+\psi^*)^3} & -\frac{u}{(\eta+\psi^*)^2} \end{pmatrix}, \quad P(\eta, u) = \begin{pmatrix} 0 \\ \frac{\beta u(\eta+\psi^*)^2}{(\eta+\psi^*)-1} \end{pmatrix}.$$

theorem

Need: Control on the C^1-norm of (η, u) on some large time interval $[0, T_0)$.

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Let

$$\begin{cases} v_i = L_i(\eta, u)(\eta, u)^T \quad (i = 1, 2), \\ w_i = L_i(\eta, u)(\partial_x \eta, \partial_x u)^T \quad (i = 1, 2), \end{cases}$$

where L_i is the i-th left eigenvector. ▸ eigenvectors

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where L_i is the i-th left eigenvector. ▸ eigenvectors
Then $v = (w_1, v_2)$ and $w = (w_1, w_2)$ satisfy the following system of diagonal form

$$\begin{cases} \partial_t v_i + \lambda_i \partial_x v_i + \kappa(v_1+v_2) = \sum_{j,k=1}^2 c_{ijk} v_j v_k + \sum_{j,k=1}^2 d_{ijk} v_j w_k \\ \\ \partial_t w_i + \lambda_i \partial_x w_i + \kappa(w_1+w_2) = \sum_{j,k=1}^2 \bar{c}_{ijk} w_j v_k + \sum_{j,k=1}^2 \bar{d}_{ijk} v_j w_k, \end{cases}$$

where $\kappa = \frac{\beta(\psi^*)^2}{2(\psi^*-1)} > 0$, $c_{ijk}, d_{ijk}, \bar{c}_{ikj}$ and $\bar{d}_{ijk}$ are continuous functions of (η, u).

The initial condition for v and w:

$$\begin{cases} v_i|_{t=0} = v_i^0 = L_i(\eta^0, u^0)(\eta^0, u^0)^T \quad (i = 1, 2), \\ w_i|_{t=0} = w_i^0 = L_i(\eta^0, u^0)(\partial_x \eta^0, \partial_x u^0)^T \quad (i = 1, 2). \end{cases}$$

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$$\text{At } x = 0,\ 1: \qquad B \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}^{-1} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

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$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = B \begin{pmatrix} \eta \\ u \end{pmatrix}_t = B\Big[- Q(\eta, u) \begin{pmatrix} \eta \\ u \end{pmatrix}_x - P(\eta, u) \Big] = -BQ(\eta, u) \begin{pmatrix} \eta \\ u \end{pmatrix}_x$$

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From local wellposedness result, we can pick some $\delta > 0$ small such that when initial data is small

$$|(v^0, w^0)| \leq \delta, \ \forall\ 0 \leq x \leq 1.$$

then

$$|(v, w)(x, t)| \leq \epsilon. \quad \text{in} \quad (0, 1) \times [0, T]. \tag{6}$$

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Let

$$\lambda_{\min} = \min\{|\lambda_1|, |\lambda_2|\}, \qquad \lambda_{\max} = \max\{|\lambda_1|, |\lambda_2|\},$$
$$T_1 = 1/\lambda_{\max}, \qquad T_2 = 1/\lambda_{\min}.$$

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Show: Solution can be pushed further to $(0, 1) \times [0, T + T_1]$ with the same estimate (6).

The i-th characteristic $\xi = f_i(\tau; x, t)$ passing through (x, t):

$$\begin{cases} \frac{d}{d\tau} f_i(\tau; x, t) = \lambda_i(\tau, f_i(\tau; x, t)), \\ \tau = t: \quad f_i(t; x, t) = x. \end{cases}$$

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Consider $v(x, t), \quad t \in [T, T + T_1]$. There are two possibilities:

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e^{\kappa t} v_1(x,t) = v_1^0(f_1(0;x,t)) - \int_0^t \kappa e^{\tau} v_2(f_1(\tau;x,t),\tau)\,d\tau \\
- \int_0^t e^{\tau} Q_1(v,w)(f_1(\tau;x,t),\tau)\,d\tau.
\end{aligned}
$$

Let

$$
\begin{aligned}
&V_i(t) = \max_{0\le x\le 1} |v_i(x,t)|, \quad W_i(t) = \max_{0\le x\le 1} |w_i(x,t)|, \quad i = 1,2, \\
&U_1(t) = \max\{V_1(t), V_2(t)\}, \qquad U_2(t) = \max\{W_1(t), W_2(t)\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
e^{\kappa t}|v_1(x,t)| \leq \; & \delta + (e^{\kappa T} - 1)\epsilon + C\epsilon^2 + \int_T^t \kappa e^{\kappa\tau} V_2(\tau) d\tau \\
& + C \int_T^t e^{\kappa\tau} \sum_{i=1}^{2} V_i^2(\tau) + W_i^2(\tau) d\tau \\
\leq \; & \delta + (e^{\kappa T} - 1)\epsilon + C\epsilon^2 + \int_T^t \kappa e^{\kappa\tau} U_1(\tau) d\tau \\
& + C \int_T^t e^{\kappa\tau} (U_1^2(\tau) + U_2^2(\tau)) d\tau,
\end{aligned}
$$

(2) The first characteristic $\xi = f_1(\tau; x, t)$ intersects the boundary $x = 1$ at point $(1, \tau_1(x, t))$ where $(\tau_1(x, t))$ satisfies

$$f_1(\tau_1(x, t); x, t) = 1.$$

$$T_2 \geq t - \tau_1(x, t) \geq 0.$$

(2a) This second characteristic intersects the interval $[0, 1]$ on the x-axis with the intersection point $(f_2(0; 1, \tau_1(x, t)), 0)$.

$$e^{\kappa t} v_1(x, t) = e^{\kappa \tau_1} v_1(1, \tau_1) - \int_{\tau_1}^{t} \kappa e^{\kappa \tau} v_2(f_1(\tau)) d\tau - \int_{\tau_1}^{t} e^{\kappa \tau} Q_1(v, w)(f_1(\tau)) d\tau$$

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Therefore also

$$\begin{aligned} e^{\kappa t}|v_1(x,t)| \leq \ & \delta + (e^{\kappa T}-1)\epsilon + C\epsilon^2 + \int_T^t \kappa e^{\kappa\tau}U_1(\tau)d\tau \\ & + C\int_T^t e^{\kappa\tau}(U_1^2(\tau)+U_2^2(\tau))d\tau. \end{aligned}$$

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▸ proof

(2b) This second characteristic intersects the boundary $x = 0$ at the intersection point $(0, \tau_{12}(x, t))$. We have

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$$T_2 \geq \tau_1(x, t) - \tau_{12}(x, t) \geq 0, \quad 2T_2 \geq t - \tau_{12}(x, t) \geq T_1.$$

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Hence for $T \leq t \leq T + T_1$,

$$0 \leq \tau_{12}(x,t) \leq T, \quad T - \tau_{12}(x,t) \leq 2T_2.$$

Then using the boundary conditions

$$\begin{aligned}
e^{\kappa t} v_1(x,t) &= e^{\kappa \tau_1} v_1(1,\tau_1) - \int_{\tau_1}^{t} \kappa e^{\kappa \tau} v_2(f_1(\tau))d\tau - \int_{\tau_1}^{t} e^{\kappa \tau} Q_1(v,w)(f_1(\tau))d\tau \\
&= e^{\kappa \tau_1} v_2(1,\tau_1) - \int_{\tau_1}^{t} \kappa e^{\kappa \tau} v_2(f_1(\tau))d\tau - \int_{\tau_1}^{t} e^{\kappa \tau} Q_1(v,w)(f_1(\tau))d\tau \\
&= e^{\kappa \tau_{12}} v_2(0,\tau_{12}) - \int_{\tau_{12}}^{\tau_1} \kappa e^{\kappa \tau} v_1(f_2(\tau),\tau)d\tau - \int_{\tau_1}^{t} \kappa e^{\kappa \tau} v_2(f_1(\tau))d\tau \\
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\end{aligned}
$$

▸ proof

Therefore in terms of U_1 and U_2 we have

$$\begin{aligned} e^{\kappa t}U_1(x,t) \le\ & \delta + (e^{\kappa T}-1)\epsilon + C\epsilon^2 + \int_T^t \kappa e^{\kappa\tau}U_1(\tau)d\tau \\ & + C\int_T^t e^{\kappa\tau}(U_1^2(\tau)+U_2^2(\tau))d\tau, \\ e^{\kappa t}U_2(x,t) \le\ & \delta + (e^{\kappa T}-1)\epsilon + C\epsilon^2 + \int_T^t \kappa e^{\kappa\tau}U_2(\tau)d\tau \\ & + C\int_T^t e^{\kappa\tau}(U_1^2(\tau)+U_2^2(\tau))d\tau. \end{aligned}$$

The control on U_1, U_2

$$U_i(t) \leq \frac{e^{-\kappa T}\left[\delta + (e^{\kappa T} - 1)\epsilon + C\epsilon^2\right]}{1 - Ce^{-\kappa T}\left[\delta + (e^{\kappa T} - 1)\epsilon + C\epsilon^2\right](t - T)}, \quad \forall\, t \in [T, T + T_1].$$

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For any given time T_0, can choose small enough $\epsilon_0 > 0$ so that for any fixed $0 < \epsilon < \epsilon_0$ there exists some $\delta > 0$ small such that

$$U_i(t) \leq \frac{e^{-\kappa T}\delta + (1 - e^{-\kappa T}/2)\epsilon}{1 - C\left[e^{-\kappa T}\delta + (1 - e^{-\kappa T}/2)\epsilon\right] T_1} \leq \epsilon, \quad \forall\ t \in [T, T + T_1].$$

Well-posedness of large-time solutions

Theorem (Calderer-C)

For any fixed $T_0 > 0$*, there exists* $\epsilon_0 > 0$ *so small that for any fixed* $0 < \epsilon < \epsilon_0$ *there exists some* $\delta = \delta(\epsilon) > 0$ *such that if*

$$\|\eta^0\|_{C^1}, \|u^0\|_{C^1} \leq \delta,$$

then the system (5) *admits a unique global* C^1 *solution* (η, u) *with the property that for all* $0 \leq t \leq T_0$*,*

$$\|\eta(\cdot, t)\|_{C^1}, \|u(\cdot, t)\|_{C^1} \leq \epsilon.$$

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The system of balance laws

$$\mathbf{u}_t + \mathbf{F}(\mathbf{u})_x + \mathbf{G}(\mathbf{u}) = 0$$

is called *hyperbolic* if the $n \times n$ matrix $D\mathbf{F}(\mathbf{u})$ has n real eigenvalues (not necessarily distinct) with n linearly independent eigenvectors. If the eigenvalues are distinct, the system is *strictly hyperbolic*.

◂ Back

Strong dissipativeness

$$\mathbf{u}_t + \mathbf{F}(\mathbf{u})_x + \mathbf{G}(\mathbf{u}) = 0$$

In a vicinity of the equilibrium state $\mathbf{u} = \mathbf{u}_e$, consider the matrix

$$A = R^{-1}(\mathbf{u}_e) D\mathbf{G}(\mathbf{u}_e) R(\mathbf{u}_e),$$

where $R(\mathbf{u})$ denotes the matrix with column vectors the right eigenvectors of $D\mathbf{F}(\mathbf{u})$.
$\mathbf{G}$ is *strongly dissipative* when $R(\mathbf{u})$ can be chosen so that A is column diagonally dominant:

$$A_{ii} - \sum_{j \neq i} |A_{ji}| > 0, \quad \forall i.$$

◂ Back

Normality condition

Consider the system of n balance laws

$$\begin{cases} \mathbf{u}_t + \mathbf{Q}\mathbf{u}_x + \mathbf{G}(\mathbf{u}) = 0, \quad 0 < x < 1, t > 0 \\ B\mathbf{u} = \begin{cases} \mathbf{H}_0(t), & \text{at } x = 0, \\ \mathbf{H}_1(t), & \text{at } x = 1, \end{cases} \quad t > 0 \\ \mathbf{u}|_{t=0} = \mathbf{u}^0, \quad 0 < x < 1. \end{cases}$$

Suppose

$$\mathbf{Q} = \text{diag}(\lambda_1, \cdots, \lambda_p, \lambda_{p+1}, \cdots, \lambda_n)$$

so that

$$\lambda_1 \geq \cdots \geq \lambda_p > 0 \geq \lambda_{p+1} \geq \cdots \geq \lambda_n.$$

Then at $x = 0$,

$$u_j(0,t) = u_j^0(-\lambda_j t) - \int_0^t g_j(-\lambda_j(t-\tau),\tau)d\tau, \quad \text{for } j \geq p+1$$

Existence

Let l be a linear form vanishing on $B(\mathbb{R}^p \times \{0_{n-p}\})$, i.e.,

$$lB = (0, \cdots, 0, L_{p+1}, \cdots, L_n).$$

Applying l on the boundary condition at $x = 0$

$$lB\mathbf{u} = \sum_{j=p+1}^{n} L_j u_j(0,t) = l\mathbf{H}_0(t)$$

when $l \neq 0$, this is a non-trivial compatibility condition for data $\mathbf{u}_0, \mathbf{G}, \mathbf{H}_0$. Such a constraint prevents a general existence result from being obtained. Therefore

Existence requires that $p \geq \mathrm{rank} B$

Uniqueness

Take $\mathbf{G} = \mathbf{u}^0 = \mathbf{H}_0 = 0$. Then $u_j = 0, \quad j \geq p+1$. Let $K \in \mathbb{R}^p$ be such that $(K, 0)^T \in \ker B$. We can choose a smooth function v with $v \equiv 0$ on $[0, +\infty]$, then define $u_j, j \leq p$ as

$$u_j(x,t) = v\Big(\frac{x}{\lambda_j} - t\Big)K_j.$$

By uniqueness, we must have $K = 0$. Therefore

Uniqueness requires that $p \leq \text{rank}B$

Hence we conclude that

$$\mathbb{R}^2 = \ker B \oplus E^u(Q) \qquad \text{at } x = 0.$$

Similarly

$$\mathbb{R}^2 = \ker B \oplus E^s(Q) \qquad \text{at } x = 1.$$

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Estimate in case (2a)

(i) If $\tau_1 \leq T$, then similar as in case (1).

Estimate in case (2a)

(i) If $\tau_1 \leq T$, then similar as in case (1).
(ii) If $\tau_1 > T$, then

$$
\begin{aligned}
e^{\kappa t}|v_1(x,t)| \leq\ & \delta + (e^{\kappa T} - 1)\epsilon + C\epsilon^2 + \int_T^{\tau_1} \kappa e^{\kappa \tau} V_1(\tau) d\tau + \int_{\tau_1}^t \kappa e^{\kappa \tau} V_2(\tau) d\tau \\
& + C \int_T^t e^{\kappa \tau} \sum_{i=1}^{2} V_i^2(\tau) + W_i^2(\tau) d\tau \\
\leq\ & \delta + (e^{\kappa T} - 1)\epsilon + C\epsilon^2 + \int_T^t \kappa e^{\kappa \tau} U_1(\tau) d\tau \\
& + C \int_T^t e^{\kappa \tau} (U_1^2(\tau) + U_2^2(\tau)) d\tau.
\end{aligned}
$$

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Estimate in case (2b)

(i) $\tau_1 \leq T$.

$$\begin{aligned} e^{\kappa t}|v_1(x,t)| \leq\ & \delta + (e^{\kappa T} - 1)\epsilon + C\epsilon^2 + \int_T^t \kappa e^{\kappa\tau} V_2(\tau)d\tau \\ & + C\int_T^t e^{\kappa\tau}\sum_{i=1}^{2} V_i^2(\tau) + W_i^2(\tau)d\tau \\ \leq\ & \delta + (e^{\kappa T} - 1)\epsilon + C\epsilon^2 + \int_T^t \kappa e^{\kappa\tau} U_1(\tau)d\tau \\ & + C\int_T^t e^{\kappa\tau}(U_1^2(\tau) + U_2^2(\tau))d\tau. \end{aligned}$$

(ii) $\tau_1 > T$. Then

$$
\begin{aligned}
e^{\kappa t}|v_1(x,t)| \le\ & \delta + (e^{\kappa T} - 1)\epsilon + C\epsilon^2 + \int_T^{\tau_1} \kappa e^{\kappa\tau} V_1(\tau)d\tau + \int_{\tau_1}^t \kappa e^{\kappa\tau} V_2(\tau)d\tau \\
& + C\int_T^t e^{\kappa\tau} \sum_{i=1}^2 V_i^2(\tau) + W_i^2(\tau)d\tau \\
\le\ & \delta + (e^{\kappa T} - 1)\epsilon + C\epsilon^2 + \int_T^t \kappa e^{\kappa\tau} U_1(\tau)d\tau \\
& + C\int_T^t e^{\kappa\tau}(U_1^2(\tau) + U_2^2(\tau))d\tau.
\end{aligned}
$$

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