The impossible tilings

In bathrooms, kitchens and beyond, house tilings are manifestations of a craft that has adorned buildings from ancient Rome to the Islamic world, from Victorian England to colonial Mexico.

In general, the word tiling refers to any pattern that covers a flat surface, like a painting on a canvas, using non-overlapping repetitions of one or more shapes, so that the design does not leave any empty spaces. Contemporary tilings can be found in African-American quilts, Indonesian batiks, molas from the San Blas de Cuna Islands, and Aboriginal paintings. Tiling was a favorite mean of expression for the Dutch artist M. C. Escher (1898-1972). He had this to say about tiling: "... I have embarked on this geometric problem again and again over the years, trying to throw light on different aspects each time. I cannot imagine what my life would be like if this problem had never occurred to me; one might say that I am head over heels in love with it, and I still don't know why." (cite: geis, page 40). In these and other words, Escher repeatedly expressed his love for this art form, acknowledging at the same time the influence on his work of the mosaics he admired and sketched at Moorish buildings in Southern Spain. In spite of this influence, Escher's art went far beyond anything seen before. He produced enigmatic tilings, with strange creatures and mutating landscapes that suggest a craft free from any worldly limitation. This perception however, owned to Escher's mastery, is far from true. Tiling is a very precise art, where no much can be left to chance. Even the simplest tilings fall to the spell of mathematical principles. We can push and turn and wiggle, but if the math is not right, it isn't going to tile.

To see how mathematics can limit the fancy of the best tile installer, we first try our hand at tiling with copies of just one regular polygon, that is to say a shape having sides of the same length and equal angles. One of these regular polygons, a square, is shown in Figure 1(a), while Figures 1(b) and 1(c) show polygons that are not regular. In Figure 1(b) not all the angles are equal, and in Figure 1(c) not all the sides have the same length.

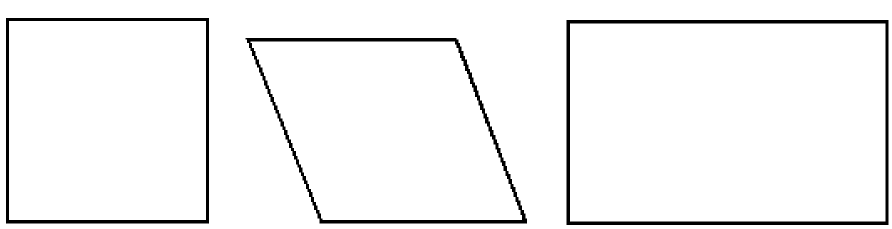


Figure 1(a) Figure 1(b) Figure 1(c)

Not only do we want to use copies of just one regular polygon, but we also want to place them vertex to vertex, that is to say, with the vertices of one copy only touching the vertices of another copy. These tilings are called regular. For instance, in Figure 2, tilings (a), (b) and (c) are regular, while tilings (d), (e) and (f) are not. In fact, the squares in tiling (d) lie in a sliding mood, while tilings (e) and (f) use more than one kind of regular polygon each, regular octagons and squares for (e), regular hexagons and equilateral triangles for (f).

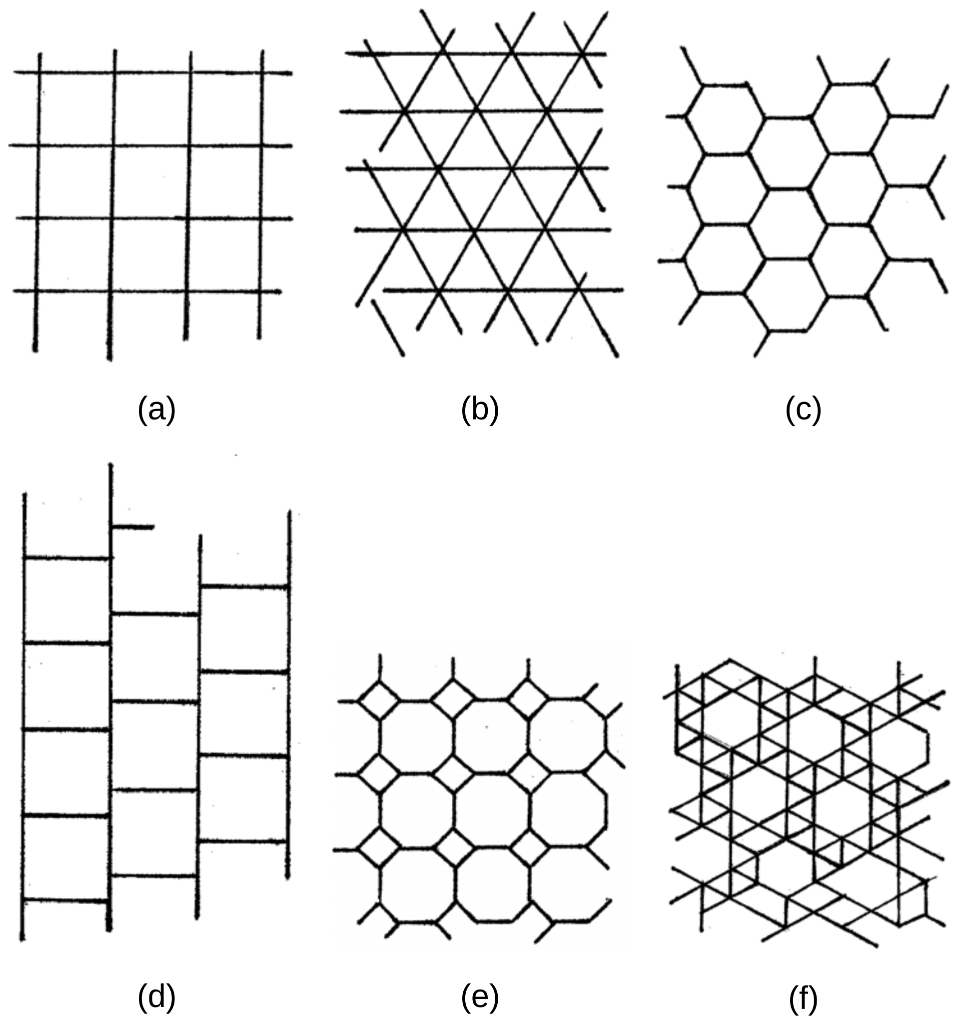


Figure 2

Observe that we are not saying anything about color or other attributes, because we are only concerned with shape and position. As the tilings (a), (b) and (c) in Figure 2 show, squares, equilateral triangles and regular hexagons do make up regular tilings, a fact that was known to Pythagoras's followers in the fifth century B.C. (see.cite: cajori, page 18). But Mother Mathematics says that no other regular polygon can claim the same. Why? If we look at the regular tilings in Figure 2, we can see that the angles meeting at each vertex make up exactly a complete revolution, or 360°. What happens with, say, regular octagons? As Figure 3 illustrates, two regular octagons fall short of a complete revolution, while three regular octagons produce some overlapping. Tiling (e) in Figure 2 shows that the perfect tiling companion of two regular octagons is a smaller square wedged between them.

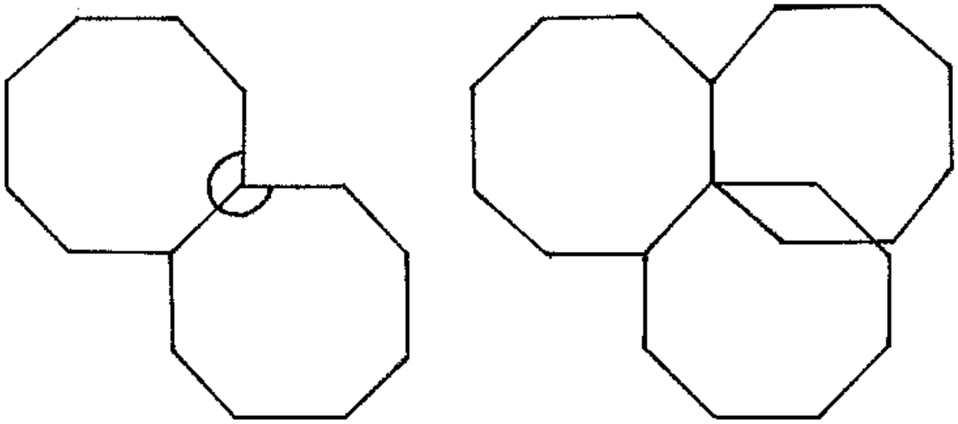


Figure 3

So, our claim about which regular polygons can tile a plane regularly, is really a claim about the size of their angles. To say that a regular polygon will produce a regular tiling of a plane, is the same as saying that the size of its angle divides exactly into 360°. In other words, that the ratio $\frac{360^\circ}{\text{angle}}$ is equal to one of the numbers $1, 2, 3, \ldots$. Let us check this claim with our successful tilings in Figure 2. In the case of the equilateral triangle, we need six of them to complete 360°. Or $360^\circ = 6 \times$ angle, or angle $= 60^\circ$. Likewise, four squares or three regular hexagons complete 360°. So, the angle of a square must be 90° and the angle of a regular hexagon must be 120°. The following table summarizes our findings so far.

Figure 2	angle	$\frac{360^\circ}{\text{angle}}$
(a)	90°	4
(b)	60°	6
(c)	120°	3

So, our claim about the angle dividing exactly into 360° works in these three cases. Now, if we want to show that only equilateral triangles, squares and regular hexagons make up regular tilings, we need to show that the angle of any other regular polygon will not divide exactly into 360°. How do we do this? One way of doing it could be to check each regular polygon, starting with a regular pentagon, continuing with a regular heptagon, regular octagon, on and on. But could this method work? Well, even if we are patient enough to check the first million regular polygons, we will still have a long way to go. More precisely, we still have to check infinitely many regular polygons. So we really haven't made much progress, have we? Instead of trying this case-by-case approach, we will see that a mix of geometry and algebra will do the trick very nicely.

We first use geometry to come up with a formula for the size of the angle of any regular polygon. Let us see how.

Figure 4(a) shows a regular polygon with n sides and angle a. The generic number of sides n can be $3, 4, 5, \ldots$. We haven't completed the picture of our polygon because we do not want to fall into thinking about a particular polygon. Whatever we do, has to work for any regular polygon. In Figure 4(b) we have highlighted one of the n isosceles triangles that compose our polygon.

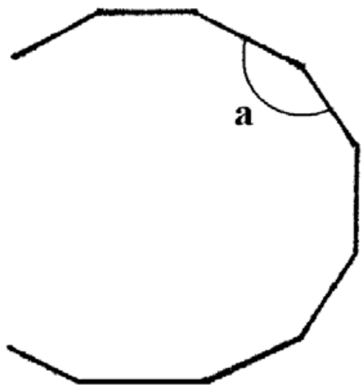


Figure 4(a)

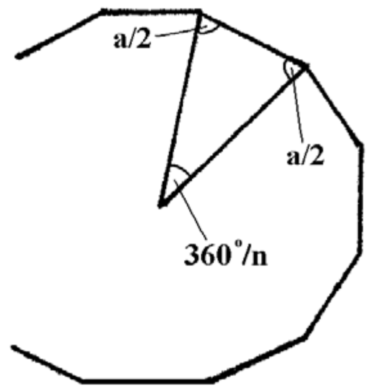


Figure 4(b)

What can we say about the angles of this isosceles triangle? Thinking a little bit about it, we can see that they must be equal to $\frac{a}{2}$, $\frac{a}{2}$ and $\frac{360^\circ}{n}$, as Figure 4(b) suggests. We also know that the sum of these three angles has to be 180°. Or,

$$\frac{a}{2} + \frac{a}{2} + \frac{360^\circ}{n} = 180^\circ$$

If we move things around in this equation, we can write it as

$$a = 180^\circ - \frac{360^\circ}{n} \qquad \#$$

This is the formula we will use for the angle of our regular polygon. Of course, it produces the values we have already found for the angle of an equilateral triangle, $n = 3$, a square, $n = 4$ and a regular hexagon, $n = 6$.

Now, let's remember that our regular polygon will make a regular tiling only when the angle a divides exactly into 360°. This means that we are looking for those regular polygons for which the ratio $\frac{360^\circ}{a}$ equals one of the numbers $1, 2, 3, \ldots$ If we substitute in this condition our newly acquired formula for a, we see that we are looking for those values of n for which $180^\circ - \frac{360^\circ}{n}$ divides exactly into 360°. In other words, the condition becomes

$$\frac{360^\circ}{180^\circ - \frac{360^\circ}{n}} \text{ must be equal to } 1 \text{ or } 2 \text{ or } 3 \text{ or } \ldots$$

We quickly realize that simplifying a bunch of things, our condition reads

$$\frac{1}{\frac{1}{2} - \frac{1}{n}} \text{ must be equal to } 1 \text{ or } 2 \text{ or } 3 \text{ or } \ldots$$

Or,

$$\frac{2n}{n-2} \text{ must be equal to } 1 \text{ or } 2 \text{ or } 3 \text{ or } \ldots$$

At this point, our geometric problem of tiling has become the following algebra problem: To show that $2n$ is divisible by $n-2$ only when $n = 3, 4, 6$. To see what is going on, we calculate the ratio $\frac{2n}{n-2}$ for a few values of n. Here are the results:

n	$\frac{2n}{n-2}$
3	6
4	4
5	$\frac{10}{3}$
6	3
7	$\frac{14}{5}$

We see that for $n = 3, 4, 6$, the mathematics agrees with the reality shown in Figure 2 (a), (b) and (c), which is always a good thing. We also see that regular polygons with 5 or 7 sides do not work. So, to complete our task, here is what we need to do: We need to show that the ratio $\frac{2n}{n-2}$ is not a counting number for any number $n \geq 8$.

Once again, substituting $n = 8, 9, \ldots$ one by one, will not do it. But some algebra will work fine. By division, we can write

$$\frac{2n}{n-2} = 2 + \frac{4}{n-2}$$

Now we only need to see that $\frac{4}{n-2}$ cannot be a counting number, for $n \geq 8$. But this is easy, because when $n = 8$ we have $\frac{4}{8-2} = \frac{2}{3}$ and as n increases, the ratio $\frac{4}{n-2}$ decreases. So there is no way that $\frac{4}{n-2}$ could ever become a counting number, for any $n \geq 8$, and the same must then be true for $2 + \frac{4}{n-2}$. And we are done. We now know for sure that the only regular tilings of the plane using one regular polygon are the first three tilings depicted in Figure 2.

It has been said that mathematicians do not know where to stop, meaning that we always find yet another wrinkle to explore. Here is my new wrinkle: Tiling (d) in Figure 2 shows that we can build a sliding tiling using copies of a square. Is this true for an equilateral triangle? How about a regular hexagon? How about other regular polygons?

Let us consider first the case of equilateral triangles. We can start with two equilateral triangles sitting as in Figure 5(a).

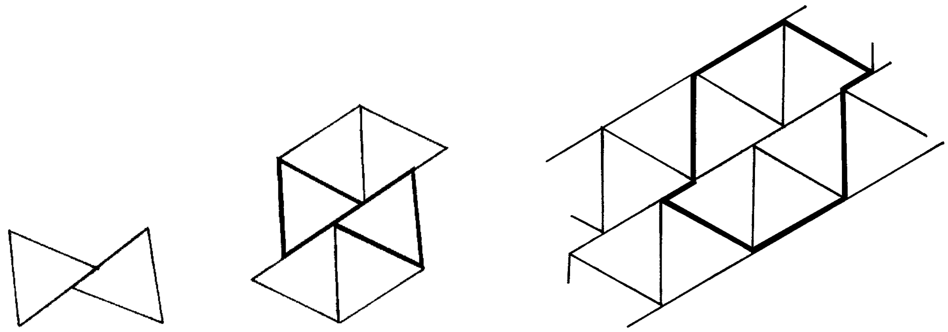


Figure 5(a) Figure 5(b) Figure 5(c)

Since $3 \times 60^\circ = 180^\circ$, we should be able to exactly fit another four copies of the triangle, two above and two below, as shown in Figure 5(b). The result is a regular hexagon broken into two halves, with one half slided along the other. Now we can see what is going on: If we break the regular tiling (b) in Figure 2 along all or some of the horizontal lines, or along all or some of the slanted lines, and if we slide the strips along the fracture lines, we obtain a tiling, like the one shown in Figure 5(c).

We could try to reason in the same way with regular hexagons, but as it happens, doing the same thing does not always guarantee the same outcome. As Figure 6 suggests, the angles refuse to cooperate, leaving annoying empty spots.

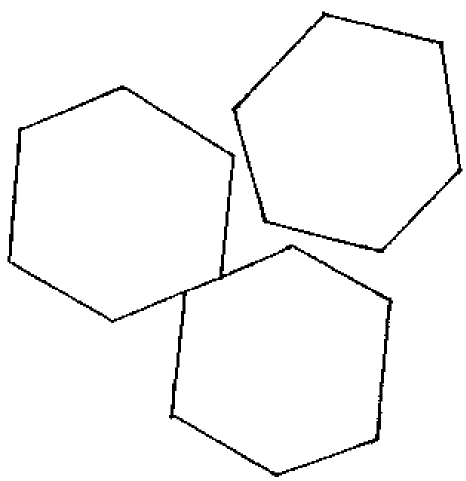


Figure 6

So, the answer for the triangular wrinkle is yes, while the answer for the hexagonal wrinkle is no. What happens with other regular polygons? A moment's reflection will show that a regular polygon will produce one of these sliding tilings only when its angle divides exactly into 180°. Or,

$$\frac{180^\circ}{180^\circ - \frac{360^\circ}{n}} \text{ must be equal to } 1 \text{ or } 2 \text{ or } 3 \text{ or } \ldots$$

Moving around this expression as before, we can see that the viability of the sliding tiling using one regular polygon, goes hand in hand with the truth of the condition

$$\frac{n}{n-2} \text{ must be equal to } 1 \text{ or } 2 \text{ or } 3 \text{ or } \ldots$$

Or, by division,

$$1 + \frac{2}{n-2} \text{ must be equal to } 1 \text{ or } 2 \text{ or } 3 \text{ or } \ldots$$

But $\frac{2}{n-2}$ is a counting number only for $n = 3, 4$. In other words, only equilateral triangles and squares can produce this kind of sliding tiling using just one regular polygon.

You can see how the rules and regulations of mathematics appear very quickly even in the simplest tiling designs. No pentagons in the bathroom floor! No pentagons? Well, thinking of it, we only know that regular pentagons do not work well. But how about if we drop the word regular from the specifications? How about if we just want to tile with copies of one convex pentagon, which means a five sided shape with all the angles less that 180°? If we do, a very different and interesting story will unfold, because there are quite a few convex pentagons that will tile a plane. For instance, how would you like to have one of these patterns in your bathroom floor?

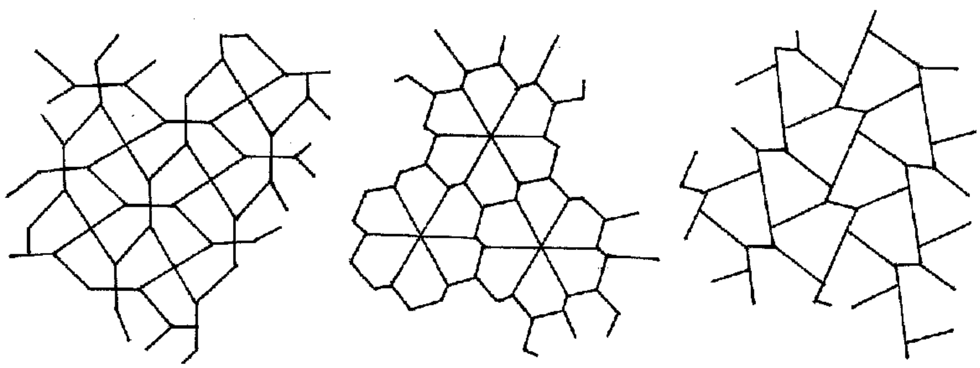
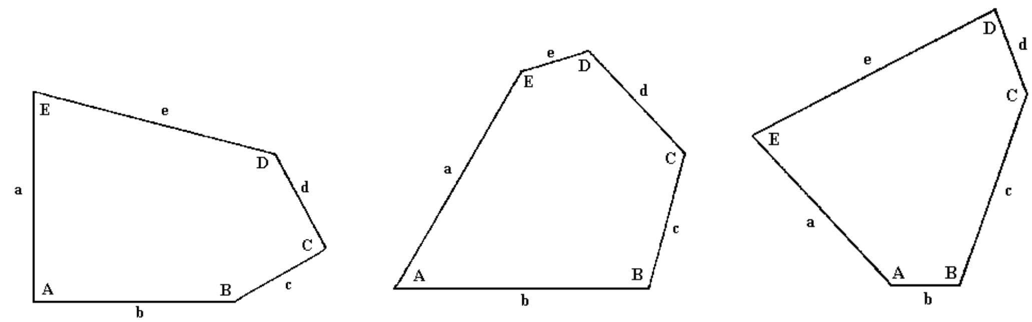


Figure 7

As you can see, each of them consists of copies of just one convex pentagon. Observe that the third tiling is not vertex to vertex, which is fine. For each of these tilings, we highlight in Figure 8 the shape of the convex pentagon tiler and the conditions on its sides and angles.



$A = C = 90°$	$A = 60°, C = 120°$	$B + E = 180°$
$a = b,\ c = d$	$a = b,\ c = d$	$a = c$

Figure 8

How many kinds of convex pentagons can tile the plane? Let me try to count them. Before 1968, it was thought that there were five types, which had been discovered by K. Reinhardt in 1918. But in an article in The American Mathematical Monthly (cite: kershner) published in 1968, the physicist R.B. Kershner presented three new types. He also announced without including a proof that there weren't other kinds of convex pentagons that tiled the plane. The matter seemed to be settled and Kershner's results appeared in the July 1975 issue of Scientific American in Martin Gardner's column *Mathematical Games*. Soon after, Richard James III, a reader and tiling aficionado, sent to Gardner a new type of convex pentagon tiler, which Gardner published in a later issue. Now we had nine. This news caught the attention of another Scientific American reader, Marjorie Rice of San Diego, California, a housewife and mother of five. With no formal mathematical training except for a general course she took in high school, Rice felt compelled to study these patterns and see if she could come up with another type. "It was like a delightful new puzzle for me", Rice recalls. The article by Doris Schattschneider, *In Praise of Amateurs* (cite: doris), chronicles Rice's investigations, which led to the discovery of four new types in the next two years, making a total of thirteen known types. Rice not only found new types, but she also made up her own way of labeling each kind of convex pentagon, which proved to be very fruitful in her investigations. Rolf Stein, a German graduate student discovered in 1985 yet another type. You can read about this new type in Doris Schattschneider's article *A New Pentagon Tiler (*cite: doris1*).* As of November 2003, no new types are known and no one has shown either that the list is complete at fourteen. In other words, the problem of tiling with copies of one convex pentagon remains open. For all of this and much more on pentagonal tilings and the like, you can look into the book *Tilings and Patterns* (cite: grunbaum) by Grünbaum and Shephard. Marjorie Rice's website, titled *Intriguing Tesselations*, has many of her Escher like tilings based on convex pentagons. By the way, tesselation is another word for tiling. It derives from the Latin word tessellare, which means to pave with small tablets, or tesserae. From Rice's website you can access examples of tilings using each of the fourteen known types. How about if you try to come up with other examples of convex pentagon tilers within the known fourteen types? Of course the really serious challenge would be to come up, if possible, with a new type.

Summing up, we know now that one regular polygon makes a regular tiling only when it is an equilateral triangle, a square or a regular hexagon. We also know that non-regular pentagons give a lot of choices and seem quite difficult to handle. How about if we get back to regular polygons, but this time we allow more than one shape and size to be used? We have seen already examples of this kind of tilings: Tiling (e) in Figure 2 uses at each vertex one copy of a square and two copies of a regular octagon, while tiling (f) uses at each vertex one copy of a regular hexagon and four copies of an equilateral triangle. Inspired by these tilings, we demand of our tilings with more than one regular polygon that polygons with the same number of sides have the same size, that vertices meet at vertices and that the same number of polygons of each shape is used at all the vertices. We will call these patterns mixed tilings. Actually, the three regular tilings at the top of Figure 2 could be considered particular cases of mixed tilings, that use at each vertex four copies of a square, or six copies of an equilateral triangle or three copies of a regular hexagon. Could we find all the possible mixed tilings? Yes, we can do it by combining geometry and algebra, with some help from a computer.

To see how, let us say that at each vertex a total of k regular polygons meet, with sides $n_1, n_2, \ldots, n_k$. The basic principle is again that the sum of all the angles meeting at a vertex has to be 360°. Using the formula we found in (ref: angle) for the angle of any regular polygon, we can write the basic principle as

$$180^\circ - \frac{360^\circ}{n_1} + 180^\circ - \frac{360^\circ}{n_2} + \ldots + 180^\circ - \frac{360^\circ}{n_k} = 360^\circ.$$

Getting rid of some common factors, we have

$$1 - \frac{2}{n_1} + 1 - \frac{2}{n_2} + \ldots + 1 - \frac{2}{n_k} = 2.$$

Or, moving things around,

$$\frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_k} = \frac{k-2}{2}. \qquad \#$$

Ouch, this formula seems to have too many unknowns. It would be nice if at least we were able to settle on how many regular polygons we can use. Let us think about this. For starters, the value $k = 1$, or one regular polygon, does not work, since we get the impossible equality

$$\frac{1}{n_1} = -\frac{1}{2}.$$

Two regular polygons, or $k = 2$, will not work either because in this case (ref: mixed) becomes

$$\frac{1}{n_1} + \frac{1}{n_2} = 0,$$

which is again impossible. So a mixed tiling has to use no fewer than three regular polygons at each vertex. Could it use any number of regular polygons, say, three, twenty, thirty, two million? Fortunately the answer is no. More precisely, six regular polygons is the maximum number we may be able to use. Let's see why: Regardless of what the combination of regular polygons might be, we know that each regular polygon will have at least three sides. This means that the numbers $n_1, n_2, \ldots, n_k$ should all be ≥ 3. Using this information in (ref: mixed), we can write

$$\frac{k-2}{2} = \frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_k} \leq \frac{1}{3} + \frac{1}{3} + \ldots + \frac{1}{3} = \frac{k}{3}.$$

So, the number k of regular polygons used at each vertex has to satisfy the condition

$$\frac{k-2}{2} \leq \frac{k}{3}.$$

Solving for k in this inequality we readily conclude that $k \leq 6$. So, it's true, we cannot have seven or more regular polygons meeting at each vertex. Having settled this point, the next step is to see which combinations of regular polygons are allowed. To do this, we need to find, for each value of $k = 3, 4, 5, 6$, all the k-tuples $(n_1, n_2, \ldots, n_k)$ of counting numbers that satisfy (ref: mixed). The answer can be obtained using pretty much any computer algebra system or, of course, checking by hand. I highly recommend the first alternative, but either way, here are all the solutions, numbered **1** to **17**.

	n_1	n_2	n_3		n_1	n_2	n_3	n_4	n_5	n_6
1	3	7	42	**10**	6	6	6			
2	3	8	24	**11**	3	3	4	12		
3	3	9	18	**12**	3	3	6	6		
4	3	10	15	**13**	3	4	4	6		
5	3	12	12	**14**	4	4	4	4		
6	4	5	20	**15**	3	3	3	4	4	
7	4	6	12	**16**	3	3	3	3	6	
8	4	8	8	**17**	3	3	3	3	3	3
9	5	5	10							

For example, this table tells me that solution 1 uses one equilateral triangle, one regular heptagon and one regular polygon with 42 sides. This makes sense because if we substitute in (ref: mixed) $k = 3$, $n_1 = 3$, $n_2 = 7$ and $n_3 = 42$, we get the correct identity

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{42} = \frac{3-2}{2}.$$

If you wish, you can verify in this way each of the solutions displayed in the table.

We notice right a way that solutions **10**, **14** and **17** produce the three regular tilings we obtained before. We could have weeded them out by requesting that not all $n_1, n_2, \ldots, n_k$ are equal. Moreover, solution **8** appears in Figure 2(e) and solution **16** appears in Figure 2(f). So we know that all these combinations are doable. That is to say, they not only work at one vertex, but they can be repeated again and again at every new vertex, producing a mixed tiling of a plane. This is not the case however, with solutions **1, 2, 3, 4, 6** and **9**, as you can easily check. I am just kidding. Regular polygons with so many sides are not that easy to draw, so you can take my word for it. Or rather, you and I can take Maurice Kraitchik's word for it, from his delightful book *Mathematical Recreations* (cite: maurice). Likewise, solution **11** cannot work by itself, but it can be used in combination with other solutions, for instance **5**, **15** and **17**. You can try your hand at drawing some of the tilings resulting from these combinations, or you can see a few samples in Kraitchick's book.

Each one of the other solutions, **5**, **7**, **12**, **13**, **15** and **16**, produces a mixed tiling of the plane. The catch is that the regular polygons can be rearranged in more than one way! For instance, Figure 9 shows two mixed tilings that use solution **12** and Figure 10 shows two mixed tilings that use solution **15**.

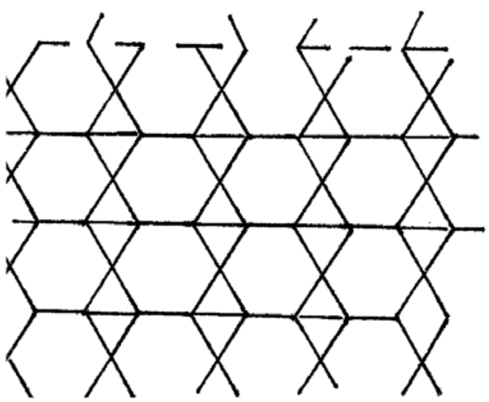


Figure 9(a)

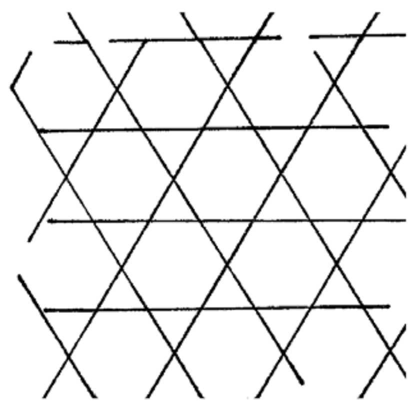


Figure 9(b)

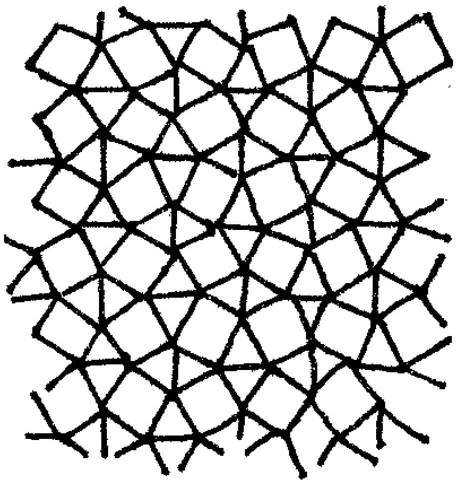


Figure 10(a)

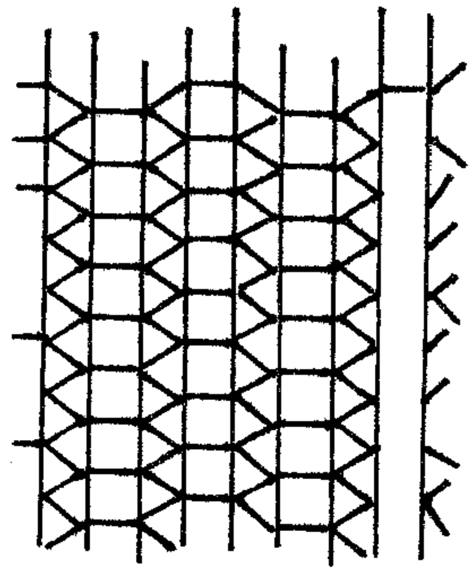
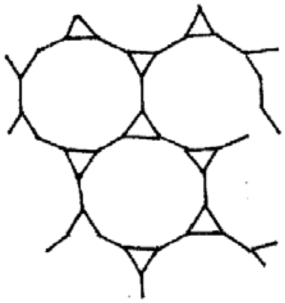


Figure 10(b)

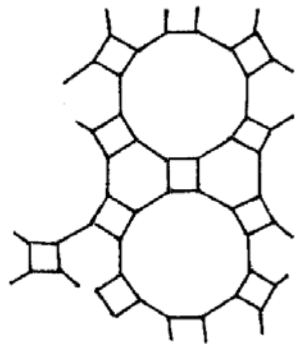
Observe that in searching for mixed tilings, it is not enough that the shapes work well one vertex at the time, but rather, they have to click at all the vertices simultaneously. Recall that we did not encounter this situation when searching for regular tilings.

If we now want to eliminate all but a few special mixed tilings, we can make the following additional requirement: At each vertex, the regular polygons should always appear in the same order. For instance, this is the case in Figure 10(a), because if we go around any vertex, say, counterclockwise, the regular polygons always appear as a chunk of length five of the sequence ...STSTTSTSTTS..., where S and T mean square and triangle, respectively. It is the case also in Figure 10(b), if we now use the sequence ...TTTSSTTTSSTTT.... The same can be said about Figure 9(b), if we consider chunks of length four of the sequence ...HTHTHTHT.... where H stands for hexagon. However, in Figure 9(a), we can see that at some vertices the arrangement is ...THTHTHTHT..., while at others the arrangement is ...TTHHTTHHTT....

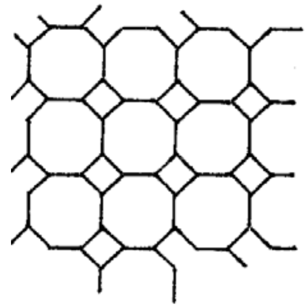
Here is the final scoop: There are eight mixed tilings that keep the same arrangement at every vertex and are not regular tilings. They are called semi-regular tilings, and they correspond to solutions **5**, **7**, **8**, **12**, **13**, **15** (two of them) and **16**. We have them all together in Figure 11.



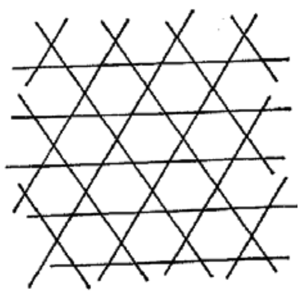
Solution 5



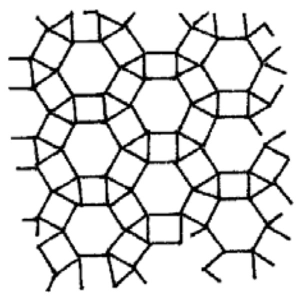
Solution 7



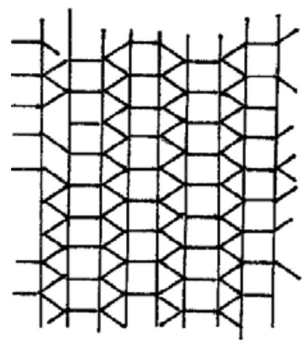
Solution 8



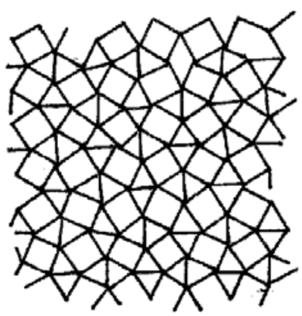
Solution 12



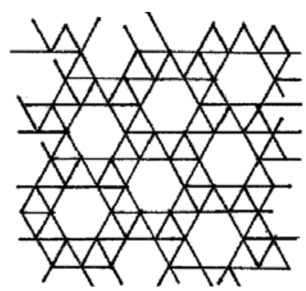
Solution 13



Solution 15



Solution 15



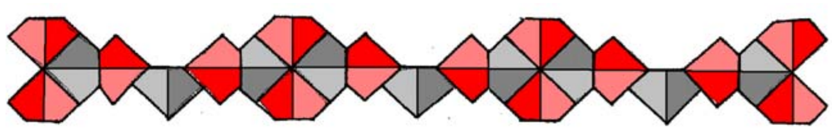
Solution 16

Figure 11

By now, you can see that we could go on forever with this very serious tiling game. What if we allow the polygons to get smaller and smaller? What if we use copies of any triangle or copies of any figure with four sides? What if we want to emulate Escher and try to draw some figurative meaning into the tiles? What if we look for patterns that, in some sense, never repeat? Each of these "what if", and pretty much any other you may imagine, will open up new fascinating possibilities. As a guide for your explorations, I suggest the excellent presentation in Chapter 20 of *For All Practical Purposes* (cite: comap). There you can read, for instance, about the endeavors of one of Escher's tiling pals, the British mathematician Roger Penrose (b. 1931). Penrose has designed non-repeating tilings that now seem to agree with the internal structure of real materials, such as some combinations using aluminum. In a lighter note, Penrose's designs have also caught the attention of a toilet paper manufacturer, because paper embossed with a non-repeating pattern can be rolled without leaving bulging spots. But Penrose had copyrighted the pattern and the manufacturer got a legal spank. Beyond the fun side of tiling and its more, or less, serious applications, Penrose's interest on tiling relates also to his interest in artificial intelligence and the works of computers. For instance, the tiling problem, that is, whether a given bunch of shapes will tile a plane, belongs to a class of mathematical problems called non-recursive. The tiling problem is answerable in each particular case, but, Penrose says, "there is no systematic procedure that, once implemented on a machine, could give an answer in any case, without requiring any more thinking." These and many other issues are discussed in Penrose's controversial books *Shadows of the Mind: A Search for the Missing Science of Consciousness* (cite: penrose1) and *The Emperor's New Mind: Concerning Computers, Minds and the Laws of Physics* (cite: penrose).

Getting back to tiling, let's mention the books *The Magic Mirror of M.C. Escher* (cite: ernst), and *Visions of Symmetry: Notebooks, Periodic Drawings and Related Work of M.C. Escher* (cite: doris2), where the authors present and explain many of Escher's tiling masterpieces. For more examples on how well mathematics and tiling play together, you can look into cite: wichmann. This CD-ROM includes many tilings classified by their artistic and mathematical traits. It also has an extensive list of references to other works on tiling. An Internet search will lead you to several nice computer programs where you can play the tiling game. I also find it very interesting to experiment with shapes cut out of sturdy paper.

From looking at pretty pictures or making them, to doing mathematics, the choice is yours. For now, we draw the line here.



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