

In the Words of Pascal

An Historical Project

We have seen that the summation of integers in an arithmetical progression has a rather concise formula in modern notation

$$\sum_{i=1}^n i = \frac{1}{2}n^2 + \frac{1}{2}n,$$

which greatly simplifies the wordy expression used by the Pythagoreans. Moreover, Archimedes' verbal paragraph for the summation of squares can be written simply as

$$\sum_{i=1}^n i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$

Al-Karaji's work on the summation of cubes, when viewed in terms of the technique of present day induction, reduces to

$$\sum_{i=1}^n i^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2.$$

Thus, the stage is set for Blaise Pascal, who in 1654 wrote¹

Given, starting with the unit, some consecutive numbers, for example 1, 2, 3, 4, one knows, by methods of the Ancients made known to us, how to find the sum of their squares, and also the sum of their cubes; but these methods, applicable only to the second and third degrees, do not extend to higher degrees. In this treatise, I will teach how to calculate not only the sum of squares and of cubes, but also the sum of the fourth powers and those of higher powers up to infinity.

¹Pascal, B., *Oeuvres*, L. Brunschvieg (ed.), Paris, 1908–14; Kraus Reprint, Vaduz, Liechtenstein, 1976, Vol. III, 347–349.

In this project we will examine Pascal's claim about the summation of the fourth, fifth, and higher powers.

(a) Find a formula for $\sum_{i=1}^n i^4$ by using the following steps. First expand

$$(n+1)^5 - n^5$$

and make careful note of the coefficients on the various powers of n . By replacing n with $n-1$ in the above expansion, find an expression for

$$n^5 - (n-1)^5$$

which involves the same coefficients. Apply the same procedure to

$$(n-1)^5 - (n-2)^5.$$

Find an expression for $3^5 - 2^5$ which involves the same coefficients as well. Finally, expand $2^5 - 1^5$ using these coefficients. Add the expansions for

$$\begin{aligned} &(n+1)^5 - n^5 \\ &n^5 - (n-1)^5 \\ &(n-1)^5 - (n-2)^5 \\ &\vdots \\ &3^5 - 2^5 \\ &2^5 - 1^5 \end{aligned}$$

to find an equation which involves $\sum_{i=1}^n i^4$, and then isolate $\sum_{i=1}^n i^4$.

(b) Find a formula for $\sum_{i=1}^n i^5$ by generalizing the ideas in (a) to the next higher power.

(c) In his treatise *Sommation des Puissances Numériques* (*Summation of Numerical Powers*), Pascal gives a lengthy verbal description for $\sum_{i=1}^n i^k$, which in modern notation becomes

$$\begin{aligned} \sum_{i=1}^n i^k &= \frac{1}{k+1} \left[(n+1)^{k+1} - (n+1) - \binom{k+1}{k-1} \sum_{i=1}^n i^{k-1} \right. \\ &\quad \left. - \binom{k+1}{k-2} \sum_{i=1}^n i^{k-2} - \binom{k+1}{k-3} \sum_{i=1}^n i^{k-3} - \dots - \binom{k+1}{1} \sum_{i=1}^n i \right]. \end{aligned}$$

Prove that Pascal's claim about $\sum_{i=1}^n i^k$ is correct by generalizing the arguments in (a) and (b) from fourth and fifth powers to k -th powers.

(d) From our work on the summation of first, second, third, fourth and fifth powers, we can guess that $\sum_{i=1}^n i^k$ is a polynomial of degree $k + 1$ with zero constant term, i.e.

$$\sum_{i=1}^n i^k = a_{k+1}n^{k+1} + a_k n^k + a_{k-1}n^{k-1} + \cdots + a_2 n^2 + a_1 n$$

for some constants $a_1, a_2, \dots, a_k, a_{k+1}$. By using the second principle of mathematical induction on k (strong induction on k), and part (c), prove that

$$\sum_{i=1}^n i^k = a_{k+1}n^{k+1} + a_k n^k + a_{k-1}n^{k-1} + \cdots + a_2 n^2 + a_1 n,$$

where a_1, a_2, \dots, a_{k+1} are constants. You do not need to find formulas for the a_i 's.