In the Words of Archimedes II

An Historical Project

Modern mathematics would write a rather concise statement for Archimedes' wordy implication[1]:

> If a series of any number of lines be given, which exceed one another by an equal amount, and the difference be equal to the least, and if other lines be given equal in number to these, and in quantity to the greatest, the squares on the lines equal to the greatest, plus the square on the greatest and the rectangle contained by the least and the sum of all those exceeding one another by an equal amount will be the triplicate of all the squares on the lines exceeding one another by an equal amount.

This statement might be written today simply as: Given a series of line segments A_1, A_2, A_3, ..., A_n with $A_{i+1} - A_i = A_1$ for $i = 1, 2, 3, \ldots, n-1$, then

$$(n+1)A_n^2 + A_1(A_1 + A_2 + \cdots + A_n) = 3(A_1^2 + A_2^2 + A_3^2 + \cdots + A_n^2).$$

The goal of this project is to prove that this implication is correct.

(a) Write out the equation $A_{i+1} - A_i = A_1$ for $i = 1$, $i = 2$, and $i = n-1$. Show that $A_j = j \cdot A_1$ for $j = 1, 2, 3, \ldots, n$.

(b) Expand $(A_1 + A_{n-1})^2$, $(A_2 + A_{n-2})^2$, ..., $(A_i + A_{n-i})^2$, ..., $(A_{n-1} + A_1)^2$, and add these equations to show that

$$(n-1)A_n^2 = 2(A_1^2 + A_2^2 + \cdots + {A^2}_{n-1}) + 2(A_1A_{n-1} + A_2A_{n-2} + \cdots + A_{n-1}A_1).$$

(c) Explain why

$$(n+1)A_n^2 = 2(A_1^2 + A_2^2 + \cdots + {A^2}_n) + 2(A_1A_{n-1} + A_2A_{n-2} + \cdots + A_{n-1}A_1).$$

[1] Dijksterhuis, E., *Archimedes*, Princeton University Press, Princeton, New Jersey, 1987, 122.

(d) Using results from part I of this project, explain why

$$(A_1 + A_2 + A_3 + \cdots + A_{n-1}) = \frac{1}{2}(n-1)A_n.$$

Use this to conclude that

$$\begin{aligned}&2(A_1A_{n-1} + A_2A_{n-2} + A_3A_{n-3} + \cdots + A_{n-1}A_1)\\&= (n-1)A_1A_n + (n-2)A_1A_{n-1} + (n-3)A_1A_{n-2} + \cdots + A_1A_2.\end{aligned}$$

Hint: Use $A_j = j \cdot A_1$ for certain A_j's.

(e) Show that the final equation in part (d) is equal to

$$(A_1^2 + A_2^2 + A_3^2 + \cdots + A_n^2) - (A_1A_n + A_1A_{n-1} + A_1A_{n-2} + \cdots + A_1A_2 + A_1A_1).$$

Hint: Write $(n-j)A_1$ in terms of the other A's and compare your result to the A_i^2's.

(f) Conclude the final result about

$$3(A_1^2 + A_2^2 + A_3^2 + \cdots + A_n^2).$$