

In the Words of Archimedes

An Historical Project

The culture of ancient Greece witnessed advances in literature, art, philosophy, and mathematics, with the study of each driven by a search for the ideal. Within mathematics, this search found expression in the development of deductive reasoning—a system of logic if you will, where each step of an argument is justified by a commonly agreed upon reason. The Greeks expressed their mathematics verbally and pictorially without the use of modern symbols such as \wedge , \vee , \rightarrow . In this project we examine two very wordy statements, the first known to Pythagoras [c. 572–497 B.C.E.], and the second attributed directly to Archimedes [c. 287–212 B.C.E.]. Both statements deal with the summation of certain integers, and would form the basis for integral calculus, not to be developed until nearly two millennia later.

Consider then the statement:

If a series of any number of lines be given, which exceed one another by an equal amount, and the difference be equal to the least, and if other lines be given equal in number to these, and in quantity to the greatest, then the sum of the lines equal to the greatest plus the greatest will be the duplicate of the sum on the lines exceeding one another by an equal amount.

To our minds, such a wordy sentence seems almost incomprehensible, yet the Pythagoreans used it with ease to calculate sums of certain integers. Help students in the coming 21-st century understand the mathematics of the first century B.C.E. by rewriting the above statement using symbolic logic. In particular:

(a) Identify (at least) five propositions, p_1 , p_2 , p_3 , p_4 , p_5 , which form the hypothesis of the above implication. Write the conclusion of the implication as an equation. You may wish to introduce the notation A_1 , A_2 , A_3 , \dots , A_n for the series of lines. What does n signify? Finally, write the above implication using the symbols \wedge , \vee , \rightarrow .

(b) Prove that the above implication is a true statement by showing that if the hypothesis holds, then the conclusion follows. You may wish to construct a proof in the tradition of ancient Greek geometry by placing the line segments $A_1, A_2, A_3, \dots, A_n$ next to each other, one unit apart. Then place the segments $A_{n-1}, A_{n-2}, A_{n-3}, \dots, A_1$ on top of $A_1, A_2, A_3, \dots, A_{n-1}$ respectively. What can you conclude about the height of each stacked segment? Consider a certain rectangle formed by all the stacked segments. Compute its area in two different ways. Explain carefully how you include the segment A_n in your rectangle.

With (b) in hand, you should have proven a formula for the sum of integers forming an arithmetical progression. Now consider the words of Archimedes¹:

If a series of any number of lines be given, which exceed one another by an equal amount, and the difference be equal to the least, and if other lines be given equal in number to these, and in quantity to the greatest, the squares on the lines equal to the greatest, plus the square on the greatest and the rectangle contained by the least and the sum of all those exceeding one another by an equal amount will be the triplicate of all the squares on the lines exceeding one another by an equal amount.

(c) Identify (at least) five propositions, p_1, p_2, p_3, p_4, p_5 , which form the hypothesis of Archimedes' implication. Write the conclusion of the implication as an equation, using the notation $A_1, A_2, A_3, \dots, A_n$ as before. Finally, write the above implication using the symbols $\wedge, \vee, \rightarrow$.

Archimedes offered the world a novel and precise proof for the above statement, which we will examine at another time. For now:

(d) Show that Archimedes' implication is correct for $n = 2$, and separately for $n = 3$.

¹Dijksterhuis, E., *Archimedes*, Princeton University Press, Princeton, New Jersey, 1987, 122.